MM5023

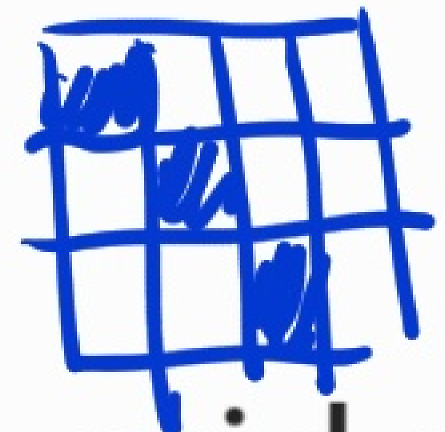
Lecture 3

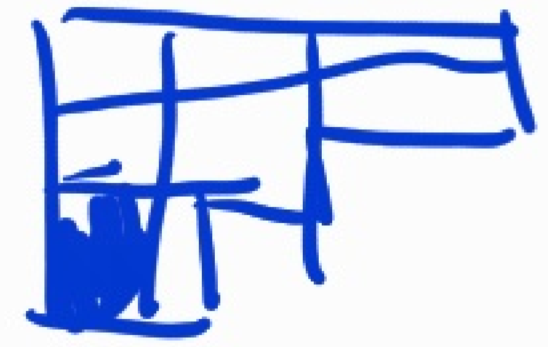
Rook polynomials

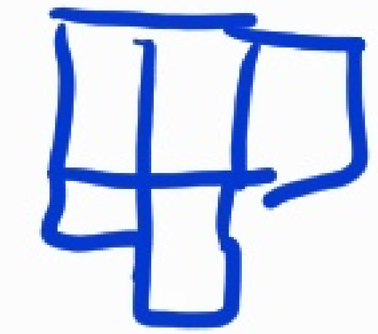
Lecture plan

- Definitions and easy examples
- Finer computational methods
- Examples (old exams)

Definition: Rook numbers







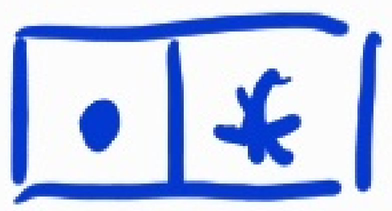
Let C be a chessboard - that is a grid made up of cells, some chells might be shared, and they are to be considered forbidden. For every integer k , the k-th rook number of C

$$r_k(C)$$

is the number of placing k rooks (place holders) on C in such away that every rows and column of C contai~~ner~~n at most 1 rook.

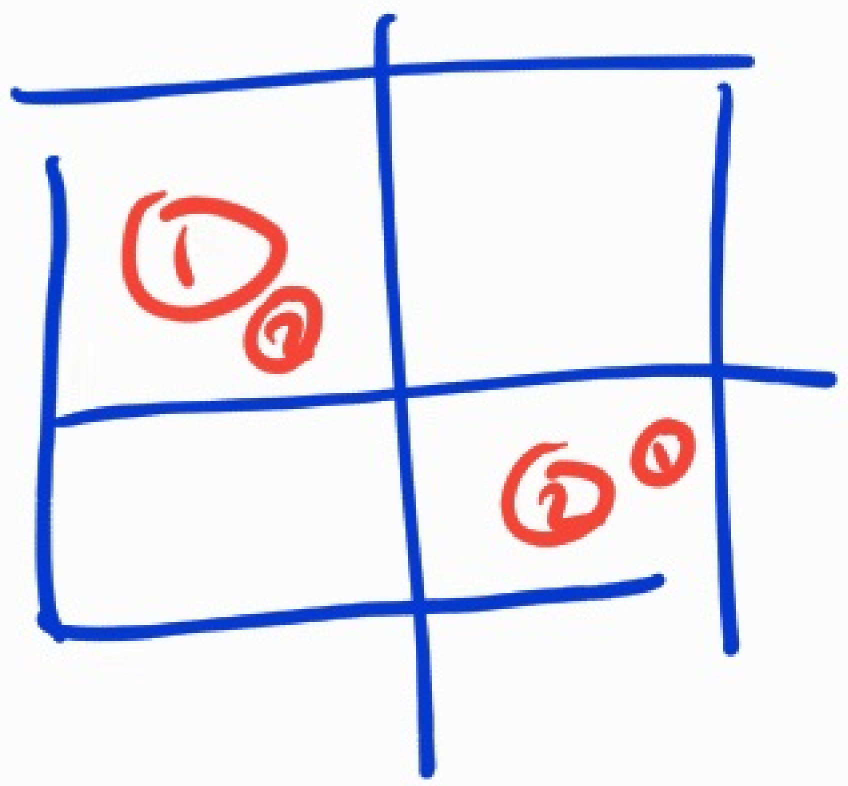
By definition we set $\boxed{r_0(C) = 1}$

Examples



$r_0(C) = 1 \qquad r_1(C) = 2$

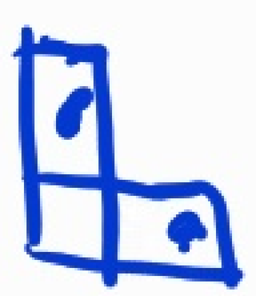
$r_k(C) = 0$ for every $k \geq 2$



$r_0(C) = 1 \qquad r_1(C) = 4$

$r_2(C) = \frac{\boxed{4}\cdot\boxed{1}}{2} = 2 \qquad r_k(C) = 0$

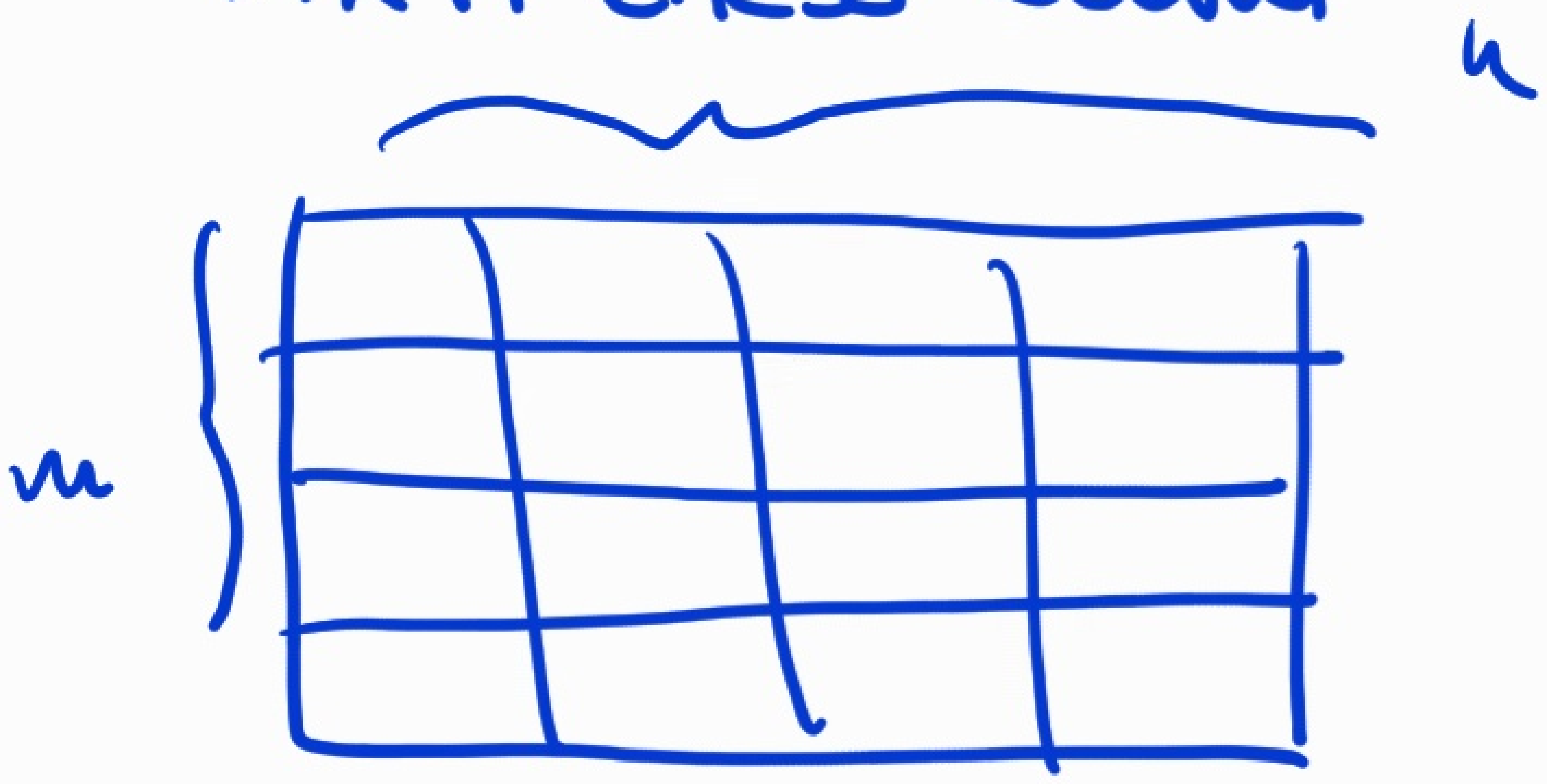
for every $k \geq 3$



$r_0(C) = 1 \qquad r_1(C) = 3 \qquad r_2(C) = 1$

$r_k(C) = 0$ for every $k \geq 3.$

m x n chess board



$$r_k(C) = \binom{n}{k}\binom{m}{k} k!$$

if $k > \min\{n, m\}$ $\quad r_k(C) = 0$ ✓

1. $r_0(C) = \binom{n}{0}\binom{m}{0} 0! = 1$ ✓

Compute $\underline{r_k(C)}$ when $k = 1, 2, \dots, \min\{n, m\}$

I consider the rook numbered

I choose the col of the

$$\left.\begin{array}{ll} \text{first rook} & n \\ 2\ \text{rook} & n-1 \\ & \vdots \\ & n-k+1 \end{array}\right\} \frac{n!}{(n-k)!}$$

I do the same thing with columns $\frac{m!}{(m-k)!}$

$$\Rightarrow \frac{n!}{(n-k)!}\frac{m!}{(m-k)!}\frac{1}{k!}$$

I am dividing by the ways of reordering the rooks

$$\binom{n}{k}\cdot\binom{m}{k}\cdot k!$$

Rook polynomial

Given a chess board C, the rook polynomial of C is

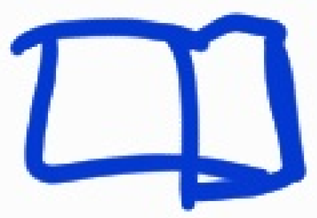
$$r(C,x) = \sum_{k=0}^{\boxed{\infty}} r_k(C)x^k = (*)$$

C is finite $\to$ finitely many rows & cols.

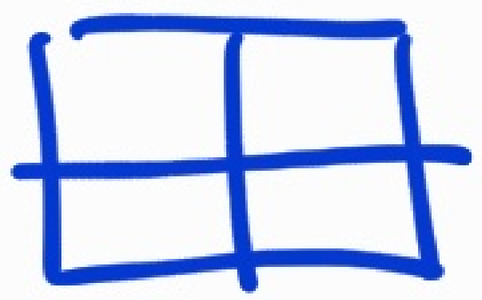
$k > \min\{\#\text{rows}, \#\text{cols}\} = d \quad r_k(C) = 0$

$$(*) = \sum_{k=0}^{d} r_k(c)x^d.$$

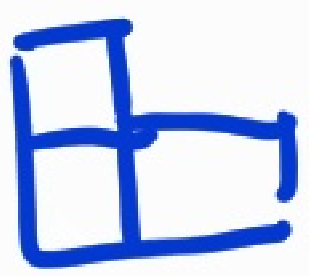
Examples



$$1 + 2x$$

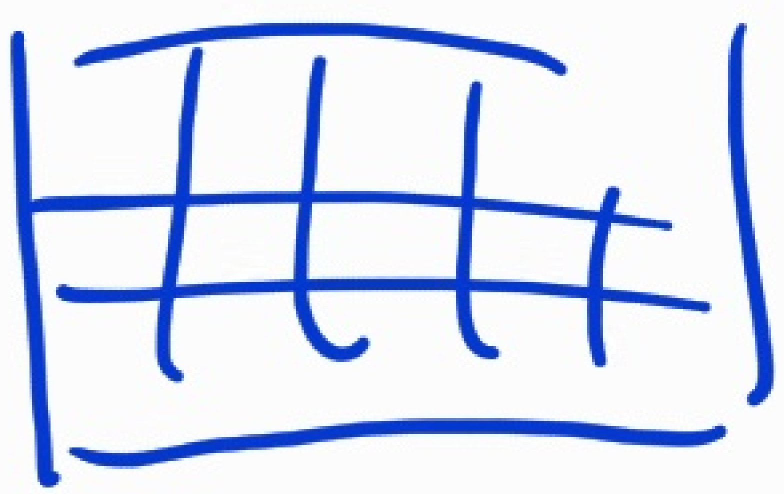


$$1 + 4x + 2x^2$$



$$1 + 3x + 1\cdot x^2$$

$m \times n$



$$\sum_{k=0}^{\min(m,n)} \binom{n}{k}\binom{m}{k}\cdot k!\; x^k$$

Disjoint chessboards

We say that a chessboard C is composed by the <u>disjoint</u> chessboards C_1 and C_2 if

$$C = C_1 \cup C_2$$

And C_1 and C_2 have no cell in the same row or column

<u>Example</u>

Rook polynomial of disjoint chessboards

<u>Proposition</u> If C is composed of C_1 and C_2, disjoint

$$r(C\ x) = r(C_1\ x) \cdot r(C_2\ x)$$

<u>Proof</u> : In order to show that two polynomial are the same we have to show that they have the same coefficient. Let us count the way to place k rooks in C $\qquad r_k(C)$

There are $k+1$ cases.

→ I place i rooks $\quad i = 0 \ldots k$ in C_1

& the other in C_2

By the rule of sum $r_k(C)$ will be the sum of the ways to place a ~~rook in~~ rooks in any of thes cases

Compute a single case

i rooks in C_1 $\rightsquigarrow r_i(C_1)$

$k-i$ ——— C_2 $\rightsquigarrow r_{k-i}(C_2)$

$\Big\}$ indep

bc the chessboard are disjoint

RULE OF PRODUCT $r_i(C_1) \cdot r_{k-i}(C_2)$

$$r_k(c) = \boxed{\sum_{i=0}^{k} r_i(c_1) \cdot r_{k-i}(c_2)}$$

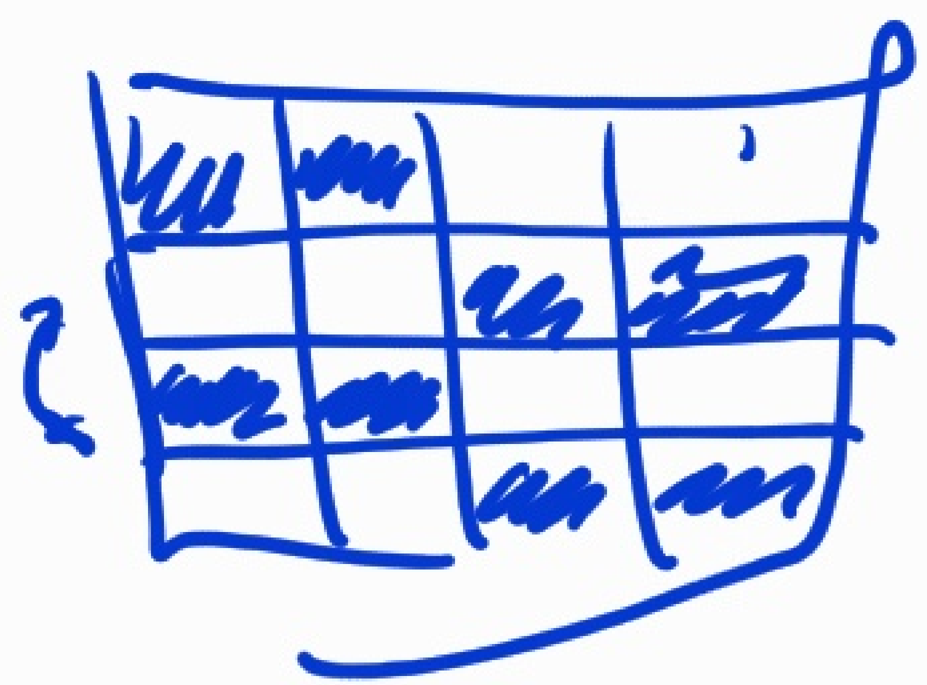
coeff of deg i $r(c_1, x)$

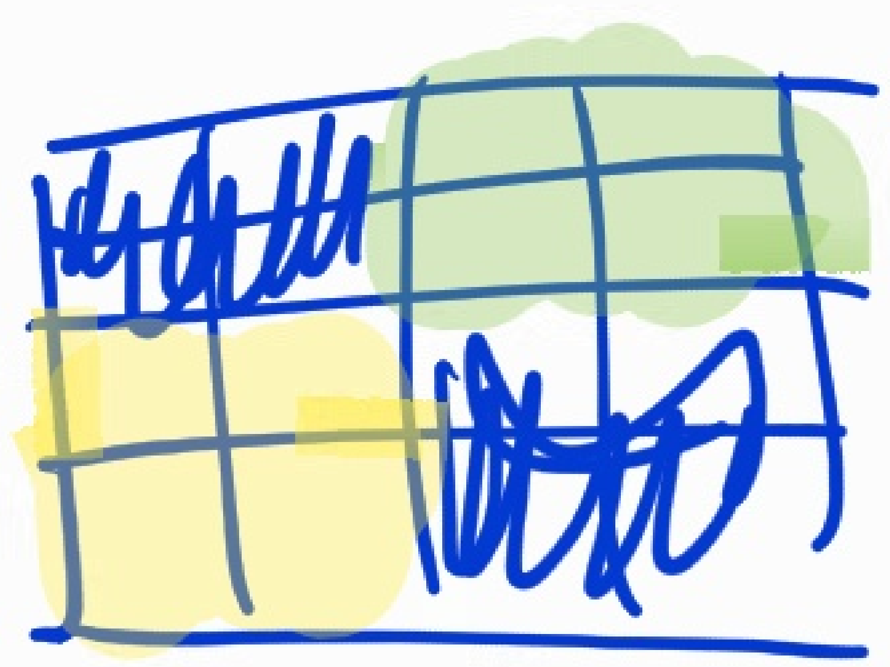
coeff of deg $k-i$ $r(c_2, x)$

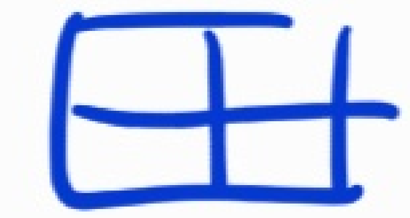
$\to$ coeff of deg k of $r(c_1, x) \cdot r(c_2, x)$!! ☺

Example

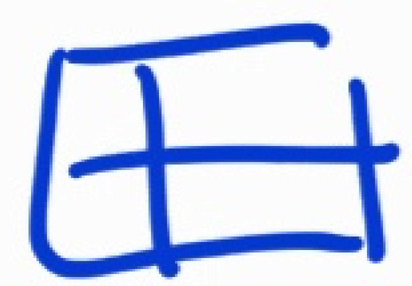
The rook polynomial (also the rook #'s) does not change if I replace C with a chessboard obtained by C by swapping rows or cols





$c_1 = c_2$ 

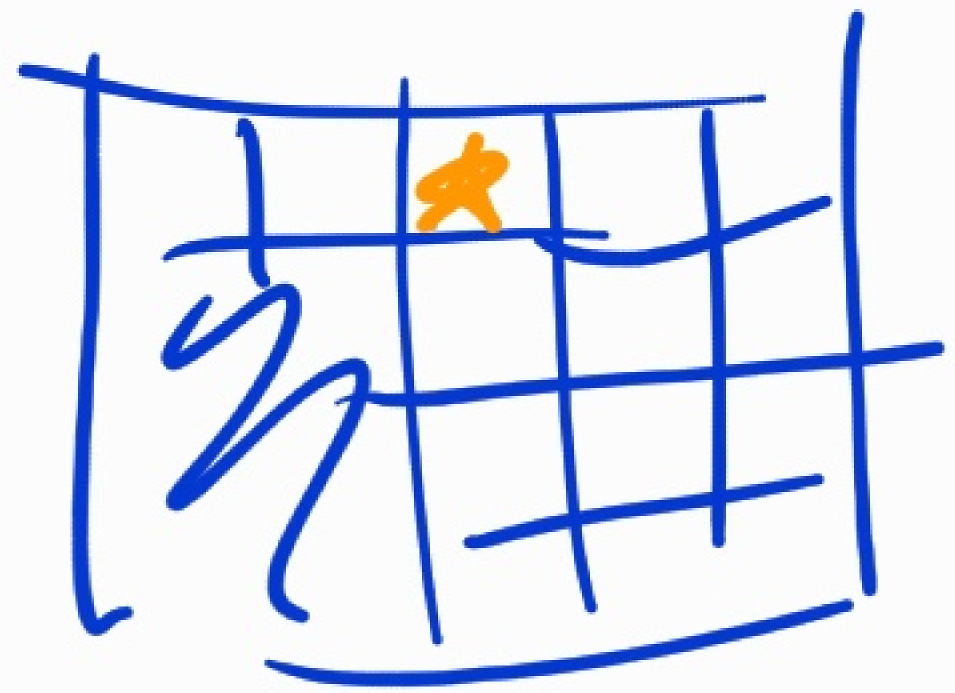
$$r(c,x) = r(c_1\, x) \cdot r(c_2\, x)$$

$= r($  $, x)^2 =$

$$= \left(1 + 4x + \underline{2x^2}\right)^2$$

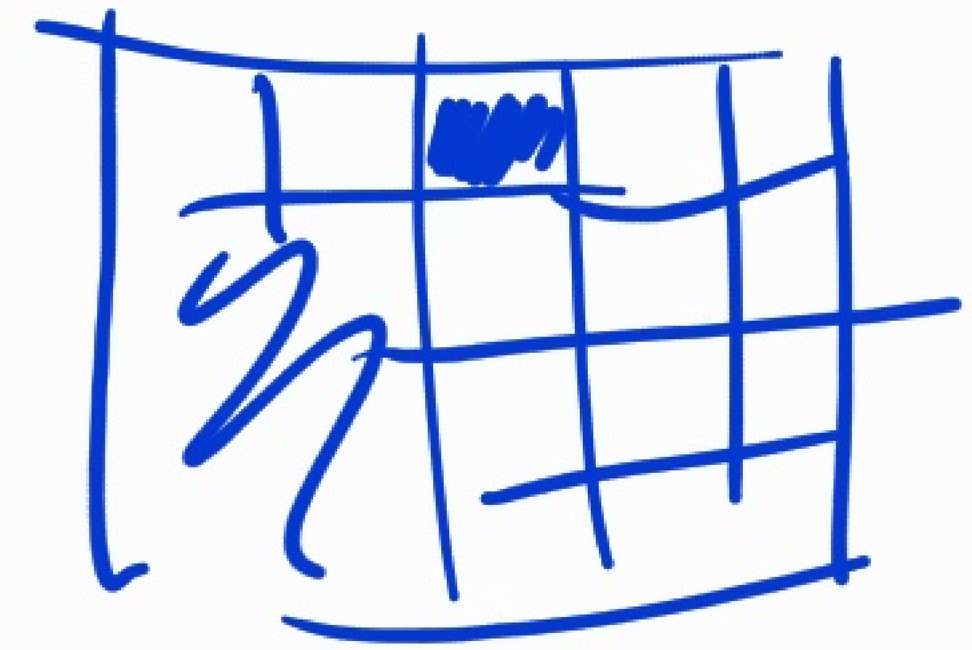
$$= 1 + \underline{16}x^2 + \underline{4}x^4 + \underline{8}x + \underline{4x}^2 + \underline{16x}^3 = \underline{1} + \underline{8}x + 20x^2 + \underline{16}x^3 + \underline{4}x^4$$

A recursive formula

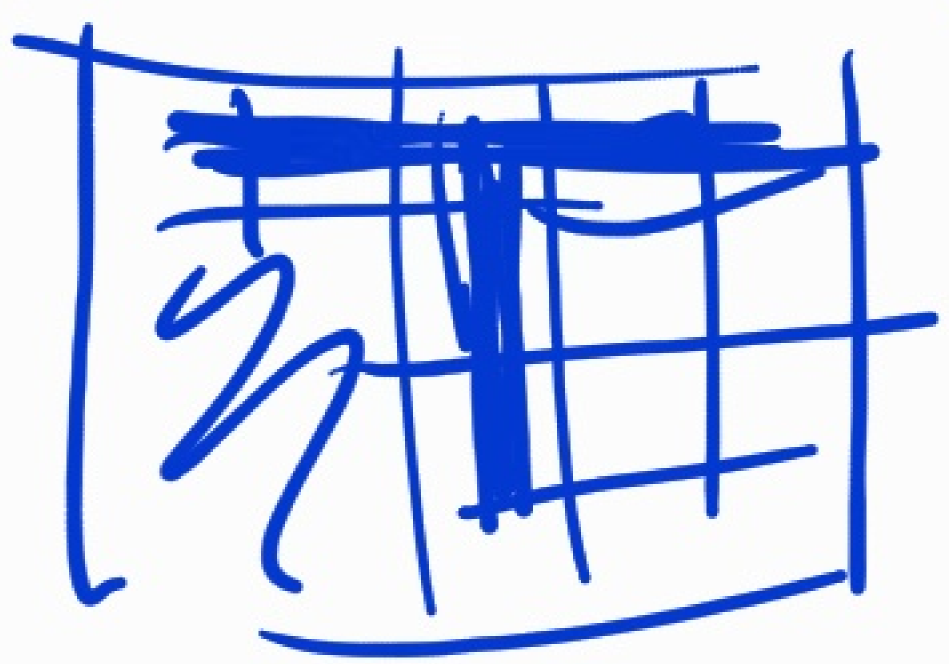


Choose a cell

we forbide the given cell C_e



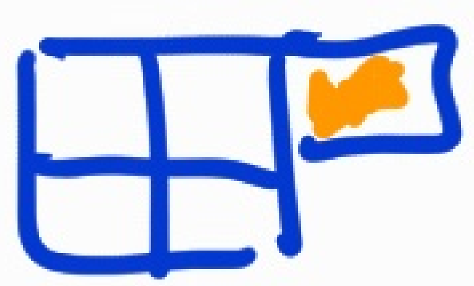
we remove from C the row and col of the cell C_s

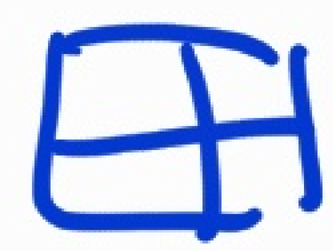


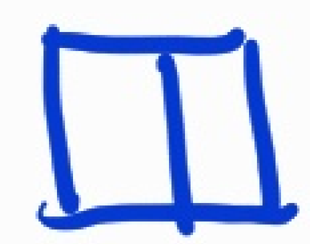
Prop

$$r(C,x) = x \cdot r(C_s, x) + r(C_e, x)$$

Example



$C_e =$  $\qquad r(C_s, x) = 1 + 4x + 2x^2$

$C_s =$  $\qquad r(C_e, x) = 1 + 2x$

$$r(\ \ , x) = x(1+2x) + 1 + 4x + 2x^2$$

$$= 1 + 5x + 4x^2$$

Proof We want to show that the poly on the left and the one of the right have the same coefficients. To this aim we want to compute

$$r_k(C) = \# ① + \# ②$$

There are two cases

① We place a rook in the 'selected cell

$+$

② — do not ———

① we place the remaining $k-1$ rooks in

$$C_s \longrightarrow r_{k-s}(C_s)$$

② It is like that cell is forbidden & we place k rooks in C_e $\longrightarrow r_k(C_e)$

$$r_k(C) = \boxed{r_{k-1}(C_s) + r_k(C_e)}$$

coeff of deg k of $x\,r(C_s\,x) + r(C_e\,x)$

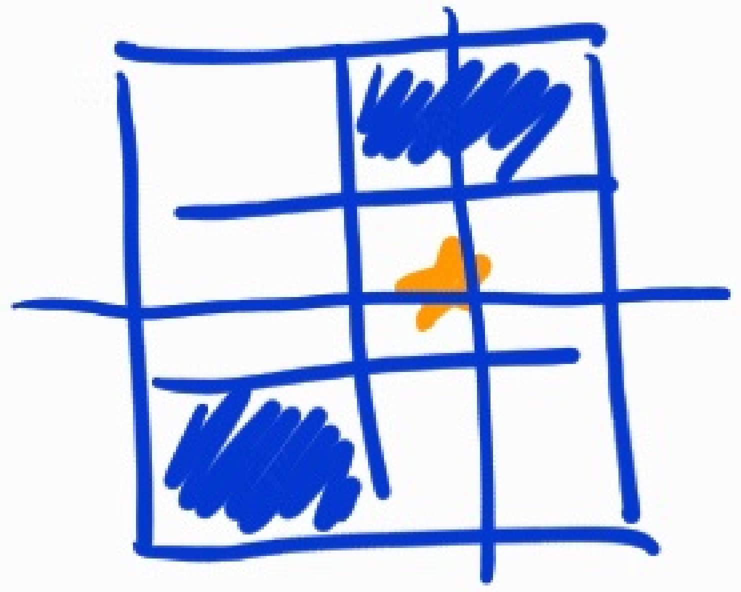
coeff. of deg $k-1$ of $r(C_s\,x)$

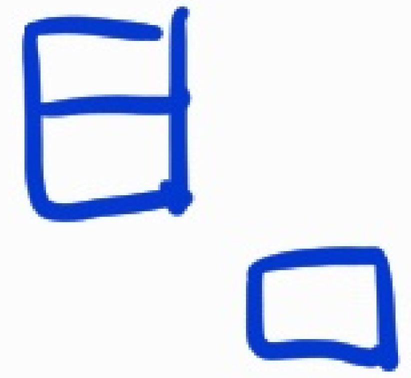
= ~~coeff~~ of deg k of $x \cdot r(C_s\,x)$

coeff of deg k of $r(C_e\,x)$

||

Example



C_e 

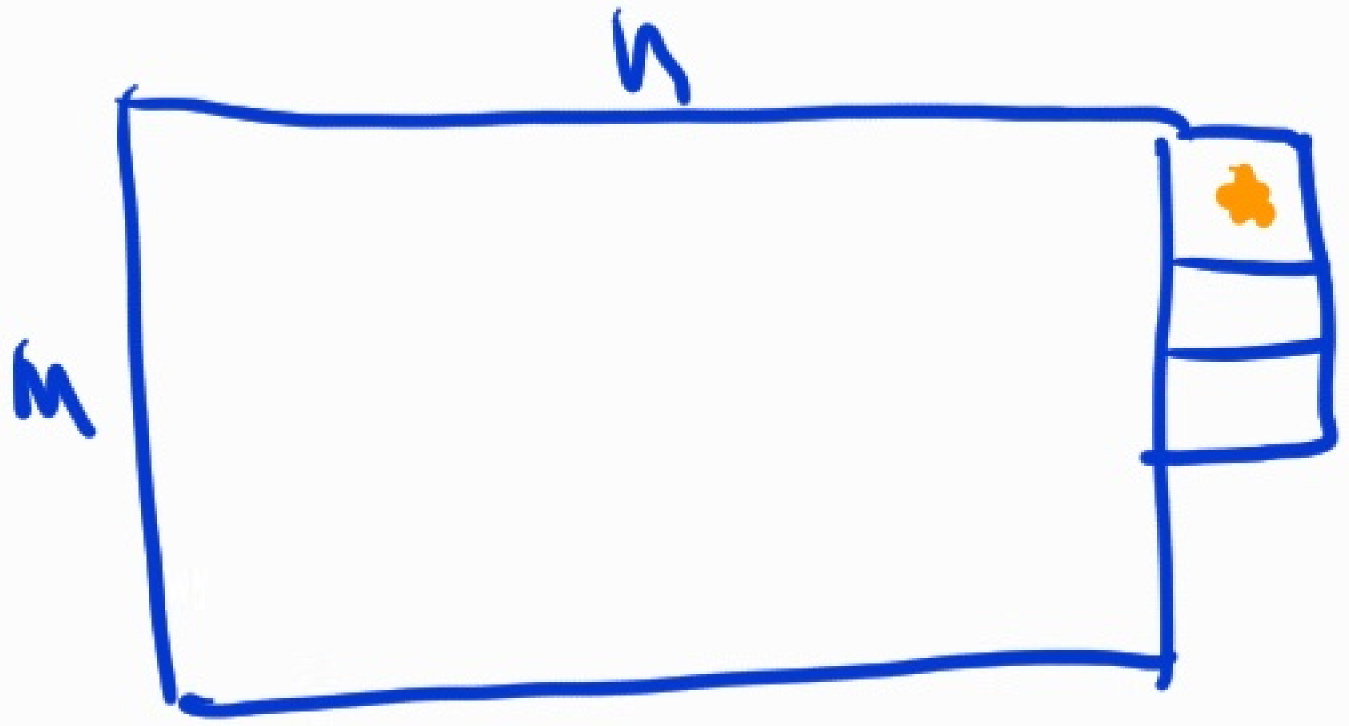
$$(1+x)(1+2x) = 1+3x+2x^2$$

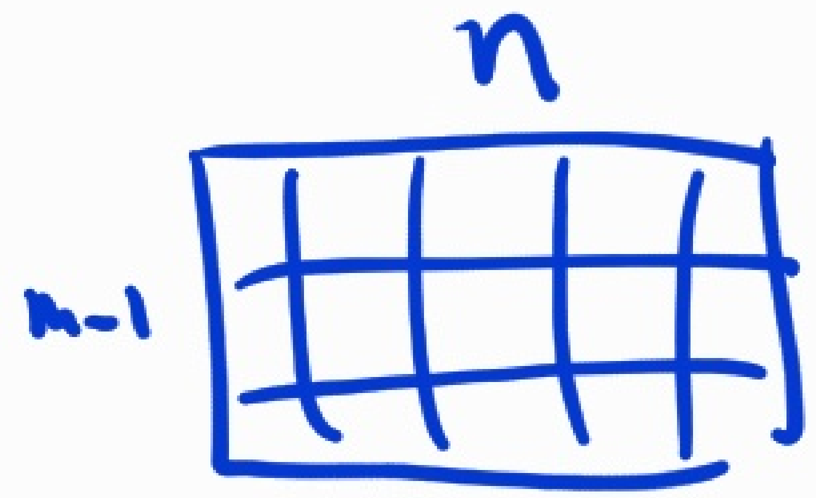
$C_s =$  $\rightsquigarrow 1+x$

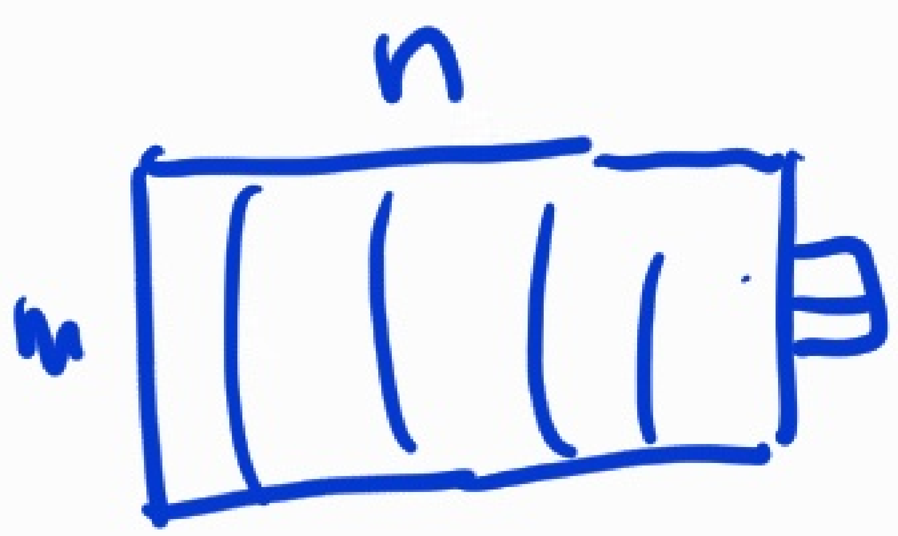
$$r(C\ x) = x(1+x) + 1+3x+2x^2$$

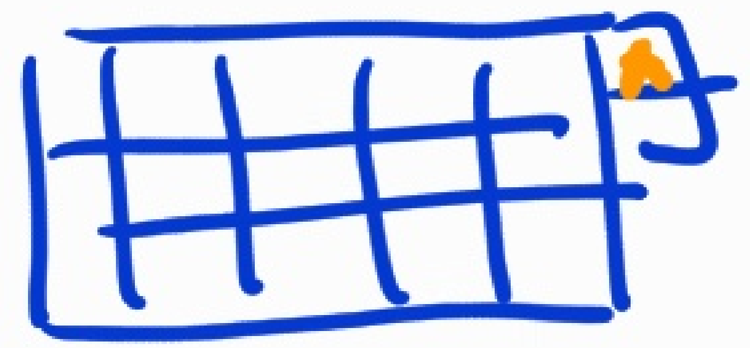
$$= \boxed{1+4x+3x^2}$$

Example (old exam)



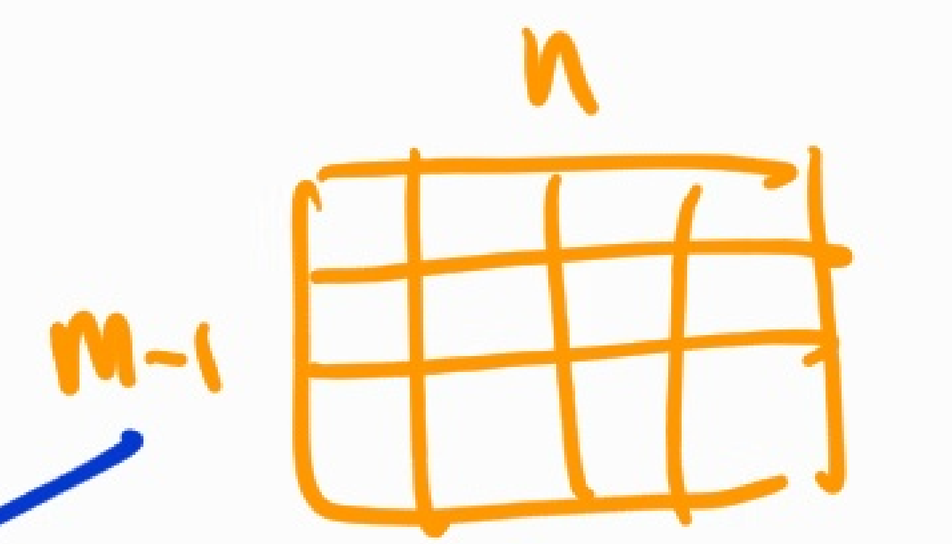
C_s 

$C_e =$ 

$=$ 

$$= x\left(\sum_{k=0}^{\infty}\binom{m-1}{k}\binom{n}{k}k!\,x^k\right) + r(C_e) = (*)$$

rook polynomial of C_e



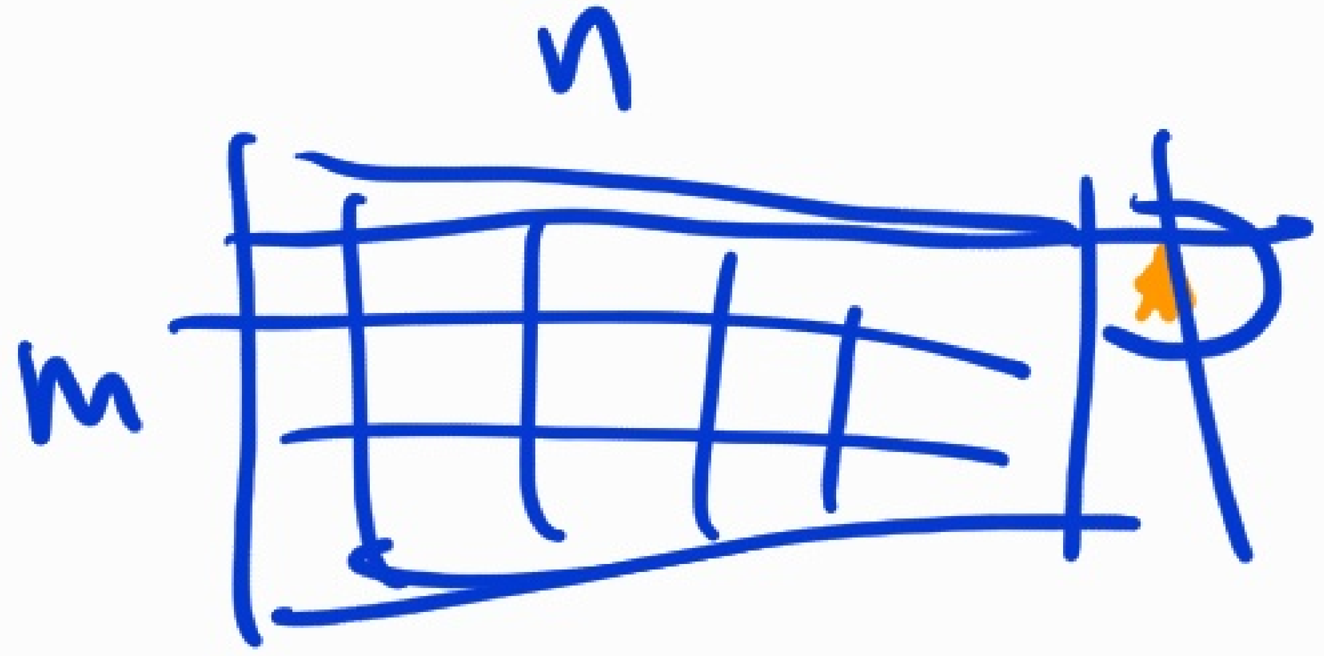
C_s'

C_e'

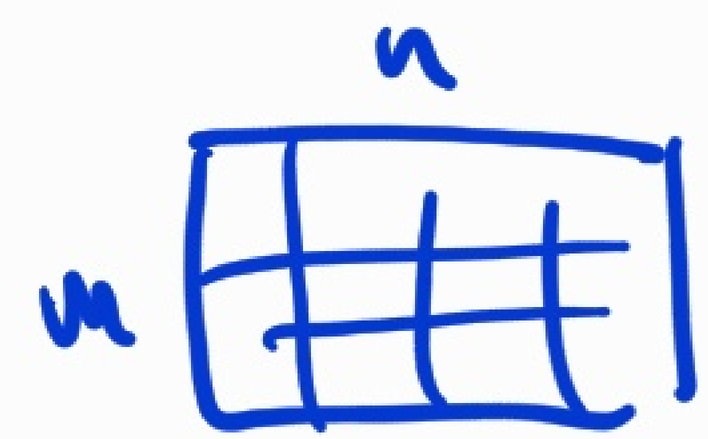
$$(*) = x\left(\sum_{k=0}^{\infty} \binom{m-1}{k}\binom{n}{k} k!\, x^k\right) + x\left(\sum_{k=0}^{\infty} \binom{m-1}{k}\binom{n}{k} k!\, x^k\right)$$

$$+\, r\left(C_e'\, x\right) = (**)$$

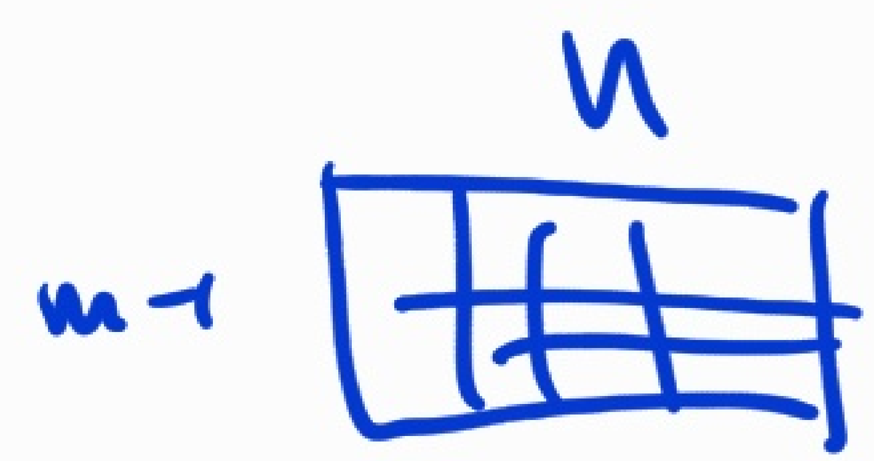
C_e'



C_e''



C_s''



$$(**) = 2x\left(\sum_{k=0}^{\infty}\binom{m-1}{k}\binom{n}{k}k!\,x^{k}\right) +$$

$$+\,x\left(\sum_{k=0}^{\infty}\binom{n-1}{k}\binom{n}{k}k!\,x^{k}\right) +$$

$$C_5''$$

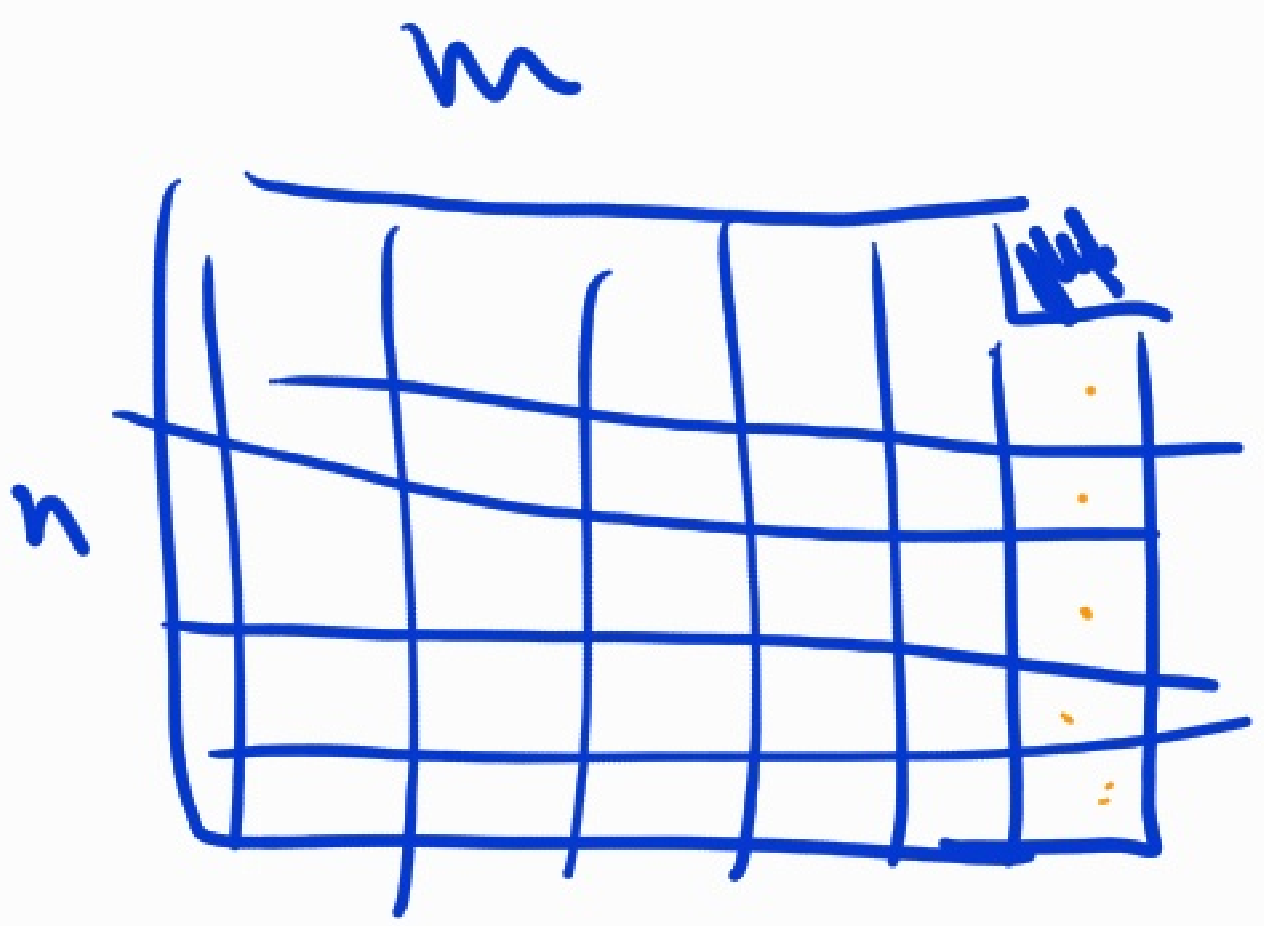
$$\sum_{k=0}^{\infty}\binom{m}{k}\binom{n}{k}k!\,x^{k}$$

$$C_6''$$

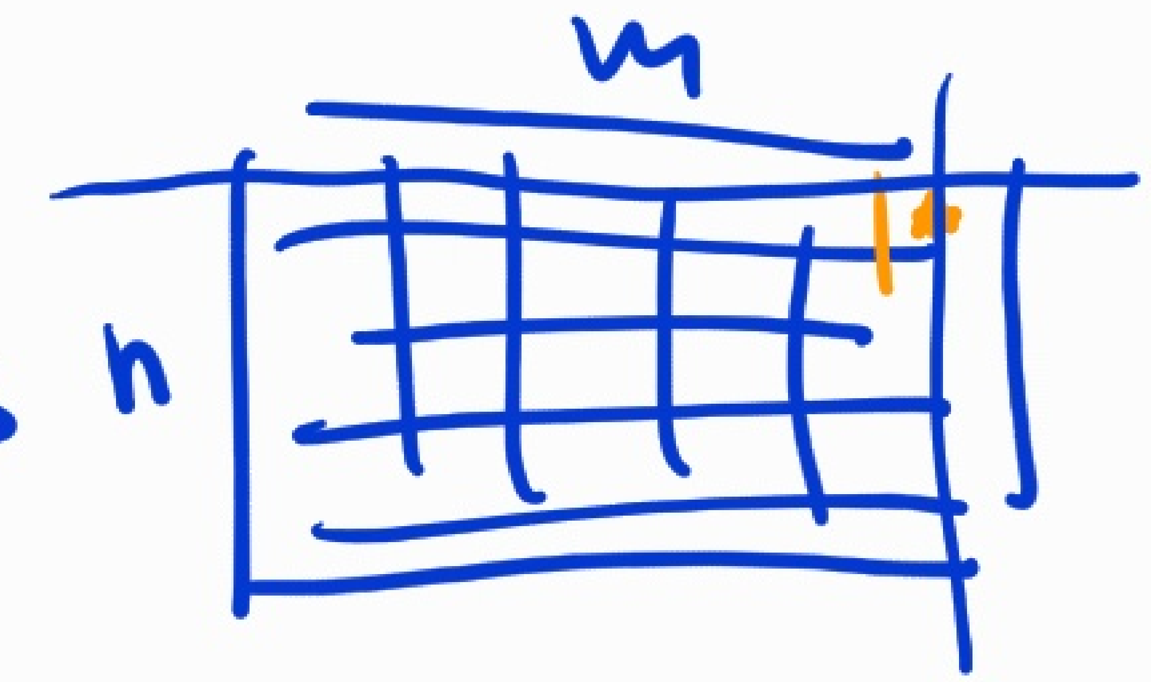
$$= \sum_{k=0}^{\infty}3\binom{m-1}{k}\binom{n}{k}k!\,x^{k+1} + \sum_{k=0}^{\infty}\binom{m}{k}\binom{n}{k}k!\,x^{k}$$

$$r_k(C) = \boxed{\binom{n-1}{k-1}\binom{n}{k-1}(n-1)! + \binom{n}{k}\binom{n}{k}k!}$$

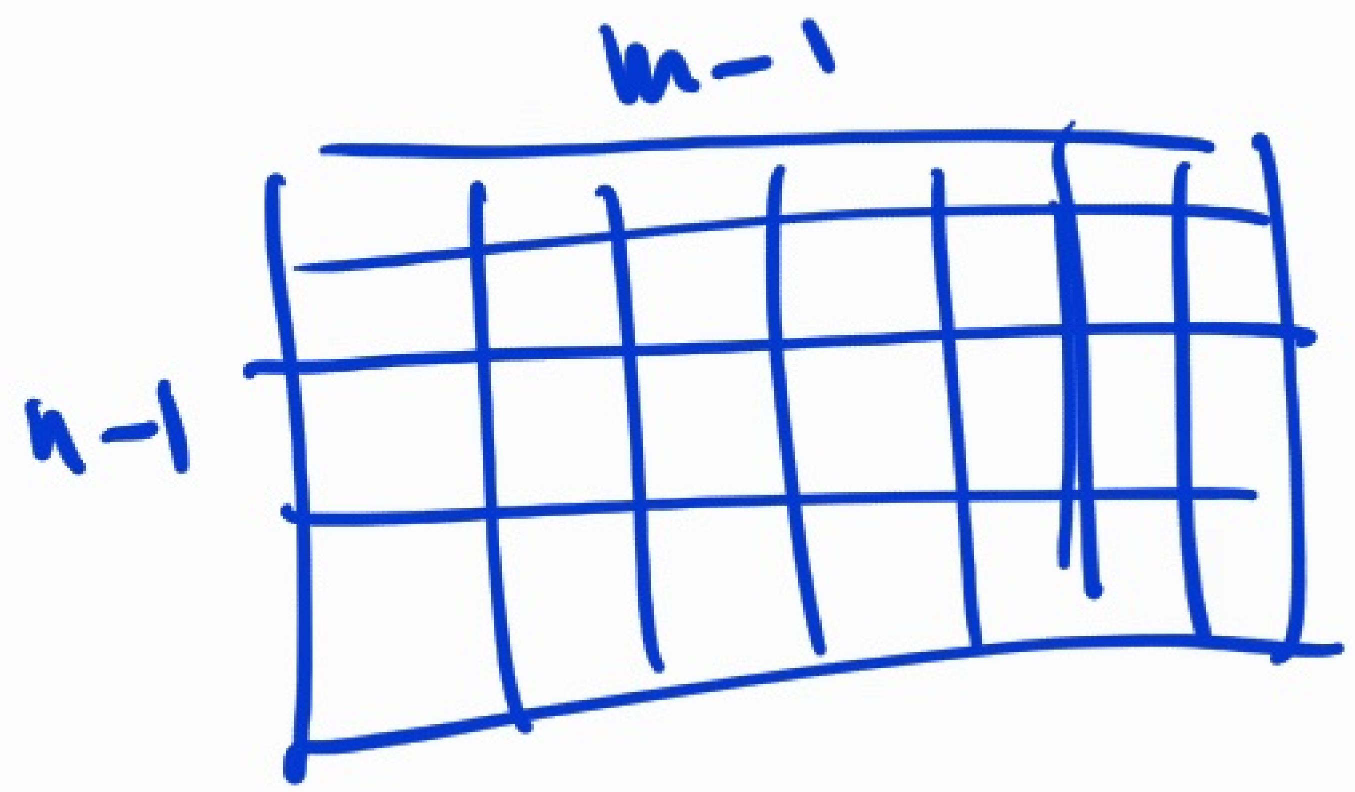
Example



$= C_e$ when C is



C_s



$$r(c, x) = x \cdot r(C_s\, x) + \boxed{r(C_e\, x)}$$

$$r(C_e\, x) = -x\, r(C_s\, x) + r(C_1\, x)$$

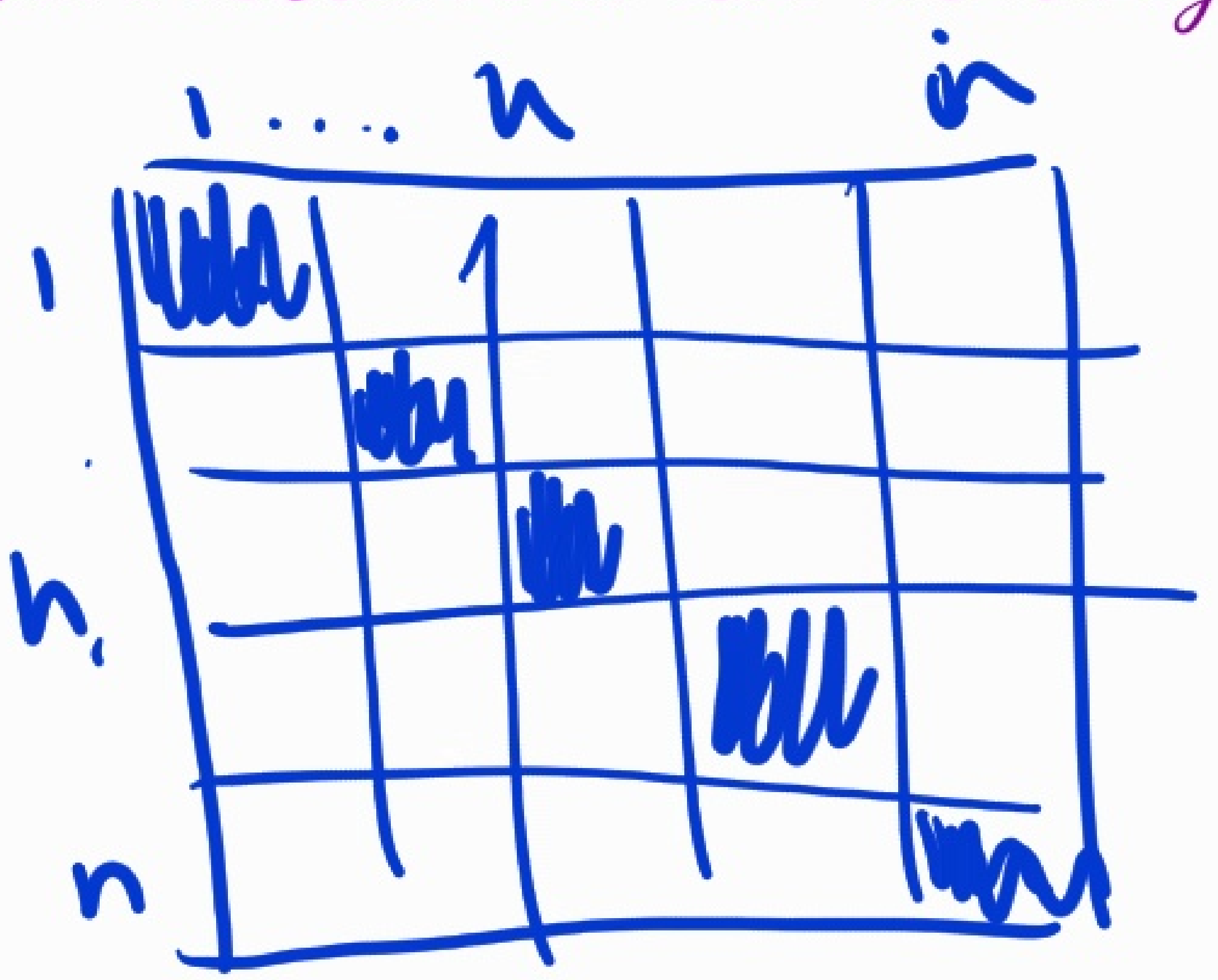
$$= -x \left(\sum_{k=0}^{\infty} \binom{m-1}{k} \binom{n-1}{k} k!\, x^k \right)$$

$$+ \sum_{k=0}^{\infty} \binom{m}{k} \binom{n}{k} k!\, x^k$$

$$= \sum_{k=0}^{\infty} \binom{m}{k} \binom{n}{k} k!\, x^k - \sum_{k=0}^{\infty} \binom{m-1}{k} \binom{n-1}{k} k!\, x^{k+1}$$

$$\boxed{r_k(C_e) = \binom{n}{k}\binom{n}{k}k! - \binom{n-1}{k-1}\binom{n-1}{k-1}(k-1)!}$$

Connection with derangements



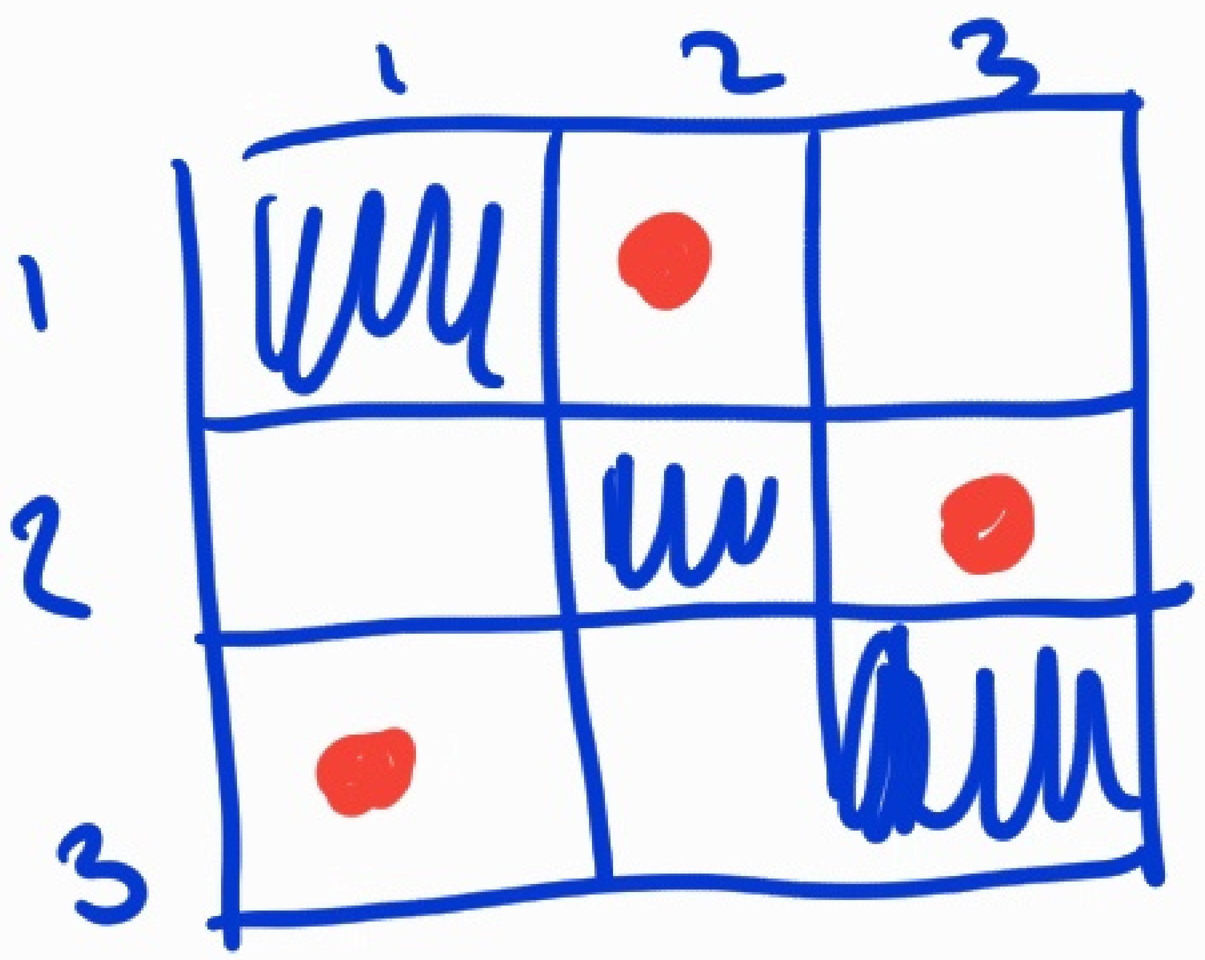
Recall a derangement is

$\sigma \in S_n$ such that $\sigma(i) \neq i$

$f : \{1 \dots n\}$ $f(i) \neq i$

is like giving a rook placement with n rooks

- place a rook in (i,j) iff $f(i) = j$

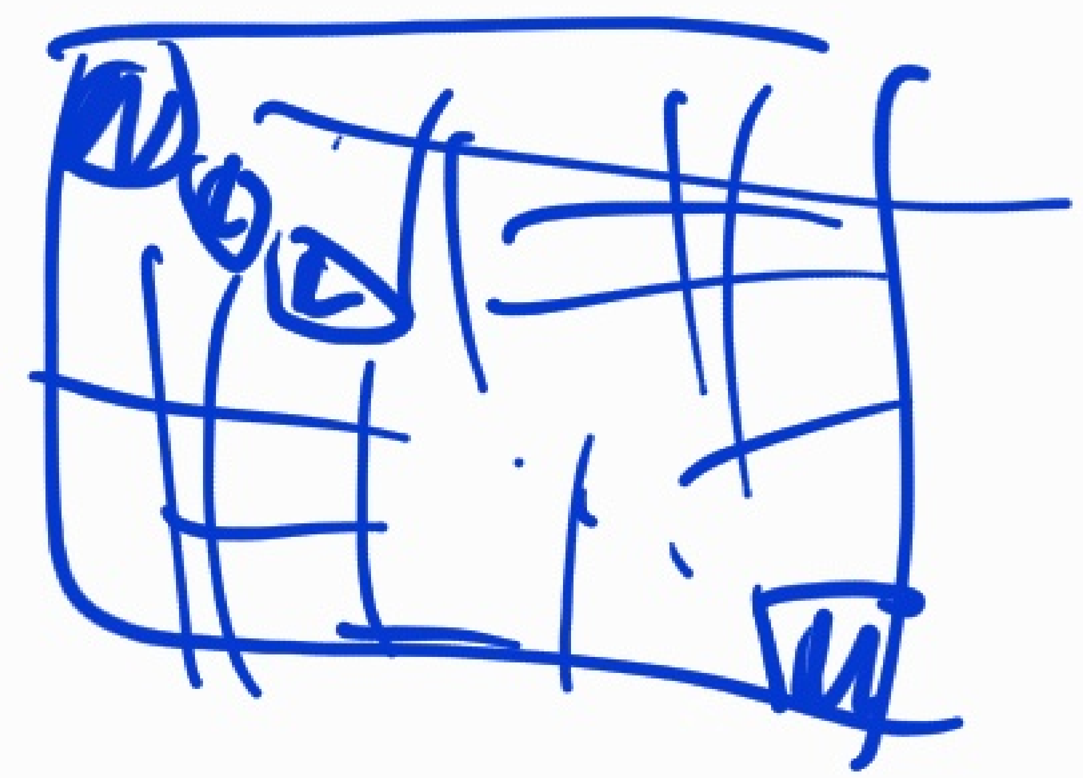


$\sigma = (1\ 2\ 3)$

\# of derangements =

\# of rook placements

in



Prop: A, B two finite sets

$n = |A| \le |B| = m$

$\{B_a\}_{a \in A}$

$B_a \subseteq B$

$$|\{\text{injective } f : A \to B \,/\, f(a) \notin B_a\}| =$$

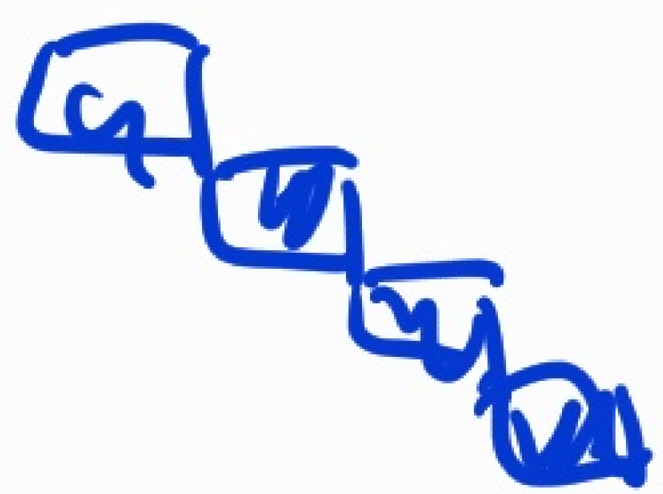
$$\sum (-1)^k r_k(C)\, P(n-k, m-k)$$

$$C = \begin{array}{c|c|c|c|c|} & b_1 & \cdots & \cdots & b_m \\ \hline a_1 & & & & \\ \hline \vdots & & & & \\ \hline \vdots & & & & \\ \hline \vdots & & & & \\ \hline a_n & & & & \\ \hline \end{array}$$

$(a\,b)$ not forbidden $\iff f(a) \in B_a$

We forbide $\{(a,b) \,/\, a\in A \quad b\notin B_a\}$

<u>Example</u> $A=B=\{1,\dots,n\}$ $\quad B_a=\{a\}$



Sketch of the proof:

Principle of exclusio/inclusion.

$A = \{a_1 \dots a_n\}$

$A_i \subseteq S = \{f : A \to B \text{ injective}\}$

$= \{f : A \to B \text{ injective } f(a_i) \in B_i\}$

$$|A_1^c \cap \cap A_n^c| = \sum_{j=0}^{n} (-1)^j \alpha_j = \underline{\sum_{j=0}^{n} (-1)^j r_j(c) P(m-j, n-j)}$$

$$\alpha_j = \sum_{|I|=j} \left| \bigcap_{i \in I} A_i \right|$$

$=$ place j rook in C , injective function from $A \setminus \{a_{i_1} \ldots a_{i_j}\}$ to $B \setminus \{f(a_{i_1}) \ldots f(a_{i_j})\}$

$r_j(C)$

$$\overline{}$$

$P(m-j, n-j)$