Combinatorics Lectures 5 - 6

- Generating functions II
  - Exponential gf
  - Summation formula

- Recursion part I

Questions ?

Exponential generating functions

(when the order does matter!)

Recall: $(a_n)_{n\in\mathbb{N}}$ a sequence. Its generating series (function) is:

$$\sum_{n\in\mathbb{N}} a_n x^n$$

Example $a_n = 1 \ \forall n$

$$f(x) = \frac{1}{1-x}$$

Def: the exponential generating series of $(a_n)_{n\in\mathbb{N}}$ is

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

Example $a_n = 1 \ \forall n$

$$f(x) = \sum_{n\in\mathbb{N}} \frac{1}{n!} x^n$$

$$= e^x$$

Other examples

$a_n = 2 \quad \forall n$

$a_0 = 1 = a_1 \quad a_i = 0 \ \forall i \geq 2$

$a_n = P(k,n)$

The oder matters!

$$\sum_{n\in\mathbb{N}} \frac{2}{n!} x^n = 2e^x$$

$$\frac{1}{0!} + \frac{1}{1!} x = 1+x$$

$$\sum_{n\in\mathbb{N}} \frac{P(k,n)}{n!} x^n$$

$$= (1+x)^k$$

<u>Example</u> A ship carries 48 flags 12 each for the colors purple yellow blue & red. There are 12 poles

① How many different signals 4^{12}

② How many signals have an even number of blue and an odd number of red. 4^{11}

$$\left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

Looking for the coefficient of degree 12. (times 12!)

$$e^x \cdot e^x \cdot \underset{\overset{\shortparallel}{\frac{e^x+e^{-x}}{2}}}{\cosh x} \cdot \underset{\overset{\shortparallel}{\frac{e^x-e^{-x}}{2}}}{\sinh x} =$$

$$= \frac{1}{4} e^{2x}\left(e^x+e^{-x}\right)\left(e^x-e^{-x}\right) = \frac{1}{4} e^{2x}\left(e^{2x}-e^{-2x}\right)$$

$$= \frac{1}{4}\left(e^{4x}-1\right) = \frac{1}{4}\left(\sum_{n\in\mathbb{N}} \frac{4^n}{n!} x^n - 1\right)$$

degree 12 $\quad \frac{1}{4} \cdot \frac{4^{12}}{12!} \rightsquigarrow$ number we are looking for is $4^{12}/4 = 4^{11}$

$$e^{x} \cdot e^{x} \cdot \cosh x \left(\frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$e^{2x} \frac{e^{x} + e^{-x}}{2} \left(\frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$\frac{1}{2}\left(e^{3x} + e^{x}\right)\left(\frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$\frac{1}{2}\left(\sum \frac{3^n x^n}{n!} + \sum \frac{x^n}{n!}\right)\left(\frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$\frac{1}{2}\left(\sum \frac{(3^n+1)}{n!} x^n\right) \cdot \left(\frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$12!\ \boxed{\frac{1}{2}\,\frac{3^9+1}{9!}\cdot\frac{1}{3!}+\frac{1}{2}\,\frac{3^7+1}{7!}\,\frac{1}{5!}}$$

$\hookrightarrow$ coefficient of deg 12

What you are looking for.

The summation operatior

We want to find an operation that sends the generating function of $(a_n)_{n\in\mathbb{N}}$ to the generating function of $\left(\sum_{k\le n} a_k\right)_{n\in\mathbb{N}}$

$\frac{1}{1-x}$ summaton operator in the contex of generating function (non exponential)

Prop: Let f be the generating function of $(a_n)_{n\in\mathbb{N}}$. Then

$$g(x) := \frac{f(x)}{1-x}$$

Summation operator

is the generating function of $\left(\sum_{k\le m} a_k\right)_{m\in\mathbb{N}}$

Proof

$$f(x) = \sum_{n \in \mathbb{N}} a_n x^n$$

$$g(x) = \frac{f(x)}{1-x} = \left(\sum_{n \in \mathbb{N}} 1 \cdot x^n \right) \cdot \left(\sum_{n \in \mathbb{N}} a_n x^n \right)$$

product of series

$$= \sum_{n \in \mathbb{N}} \left(\sum_{k=0}^{n} 1 \cdot a_{n-k} \right) x^n \qquad h = n-k$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{h=0}^{n} a_h \right) x^n$$

generating f. of $\left(\sum_{h=0}^{n} a_h \right)_{n \in \mathbb{N}}$

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Example find a closed formula for

$$\sum_{k=1}^{n} k = \frac{(n+1)\cdot n}{2}$$

$$(a_n = n)_{n\in\mathbb{N}} \rightsquigarrow \left(\sum_{k=0}^{n} k\right)_{n\in\mathbb{N}} = \left(\sum_{k=0}^{n} a_k\right)_{n\in\mathbb{N}}$$

Prop generating function is

$$\frac{\sum n x^n}{1-x} = \frac{1}{(1-x)^2} = x \cdot \frac{1}{(1-x)^3}$$

$$\sum_1 n x^n = x \sum_1 n x^{n-1} = x \cdot \sum \frac{d}{dx}(x^n)$$

$$= x \frac{d}{dx}\left(\sum_1 x^n\right) = x \frac{d}{dx} \frac{1}{1-x}$$

$$= x \frac{1}{(1-x)^2}$$

$$\frac{x}{(1-x)^3} = x\left(\sum_{k=0}^{\infty}\binom{3+k-1}{k}x^k\right)$$

$$= \sum_{k=0}^{\infty}\binom{3+k-1}{k}x^{k+1}$$

$n = k+1$

$$= \sum_{n=1}^{\infty}\binom{3+n-2}{n-1}x^n$$

coefficient of deg n

$$\binom{3+m-2}{m-1} = \binom{m+1}{m-1} = \frac{\cancel{(m+1)!}}{\cancel{(m-1)!}\cdot 2!} \qquad (m+1)\cdot m$$

$$= \frac{(m+1)\cdot m}{2!} = \frac{(m+1)\cdot m}{2}$$

Example find a closed formula for $\sum_{k=1}^{n} k^2$:

Recurrence relations I

Remember matte 1 seminar: you had a sequence given by recursion you had to guess a closed formula and then prove it

Today We eliminate the guessing part.

• Def a first order linear homogeneous recurrence relation with constant coefficient is a relation of the form

homogene.

$$a_{n+1} = d\, a_n \qquad n \geq 0$$

constant coeff

first order.

Many sequences can satisfy the same relation

Ex $2 \quad 6 \quad 18 \quad \ldots$

$7 \quad 21 \quad 63 \quad \ldots$

satisfy $a_n = 3a_{n-1}$

Values as $a_0 = A$ or $a_1 = B$ are called boundary conditions ($a_0 = A$ is also called initial conditions)

Idea Recurrence relation + boundary conditions identify a unique sequence.

Example $a_n = 7a_{n-1}$ $\quad a_2 = 98$

closed formula

$$a_n = 7a_{n-1} = 7^2 a_{n-2} = \quad = 7^n \cdot a_0$$

$$98 = a_2 = 49 \cdot a_0 \Rightarrow \boxed{a_0 = 2}$$

Closed formula $\boxed{a_n = 2 \cdot 7^n}$

Proof If b_n is another sequence satisfying the recurrence relation with initial condition

$$\frac{b_{n+1}}{a_{n+1}} = \frac{\alpha\, b_n}{\alpha\, a_n} = \quad = \boxed{\frac{b_0}{a_0}} = 1$$

a_n, b_n satisfy $\quad a_n = \alpha\, a_{n-1} \qquad b_n = \alpha\, a_{n-1}$

$a_0 = b_0$

$$\Rightarrow \quad b_{n+1} = a_{n+1} \quad \checkmark$$

Applications to counting: Combinations

We have 1 combination for $n=1$ ($a_1 = 1$)

A combination for n can be obtained in the following two ways

1) We add $+1$ to a combination of $n-1$
2) we rise the last summand to a combination of $n-1$

$$a_n = 2a_{n-1} = 2^2 a_{n-2} \dots 2^{n-1} a_1$$

$$\boxed{a_n = 2a_{n-1}} \rightsquigarrow \boxed{a_n = 2^{n-1} \cdot a_1 = 2^{n-1}}$$

Application to complexity

Input $(x_1 \dots x_n)$

For $i = 1 \dots n-1$ do

For $j = n \dots i+1$

if $x_{j-1} < x_j$ swap x_j and x_{j-1}

$j = j-1$

$i = i+1$

How many comparison?

$a_1 = 0$

now nowag.

$a_n = (n-1) + a_{n-1}$

compare x_k with each

compare the other.

Second order recurrence relations with constant coefficients

A linear recurrence relation with constant coefficients is a relation of the form

$$C_k \cdot a_{n+k} + C_{k-1} a_{n+k-1} + \quad + C_0 a_n = f(n)$$

k - order of the relation

if $f = 0$ we say that the relation is homogeneous linear no powers of a_n appears

constant coeff the C_k's do not depend on n

Lemma if $a_n^{(p)}$ is a particular solution of the recurrence relation and $a_n^{(h)}$ is the general solution to the homogeneous relation then the general solution to the relation is

$$\boxed{a_n = a_n^{(p)} + a_n^{(h)}}$$

We split the problem in 2

- Solution of the homogeneous
- Solution of the non homogeneous

Today: general solution for the homogeneous problem !

Homogenous case :

Rmk $C_k a_{n+k} + \quad + C_0 a_n = 0$

Suppose that $\boxed{A r^n = a_n}$ is a solution then

always a solution

$$C_k A r^{n+k} + \quad + C_0 A r^n = 0$$

$$A r^n \left(C_k r^k + \quad + C_0 \right) = 0$$

$A r^n = 0$

$a_n = 0 \ \forall n$

r is a root

$$C_k r^k + C_{k-1} r^{k-1} + \quad + C_0 = 0$$

is the characteristic polynomial of the relation

Theorem Let $k = 2$ $f = 0$ → homogeneous
↳ second order

Let λ_1 and λ_2 to be the two roots of the characteristic polynomial / equation

$$\boxed{C_2 r^2 + C_1 r + C_0 = 0}$$

① If $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ then the general solution for the recurrence relation is

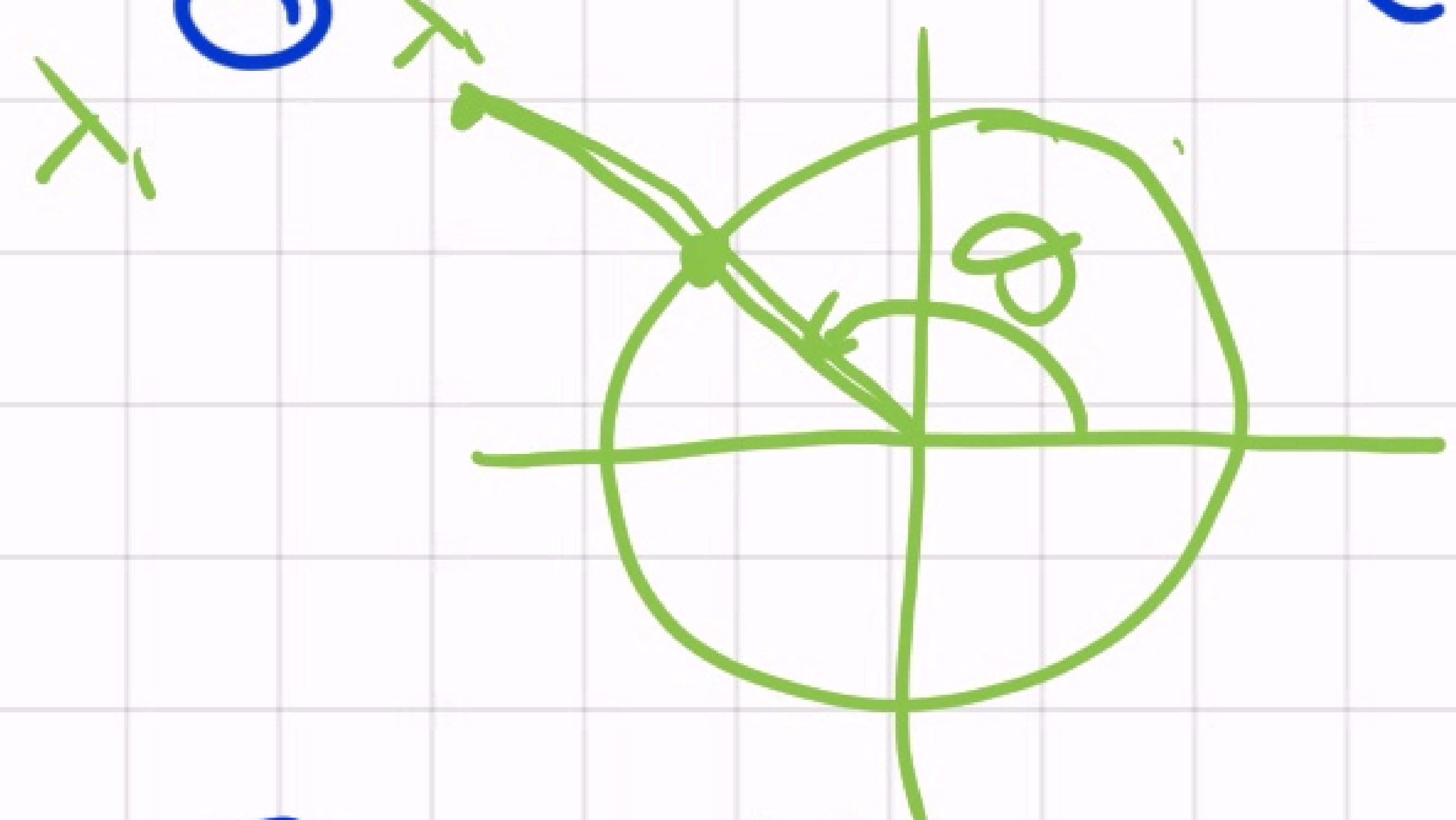
$$\boxed{a_n = \alpha \lambda_1^n + \beta \lambda_2^n} \qquad \alpha, \beta \in \mathbb{R}$$

② $\lambda_1, \lambda_2 \notin \mathbb{R}$ then $\lambda_1 = a + ib$ $\lambda_2 = a - ib$ and the general solution of the recurrence relation is

$$\boxed{a_n = \sqrt{a^2+b^2}\left(\alpha\cos(n\theta) + \beta\sin(n\theta)\right)}$$

where $\theta = \arg(\lambda_1) = \cos^{-1}\left(\frac{\lambda_1}{|\lambda_1|}\right) \in [0, \pi]$

$\alpha, \beta \in \mathbb{R}$



③ If $\lambda_1 = \lambda_2 \in \mathbb{R}$ then

$$\boxed{a_n = (\alpha + \beta n)\lambda_1^n}$$

$\alpha, \beta \in \mathbb{R}$.

Example The Fibonacci sequence

$$F_{n+2} = F_n + F_{n+1} \qquad F_0 = 0 \qquad F_1 = 1$$

→ This is not in "working form"

$$\Rightarrow \quad F_{n+2} - F_{n+1} - F_n = 0$$

the characteristic polynomial / equation

$$r^2 - r - 1 = 0$$

roots

$$r = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4a \cdot c}}{2a}$$

we are in case 1 : two distinct real roots

$$F_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$0 = F_0 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^0 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha+\beta$$

$$1 = F_1 = \alpha\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right) + \beta\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)$$

$\alpha = -\beta$

$$1 = \alpha\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right) - \alpha\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)$$

$$= \cancel{\frac{\alpha}{2}} + \frac{\alpha\sqrt{5}}{2} - \cancel{\alpha\frac{1}{2}} + \frac{\alpha\sqrt{5}}{2}$$

$$= \frac{\cancel{2}\alpha\sqrt{5}}{\cancel{2}}$$

$$\Rightarrow \quad \alpha = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\beta = -\frac{\sqrt{5}}{5}$$

$$F_n = \frac{\sqrt{5}}{5}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{2}\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n$$

$$= \frac{\sqrt{5}}{5}\left[\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n - \left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n\right]$$

Example

Explicit formula for

$$a_n = \det \begin{array}{c} n \\ \begin{pmatrix} b & b & 0 & \cdots & \cdots & \cdots & 0 \\ b & b & b & 0 & \cdots & \cdots & 0 \\ 0 & b & b & b & 0 & \cdots & 0 \\ \vdots & 0 & & & & & \vdots \\ \vdots & \vdots & 0 & \cdots & b & b & b \\ 0 & 0 & & & & b & b \end{pmatrix}_n \end{array} \qquad b \in \mathbb{R}_{\geq 0}$$

$$a_1 = \det(b) = b \qquad a_2 = 0$$

$$a_n = b\, a_{n-1} - b \cdot b\, a_{n-2} = b a_{n-1} - b^2 a_{n-1}$$

$$\begin{pmatrix} b & b & \\ b & b & b \\ & b & b \end{pmatrix}$$

$$b\begin{pmatrix} b & b \\ b & b \end{pmatrix} - b\begin{pmatrix} b & b \\ b & b \end{pmatrix}$$

$$= -b \cdot b \det(b)$$