8.4+8.5: Exercise 4:

Find the rook polynomial for the following chessboards:

C_1: 

C_2: 

Sol: We see that C_2 is composed of two disjoint parts:  and 

C_2' C_2''

so $r(C_2, x) = r(C_2', x) \cdot r(C_2'', x)$

$r(C_2', x) = 1 + x$

$r(C_2'', x) = 1 + 3x$

$r(C_2, x) = (1+x)(1+3x) = 1 + 4x + 3x^2$

For C_1 look at the top-left square: if we place a rook there, we are left with

, otherwise we are left with

$r(C_1,x) = x \cdot r($  $, x) + r($  $, x)$

$$= x \cdot (1+2x) + (1+3x+x^2)$$

$$= x + 2x^2 + 1 + 3x + x^2$$

$$= 1 + 4x + 3x^2$$

Exercise 6:

a) C is a chessboard with m rows and n columns, with $m \leq n$. For $k \leq m$, in how many ways can we arrange k nontaking rooks.

b) Determine $r(C,x)$.

Sol: C is a very easy chessboard, so we can compute directly:

a) If we want to place k nontaking rooks, it is the same as choosing k distinct rows and associate k distinct columns.

First order the rooks:

So we want to pick k ordered columns without repetition, and k ordered rows without repetition. There are $\frac{n!}{(n-k)!}$ possibilities for the first one and

$\frac{m!}{(m-k)!}$ for the second one.

So in total if we order the rooks there are $\frac{n!}{(n-k)!} \cdot \frac{m!}{(m-k)!}$ possibilities.

Since the rooks are actually not ordered, we need to divide by the number of their permutations

$$\rightsquigarrow \boxed{\frac{n! \cdot m!}{(n-k)!(m-k)! \cdot k!}}$$

b) $$r(C,x) = \sum_{k=0}^{n} \frac{n! \cdot m!}{(n-k)!(m-k)! \cdot k!} \cdot x^k$$

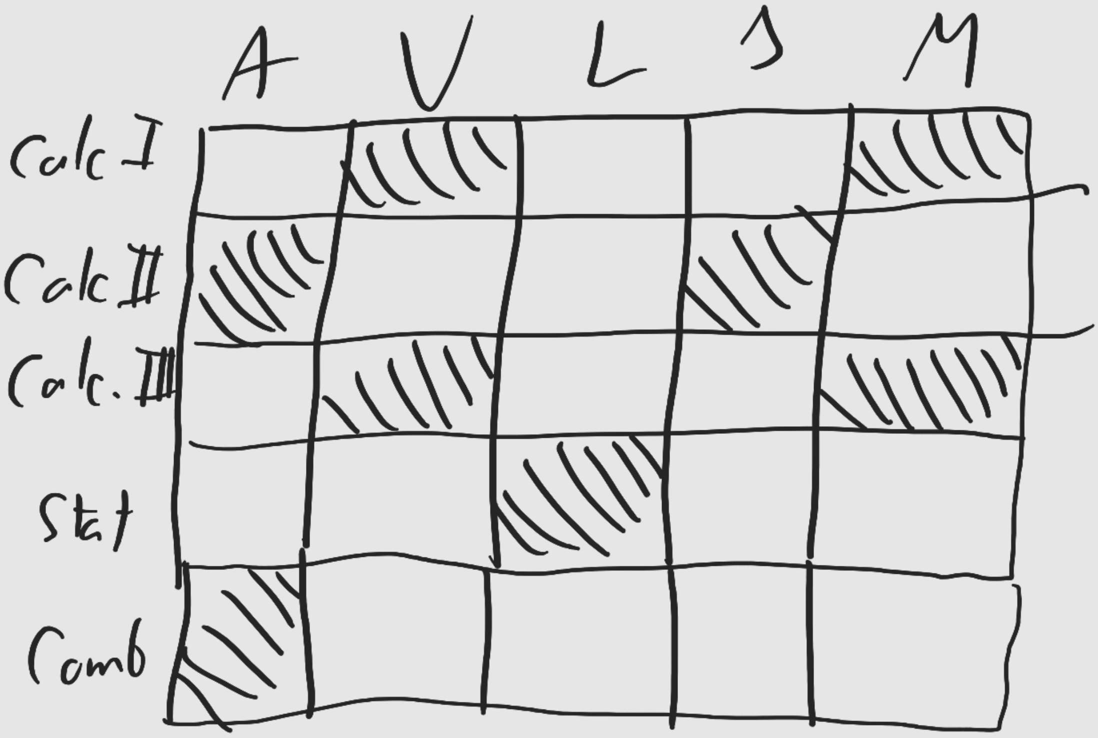
there can be at most n rooks since each rook occupies a row.

$$= \sum_{k=0}^{n} \frac{n! \cdot m!}{(n-k)!(m-k)!} \cdot \frac{x^k}{k!}$$

Exercise 9:

Professors: A, V, L, J, M

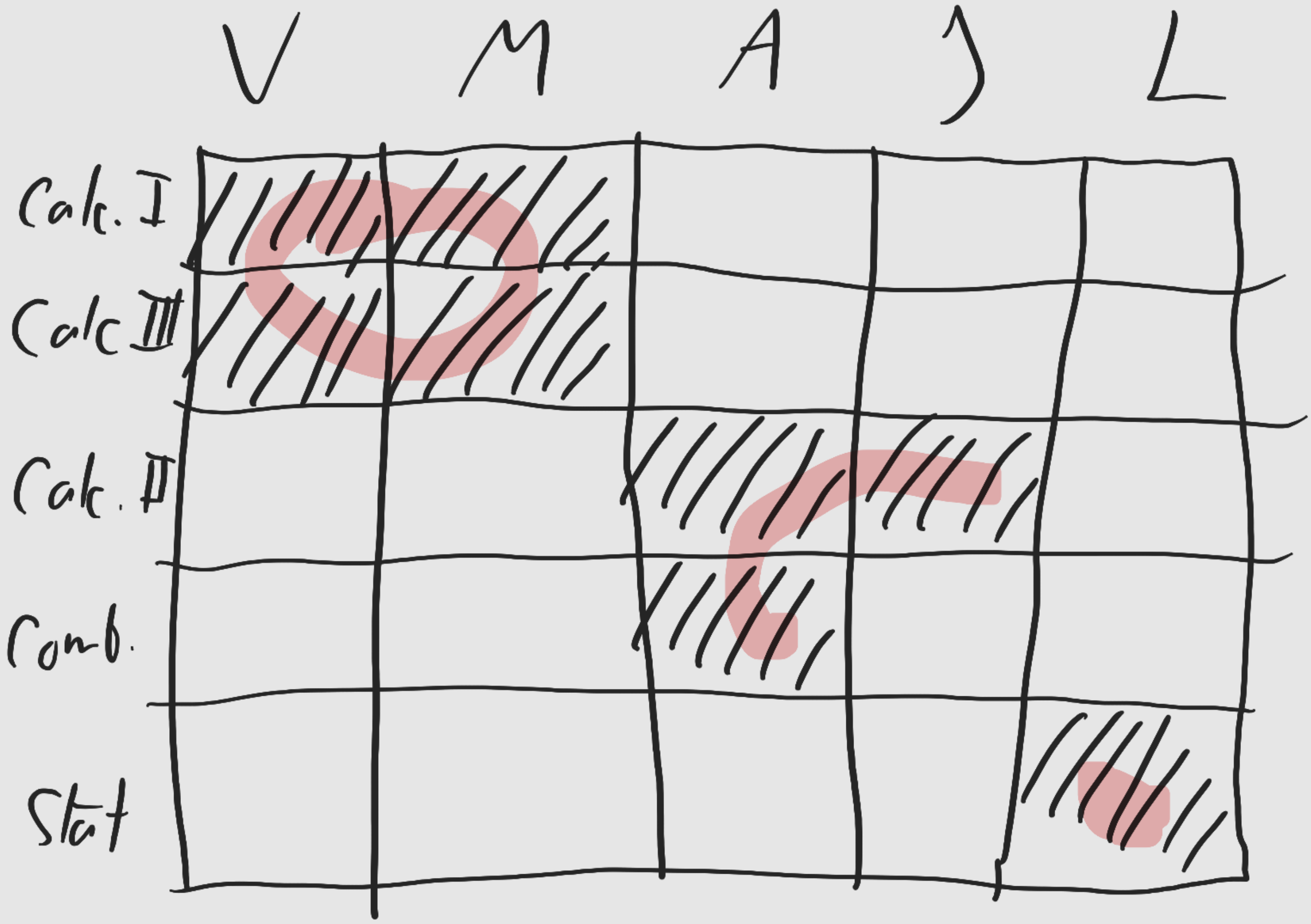
Classes : Calc. I, Calc. II, Calc. III, stat, Comb.



a) In how many ways can the five courses be assigned to the five professors?

b) For those assignments, what is the probability that Violet teaches combinatorics?

<u>Solution</u>: We rearrange the chessboard.



We compute the rook polynomial of the chessboard, and we start with its complement:

$$r(\overline{C}, x) = r(\boxplus, x)\cdot r(\boxplus, x)\cdot r(\square, x)$$

$$= (1+4x+2x^2)(1+3x+x^2)(1+x)$$

$$= (1+4x+2x^2)(1+4x+4x^2+x^3)$$

$$= 1+8x+22x^2+25x^3+12x^4+2x^5$$

By inclusion-exclusion, we therefore obtain the number of possibilities:

$$\sum_{i=0}^{5} (-1)^i (5-i)! \cdot r_i$$

coefficients of $r(\bar{C}, x)$

$$= 5! \cdot 1 - 4! \cdot (8) + 3! \cdot (22) - 2!(25) + 1!(12) - 0! \cdot (2)$$

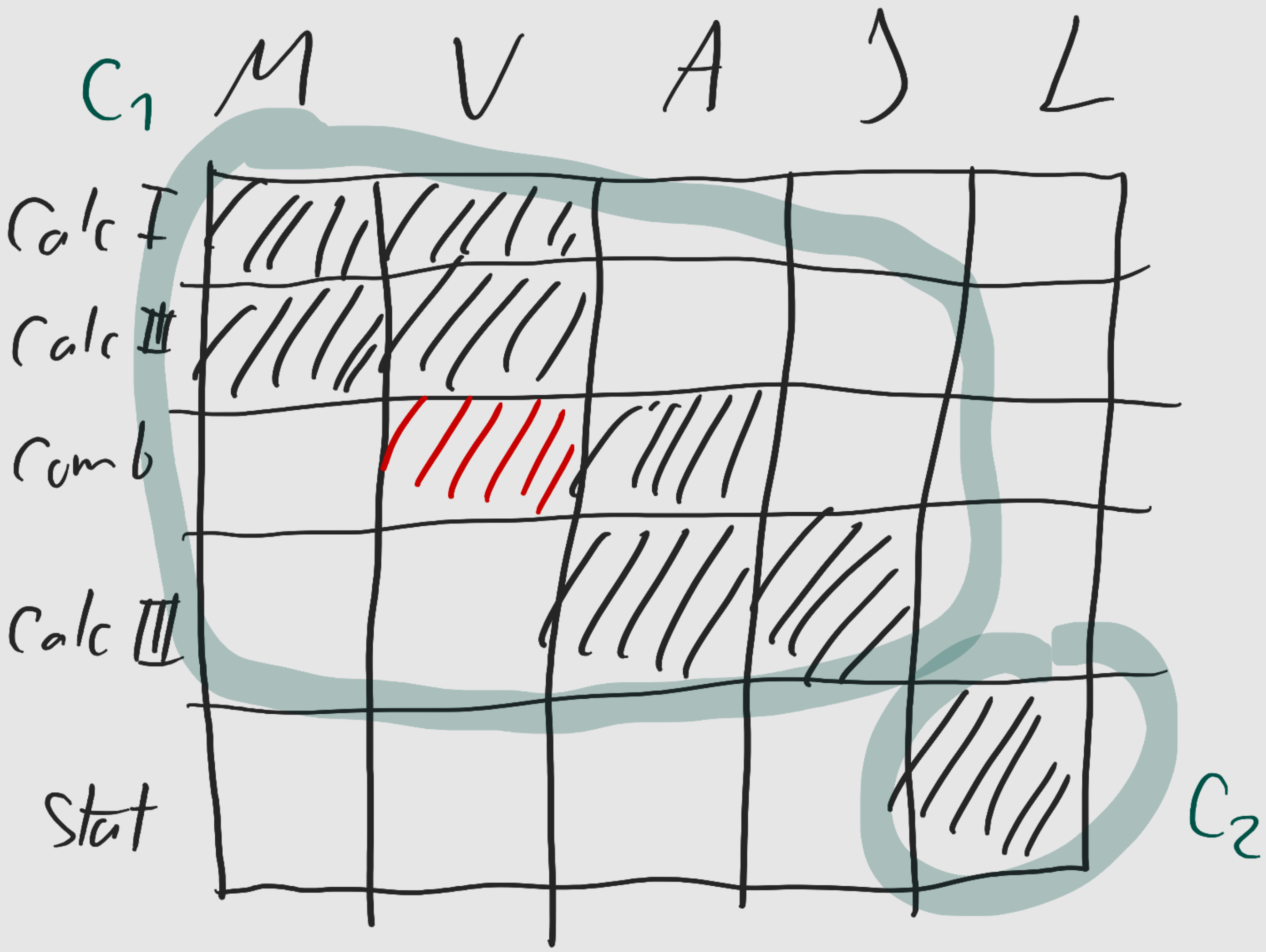
$$= 120 - 8 \cdot 24 + 22 \cdot 6 - 25 \cdot 2 + 12 - 2$$

$$= 120 - 192 + 132 - 50 + 12 - 2$$

$= \boxed{20}$ possibilities

b) Out of those 20 assignments, in how many of them does Violet teaches combinatorics?

Idea: We add the square (Comb, V) to the forbidden squares, and we see how many possibilites remain:



we compute again $r(\bar{C}, x)$ we can decompose $\bar{C}$ into two disjoint parts, C_1 and C_2,

$r(C_2, x) = 1+x$

In order to compute $r(C_1, x)$, we decompose according to whether there is a rook on the red square: if there is no rook we are in the previous scenario and we obtain the polynomial

$$(1+4x+2x^2)(1+3x+x^2) = (1+7x+15x^2+10x^3+2x^4)$$

otherwise if there is one we obtain

$$x \cdot r\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, x\right) \cdot r(\square\square, x)$$

$$= x(1+2x)(1+2x)$$

$$= x+4x^2+4x^3.$$

In total $r(C_1, x) = (x+4x^2+4x^3)+(1+7x+15x^2$
$+10x^3+2x^4)$

$$= 1+8x+19x^2+14x^3+2x^4$$

and so $r(\bar{C}, x) = (1+8x+19x^2+14x^3+2x^4)\cdot(1+x)$

$$= 1+9x+27x^2+33x^3+16x^4+2x^5.$$

The number of possibilities in which Violet does <u>not</u> teach combinatorics is therefore

$$5!(1) - 4!\cdot(9) + 3!(27) - 2!(33) + 1!(16) - 0!\cdot 2$$

$= 120 - 9 \cdot 24 + 27 \cdot 6 - 33 \cdot 2 + 16 - 2$

$= 120 - 216 + 162 - 66 + 16 - 2$

$= 14.$

In other words there are 6 assignments where Violet teaches combinatorics, so the probability is $\boxed{\frac{6}{20}} = 30\%$