9.2 Exercise 8:

For $n \in \mathbb{Z}^+$, find in $(1+x+x^2)(1+x)^n$ the coefficient for a) x^7; b) x^8; c) x^r for $0 \le r \le n+2$, $r \in \mathbb{Z}$.

Sol: $$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^i$$

$$(1+x+x^2)(1+x)^n =$$

$$(1+x+x^2)\left(1+nx+\sum_{i=2}^{n-2} \binom{n}{i} \cdot x^i + nx^{n-1} + x^n\right)$$

$$= 1+(n+1)x + \sum_{i=2}^{n} x^i \left[\binom{n}{i}+\binom{n}{i-1}+\binom{n}{i-2}\right]$$

$$+ (n+1) \cdot x^{n+1} + x^{n+2}$$

we read from there the answer for c); a) and b) are special cases.

Exercise 10:

How many ways to assign 24 identical robots to four assembly lines with a) at least three robots assigned to each line

b) at least three robots, but no more than nine, assigned to each line.

Sol: a) Look at a fixed line. Since the robots are identical, the generating function for that line is $F_1(x) = x^3 + x^4 + x^5 + \dots$

$$= \sum_{i=3}^{\infty} x^i = \frac{x^3}{1-x}$$

we have the same function for each line, so the generating function for this problem is

$F(x) = F_1(x) \cdot F_2(x) \cdot F_3(x) \cdot F_4(x)$ and we will want the coefficient of x^{24}.

$$F(x) = \frac{x^{12}}{(1-x)^4} = x^{12} \cdot \sum_{i=0}^{\infty} \binom{3+i}{i} \cdot x^i$$

therefore, the coefficient for x^{24} is

$$\binom{15}{12} = \frac{15 \cdot 14 \cdot 13}{6} = 35 \cdot 13 = \boxed{455}$$

b) We use the same idea as previously, but now $F_1(x) = \sum_{i=3}^{9} x^i = \frac{x^3 - x^{10}}{1-x}$

$$F(x) = F_1(x)^4 = \frac{(x^3 - x^{10})^4}{(1-x)^4} =$$

$$\frac{x^{12} - 4 \cdot x^{19} + 6x^{26} - 4x^{33} + x^{40}}{(1-x)^4}$$

$$= (x^{12} - 4x^{19} + 6x^{26} - 4x^{33} + x^{40}) \cdot \sum_{i=0}^{\infty} \binom{3+i}{i} \cdot x^i$$

again we want the coefficient of x^{24}:

$$\binom{15}{12} - \binom{8}{5} = \frac{15 \cdot 14 \cdot 13}{6} - \frac{8 \cdot 7 \cdot 6}{6} =$$

$$= 455 - 56 = \boxed{399}$$

9.3 Exercise 2 :

Determine the generating function for the sequence $a_0, a_1, a_2, \dots$ where a_n is the number of partitions of the nonnegative integer n into

a) even summands

b) distinct even summands

c) distinct odd summands

Sol : a) look at the function

$$f_{even}(x) = 1 + x^2 + x^4 + \dots = \sum_{i=0}^{\infty} x^{2i}$$

If we want a partition into even summands, what we need to do is choose the number of summands and then chose the size of each summand.

what we need to look at is the product

$(1+x^2+x^4+\dots)\cdot(1+x^4+x^8+\dots)\cdot$

blocs of 2   blocs of 6   blocs of 4

$(1+x^6+x^{12}+\dots)\cdot\dots$

$$= \prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} x^{2ij}\right) = \boxed{\prod_{i=1}^{\infty} \frac{1}{1-x^{2i}}}$$

b) We only want at most one bloc of each size, so the generating function is now

$$(1+x^2)(1+x^4)\cdot(1+x^6)\cdot\dots = \prod_{i=1}^{\infty}(1+x^{2i})$$

c) Similarly, the generating function is

$$(1+x)(1+x^3)(1+x^5)\dots = \prod_{i=0}^{\infty}(1+x^{2i+1})$$

Exercise 3 :

In $f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3}$, the coefficient of x^6 is 7. Interpret this in terms of partitions of 6.

Sol: $f(x)$ is the generating series for partitions of n into summands of size at most 3. Therefore, this means that there are 7 ways to write 6 as a sum of 1's, 2's and 3's.

Exercise 4:

Find the generating function for the number of integer solutions of

a) $2w + 3x + 5y + 7z = n$, $0 \leq w, x, y, z$

b) $2w + 3x + 5y + 7z = n$, $0 \leq w$, $4 \leq x, y$, $5 \leq z$

Sol:

a) This problem is actually equivalent to finding the number of partitions of n into blocs of size 2, 3, 5 or 7. Therefore the generating function is $(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\cdot(1+x^5+x^{10}+\dots)\cdot(1+x^7+x^{14}+\dots) =$

$$\boxed{\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}}$$

b) Since x, y and z don't start at 0, the generating function is now

$$(1+x^2+x^4+\dots)(x^{12}+x^{15}+x^{18}+\dots)(x^{20}+x^{25}+x^{30}+\dots)(x^{35}+x^{42}+x^{49}+\dots) =$$

$$\boxed{\frac{1}{1-x^2}\cdot\frac{x^{12}}{1-x^3}\cdot\frac{x^{20}}{1-x^5}\cdot\frac{x^{35}}{1-x^7}}$$

Proposition: $\dfrac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n-1+i}{i} \cdot x^i$

Proof: by induction:

- if $n=1$, $\dfrac{1}{1-x} = \sum_{i=0}^{\infty} x^i = \sum_{i=0}^{\infty} \binom{0+i}{i} \cdot x^i$

- if the formula is true for k, then

$$\frac{1}{(1-x)^{k+1}} = \frac{1}{1-x} \cdot \frac{1}{(1-x)^k} =$$

$$\sum_{i=0}^{\infty} x^i \cdot \sum_{j=0}^{\infty} \binom{k-1+j}{j} \cdot x^j$$

$$= \sum_{n=0}^{\infty} x^n \sum_{j=0}^{n} \binom{k-1+j}{j}$$

I want to prove: $\sum_{j=0}^{n} \binom{k-1+j}{j} = \binom{k+n}{n}$

we start a new recurrence on n:

- $n=0$ $\quad \sum_{j=0}^{0} \binom{k-1+0}{0} = \binom{k-1}{0} = 1 = \binom{k+0}{0}$ ✓

$$- \sum_{j=0}^{n+1} \binom{k-1+j}{j} = \underbrace{\sum_{j=0}^{n} \binom{k-1+j}{j}} + \binom{k+n}{n+1}$$

$$= \binom{k+n}{n} + \binom{k+n}{n+1} = \binom{k+n+1}{n+1} \checkmark$$

therefore we obtain indeed that

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} x^n \cdot \sum_{j=0}^{n} \binom{k-1+j}{j}$$

$$= \sum_{n=0}^{\infty} x^n \cdot \binom{k+n}{n}$$ which is

what I wanted to prove.