MM5023
Lecture 4
Generating functions

Plan

- Review of Analys ~~A~~B (Convergent power series)
- Generating functions / generating series (Examples)
- Application to partitions

Power series

A formal power series with real coefficient in $\mathbb{R}$ is a symbol

$$\sum_{n=0}^{\infty} a_n x^n \qquad a_n \in \mathbb{R}$$

We say that it has positive radius of convergence ρ if the sequence of functions

$$f_k(x) := \sum_{n=0}^{k} a_n x^k$$ converges absolutely to

$$f : (-\rho, \rho) \longrightarrow \mathbb{R}$$

Rmk $\sum a_n x^n$ absolutely convergent to f

Near 0 $\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$\parallel$

a_n

$|x| < \rho$

Theorem $\sum a_n x^n$ $\sum b_n x^n$ power series absolutely converging to $f(x)$, $g(x)$ with radii ρ_1 ρ_2 respectively then $x < \min\{\rho_1\ \rho_2\}$

↙ relies on commutativity of +

1) $\sum (\underline{a_n+b_n}) x^n \longrightarrow f(x)+g(x)$ absolutely

↙ relies on commutativity of +

2) $\underline{c_n = \sum_{k=0}^{n} a_k b_{n-k}}$ $\sum c_n x^n \longrightarrow \underline{f(x)g(x)}$ abs

3) $\sum_n n a_n x^{n-1} \longrightarrow \frac{d}{dx} f(x)$ abs

$$\frac{d}{dx} \sum a_n x^n = \sum \frac{d}{dx} n a_n x^{n-1}$$

Where does it converge?

$$\boxed{\sum_{n=0}^{\infty} a_n x^n}$$

"easy for a power series to be absolutely conv."

If $|a_n| < c^n$ for some $c \in \mathbb{R}$

$$\boxed{n >> 0}$$

$\Rightarrow$ absolute convergence for $\boxed{|x| < \frac{1}{c}}$

Generating series and functions

Given a sequence $(a_n)_{n\in\mathbb{N}}$ we say that the (formal) power series $\sum_{n=0}^{\infty} a_n x^n$ is the generating series of the sequence.

if $S_k(x) = \sum_{0}^{k} a_n x^n$ does not converge absolutely

this is not a function

BOOK: this is the generating f.

If this has a positive radius of (absolute) convergence then $\sum a_n x^n$ converges to some function f which we call the generating function of $(a_n)_{n\in\mathbb{N}}$

→ Rook poly are gf of $r_k(C)$

Examples

- **$a_n = 1$ for every n**

Generating series $\sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n$

This has radius of convergence 1 This is a generating function f
The law for the generating function is

$$f(x) = \frac{1}{1-x}$$

$a_n = \binom{k}{n}$ for k fixed

Generating series $\displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = \sum_{n=0}^{k} \binom{k}{n} x^n$

$$\| $$

$$(1+x)^n$$

It converges absolutely everywhere

$$a_n = \begin{cases} 1 & n \le k \\ 0 & n > k \end{cases}$$

k fixed

Generating series $\displaystyle\sum_{n=0}^{k} x^n$ Converges everywhere

$$\|$$

$x \neq 1$

$$\begin{cases} \dfrac{1-x^{k+1}}{1-x} & x \neq 1 \\ k+1 & x = 1 \end{cases}$$

$a_n = n+1$

$$\sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$$

we take derivative
~~on both s~~

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

$j = n-1$

$$= \sum_{j=0}^{\infty} (j+1) x^j$$

$n = j$

$$= \sum_{n=0}^{\infty} \underline{(n+1)x^n}$$

generating function of $(n+1)$

$|x| < 1$

$$= \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\frac{1}{1-x} = +(1-x)^{-2} = \underline{\frac{1}{(1-x)^2}}$$

$a_n = n$

$$\underset{\substack{\downarrow \\ x}}{x}\left(\underset{\substack{\downarrow \\ \frac{1}{(1-x)^2}}}{\sum_{n=0}^{\infty} (n+1)\, x^{n}}\right) = \sum_{n=0}^{\infty} (n+1)\, x^{n+1}$$

$$= \sum_{j=1}^{\infty} j\, x^{j}$$

$$= \sum_{j=1}^{\infty} j\, x^{j} + 0\, x^{0}$$

$$= \sum_{j=0}^{\infty} j\, x^{j} \longrightarrow \frac{x}{(1-x)^2}$$

Generating function.

$a_n = n^2$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} n x^{n}\right) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$x \cdot \frac{d}{dx}\sum_{n=0}^{\infty} n x^{n} = \sum_{n=1}^{\infty} n^2 x^{n} = \sum_{n=1}^{\infty} n^2 x^{n} + 0^2 \cdot x^0$$

$\downarrow$

$$x \frac{d}{dx}\frac{x}{(1-x)^2} \qquad = \sum_{n=0}^{\infty} n^2 x^{n}$$

Generating series

the generating function is $x \cdot \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) =$

$$x \frac{(1-x)^2 + x\,2(1-x)}{(1-x)^4}$$

$$= x \frac{1 \cancel{-2x} + x^2 \cancel{+2x} - 2x^2}{(1-x)^4} = x \frac{1-x^2}{(1-x)^4}$$

$$= x \frac{1+x}{(1-x)^3}$$

Generating function of $a_n = n^3$

Prop If $f(x)$, $g(x)$ are generating functions of $(a_k)_{k\in\mathbb{N}}$ $(b_k)_{k\in\mathbb{N}}$ than

- $(f+g)(x)$ is the generating function of $(a_k+b_k)_{k\in\mathbb{N}}$

- $(f\cdot g)(x)$ is the generating function of $\boxed{\left(\sum_{n=0}^{k} a_n b_{k-n}\right)_{k\in\mathbb{N}}}$

formula of coeff of the product of power series

Example:

Compute the number of solutions of the equation

$$c_1 + c_2 + c_3 + c_4 = 25 \qquad c_i \in \mathbb{Z} \quad c_i \geq 0$$

We already know the solution: the number of ways to give 25 identical candies to 4 kids $\binom{25+4-1}{25}$

$$\begin{array}{ccccccc} c_i & 0 & 1 & 2 & 3 & \cdots & 25 \\ & 1 & + x & + x^2 & + & \cdots & x^{25} \end{array}$$

$$\left(\sum_{k=0}^{25} x^k \right)^4$$

The number we are looking for is the coefficient of deg 25 of this polynomial.

$$(\overset{\bullet}{1} + \overset{\times}{x} + \overset{\heartsuit}{x^2})(\overset{\heartsuit}{1} + \overset{\times}{x} + \overset{\bullet}{x^2})$$

$$= 1 + \underset{\cdot}{x} + \underset{\bullet}{\underline{x^2}} + \underset{\cdot}{x} + \underset{\times}{\underline{\underline{\underline{x^2}}}} + x^3 + \underset{\heartsuit}{\underline{\underline{x^2}}} + x^3 + x^4$$

$$= 1 + 2x + 3x^2 + 2x^3 + x^4$$

to get degree 25 $\quad x^{n_1} \quad x^{n_2} \quad x^{n_3} \quad x^{n_4}$

from factor 1 $\quad$ factor 2 $\quad$ 3 factor $\quad$ 4th

$$n_1 + n_2 + n_3 + n_4 = 25$$

the coefficient of $\boxed{\text{deg } 25}$ $\left(\sum_{k=0}^{25} x^k\right)^4$ is actually the

______ of the (formal) power series

$$\left(\sum_{k=0}^{\infty} x^k\right)^4$$ product of converging power series

$$\downarrow$$

$$f(x) = \left(\frac{1}{1-x}\right)^4$$

The number we are looking for is the coeff of deg 25 of the McLaurin expansion of $\left(\frac{1}{1-x}\right)^4 = \binom{25+4-1}{25}$

deg 0 $\quad f(0) = 1$

deg 1 $\quad f'(x) = (1-x)^{-4} = \overset{1+3}{4}(1-x)^{-5}$ $\qquad \frac{f'(0)}{1!} = 4$

deg 2 $\quad f''(x) = \overset{2+3}{5 \cdot 4}(1-x)^{-6}$ $\qquad \frac{f''(0)}{2!} = \frac{5 \cdot 4}{2!}$

deg 3 $\quad \frac{f^{(3)}(x)}{3!} = \frac{\overset{3+3}{4 \cdot 5 \cdot 6}(1-x)^{-7}}{3!}$ $\qquad \frac{6!}{3!3!}$

deg 25

$$\frac{f^{(25)}(x)}{25!} = \frac{3!}{3!}\,\frac{4\cdot 5\cdots(25+3)}{25!}\,(1-x)^{-29}$$

$$\frac{f^{(25)}(0)}{25!} = \frac{28!}{3!\,25!} = \binom{28}{25} = \binom{25+4-1}{25} = \binom{25+4-1}{3}$$

Generalization the # of solutions to the equation

$$c_1 + c_2 + \quad + c_k = n$$

when $c_i \in \mathbb{Z}$ $c_i \geqslant 0$ is given by

$$\binom{n+k-1}{k-1} = \frac{f^{(k)}(0)}{k!}$$

the generating function $f(x)$ $\left(\frac{1}{1-x}\right)^n$ coeff of deg k

Compositions

A composition of a natural number n is the number of ways to write $n = n_1 + \quad + n_k$ $C(n) = \#$ of com. with $n_i > 0$ in which the $\boxed{\text{order counts}}$

Example 3

$1+1+1$

$1+2$ 4 composition

$2+1$

3

$C(n) := \#$ of compositions of n (with as many or few summand as you want)

$$\boxed{C(n) = 2^{n-1}}$$

$C(n)$

\# of compositions with 1 summand 1

\# 2 summands

\# 3 summands

\# of compositions with n summands 1

rule of sum.

the generating function f_k

① we compute the number of ways to write n as the sum of k positive integers

② The generating function of $C(n)$ will be

$$\sum_{k=0}^{\infty} f_k(x)$$

(Check convergence!)

① The generating function of the number of ways to write n as the sum of k positive integers

$$(x + x^2 + x^3 + \quad + x^k)^k$$

Want the coefficient of deg k.

This is the coefficient of deg k of

$$\left(\sum_{j=1}^{\infty} x^j\right)^k = \left(x \sum_{j=0}^{\infty} x^j\right)^k \longrightarrow \left(\frac{x}{1-x}\right)^k =: f_k(x)$$

② The generating function of $C(n)$

$$\sum C(n) x^n = \sum_{k=1}^{\infty} \left(\frac{x}{1-x}\right)^k = (*)$$

if $|x| < \frac{1}{2} \Rightarrow \left|\frac{x}{1-x}\right| < 1$

$y = \frac{x}{1-x}$

$$(*) = \sum_{k=1}^{\infty} y^k = y \sum_{k=0}^{\infty} y^k = y \frac{1}{1-y} = \frac{x}{1-x} \frac{1}{1 - \frac{x}{1-x}}$$

$$= \frac{x}{\cancel{1-x}} \frac{1}{\frac{1-x-x}{\cancel{1-x}}} = \frac{x}{1-2x}$$ generating function of $C(n)$

Now we compute $c(n)$ from its generating function

$$f(x) = \frac{x}{1-2x} = x\sum_{n=0}^{\infty} (2x)^n$$

$$= \sum_{n=0}^{\infty} 2^n x^{n+1}$$

$$= \sum_{j=1}^{\infty} \underset{\substack{\uparrow \\ C(j)}}{2^{j-1}} x^j$$

$$\Rightarrow \quad C(j) = 2^{j-1}$$

Partitions

A partition of n is an expression of n as the sum of positive integer

$$n = k_1 + \quad + k_j \qquad k_i > 0$$

$$k_1 \le k_2 \le k_3 \quad \ldots$$

(the order does not matter)

$p(n) = \#$ of partitions.

Examples

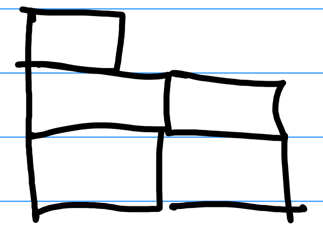
$p(1) = 1$				
$p(2) = 2$	2	$1+1$		
$p(3) = 3$	3	$1+2$	$1+1+1$	
		$1\cdot 1 + 1\cdot 3$		
$p(4) = 5$	4	$1+3$	$1+1+2$	
		$1+1+1+1$	$2+2$	
		$4\cdot 1$	$2\cdot 2$	
$p(5) = 7$				

Young diagram Given a partition $n = k_1 + \quad + k_j$

j rows row i has k_i blocks

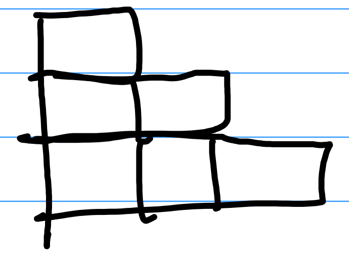
$5 = 1 + 2 + 2$

$= 1 \cdot 1 + 2(2)$



$6 = 1 + 2 + 3$

$1 \cdot 1 + 12 + 1 \cdot 3$



# of partitions of n = the number of Young diagr.

———— of n with k summands

⇝ # Young diagrams with k rows.

⇝ # Young ———— with k cols

→ # partition with summand $\leq k$.

Prop The generating function of $(p(n))_{n \in \mathbb{N}}$ is

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Proof m_i = # of times i appears as a summand of n

$$n = m_1 \cdot 1 + m_2 \cdot 2 + \quad + m_n \cdot n + 0(n+1)$$

$$m_k \cdot k \qquad 1 + x^k + x^{2k} + x^{3k} + \cdots$$

$$= \sum_{k=0}^{\infty} x^{ik} \longrightarrow \frac{1}{1-x^k}$$

$p(n)$ is the coefficient of x^n $\prod_{k=1}^{\infty} \left(1 + x^k + x^{2k} + \cdots\right)$

$$p(2) \qquad \underbrace{\left(\overset{0\cdot 1}{\underline{1}} + x + \underline{x^2} + \cdots\right)\left(\overset{0\cdot 2}{1} + \overset{1\cdot 2}{\underline{x^2}} + \overset{2\cdot 2}{x^4} + \cdots\right)}_{\substack{2\cdot 1 + 0 = 1+1 \\ 0\cdot 1 + 2 = 2}} \left(1 + x^3 + x^6 + \cdots\right)$$

$$\sum p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$ as formal objects

$$0 \le p(n) \le C(n) \doteq 2^{n-1}$$

the lef side has radius of converga $\frac{1}{4} > 0$

We have to see ~~that~~ the right side is a function.

$$\ln\left(\prod_{k=1}^{\infty} \frac{1}{1-x^k}\right) \overset{\text{formal.}}{=} \sum_{k=1}^{\infty} \ln\left(\frac{1}{1-x^k}\right)$$

this is a functio

$$\overset{\text{form}}{=} -\sum_{k=1}^{\infty} \ln(1-x^k)$$

If this sereis of function converge uniformly

$$-\lim_{n\to\infty} \sum_{k=1}^{n} \ln(x-x^k) = -\lim_{n\to\infty} \ln\left(\prod_{k=1}^{\infty}(1-x^k)\right)$$

$$= -\ln\left(\prod_{k=1}^{\infty}(1-x^k)\right.$$

$|x| \le c < 1$

$$\left|\frac{d}{dx} \ln(1-x^k)\right| = \left|-kx^{k-1}\frac{1}{1-x^k}\right| \le k c^{k-1}\frac{1}{1-c}$$

$$\sum_{k=1}^{\infty} |\ln(1-x^k)| \overset{\text{MVT}}{\le} \sum_{k=1}^{\infty} k c^{k-1}\frac{1}{1-c}|x|^k$$

↳ this converges if x is smal