9.4 Exercise 2:

Determine the sequence generated by each of the following exponential generating functions:

a) $f(x) = 3 \cdot e^{3x}$

b) $f(x) = 6 \cdot e^{5x} - 3e^{2x}$

c) $f(x) = e^{x} + x^{2}$

d) $f(x) = e^{2x} - 3x^{3} + 5x^{2} + 7x$

e) $f(x) = \frac{1}{1-x}$

f) $f(x) = \frac{3}{1-2x} + e^{x}$

Sol:

a) $3 \cdot e^{3x} = 3\left(1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right)$

$$= 3 \sum_{i=0}^{\infty} \frac{(3x)^i}{i!} = \sum_{i=0}^{\infty} 3 \cdot 3^i \cdot \frac{x^i}{i!}$$

$\rightsquigarrow a_n = 3^{n+1}$

b) $6 \cdot e^{5x} - 3 \cdot e^{2x}$

$$= 6 \cdot \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} - 3 \cdot \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(6 \cdot 5^n - 3 \cdot 2^n\right) \cdot \frac{x^n}{n!}$$

$$\leadsto b_n = 6 \cdot 5^n - 3 \cdot 2^n$$

c) $e^x + x^2 = \frac{2 \cdot x^2}{2!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\leadsto c_n = \begin{cases} 3 & \text{if } n=2 \\ 1 & \text{otherwise} \end{cases}$$

d) same idea $d_n = \begin{cases} 9 \text{ if } n=1 & -10 \text{ if } n=3 \\ 14 \text{ if } n=2 & 2^n \text{ otherwise} \end{cases}$

e) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n! \cdot \frac{x^n}{n!}$

$$\leadsto e_n = n!$$

f) same idea $f_n = 3 \cdot 2^n \cdot n! + 1$

Exercise 6:

a) Find the exponential generating function for the number of ways to arrange n letters selected from each of the following words:

i) HAWAII

ii) MISSISSIPPI

iii) ISOMORPHISM

b) For a) ii) what is the exponential generating function if the arrangement contains at least two I's.

Sol: a) i) We multiply the exponential generating functions for H, A, W, I:

$$\left(1+x+\frac{x^2}{2}+\cdots\right)^4 = (e^x)^4 = e^{4x}$$

$$= \sum_{n=0}^{\infty} 4^n \cdot \frac{x^n}{n!}$$

ii) We have the letters M, I, S, P, so the exponential generating function is again $\left(1+x+\frac{x^2}{2}+\dots\right)^4 = e^{4x}$

iii) Now we have the letters I, S, O, M, R, P, H so the EGF is e^{7x}.

b) the EGF for I starts at $\frac{x^2}{2}$, so we obtain

$$\underbrace{\left(1+x+\frac{x^2}{2}+\dots\right)}_{M}\cdot\underbrace{\left(1+x+\frac{x^2}{2}+\dots\right)}_{S}\cdot\underbrace{\left(1+x+\frac{x^2}{2}+\dots\right)}_{P}\cdot\underbrace{\left(\frac{x^2}{2}+\frac{x^3}{6}+\dots\right)}_{I}$$

$$= (e^x)^3\cdot\left(e^x-1-x\right) = e^{4x}-(1+x)e^{3x}$$

9.5: Exercise 2:

a) Find the generating function for the sequences

i) 0, 1, 0, 0, ...

ii) 0, 1, 1, 1, ...

iii) 0, 1, 2, 3, 4, ...

iv) 0, 1, 3, 6, 10, ...

b) Use these to find a formula for $\sum_{k=1}^{n} k$.

Sol:

i) $f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n \boxed{= x}$

ii) $f(x) = \sum_{n=0}^{\infty} b_n \cdot x^n = \sum_{n=1}^{\infty} x^n \boxed{= \frac{x}{1-x}}$

iii) $f(x) = \sum_{n=0}^{\infty} c_n \cdot x^n = \sum_{n=0}^{\infty} n \cdot x^n$

$(1-x) \cdot f(x) = \sum_{n=0}^{\infty} n \cdot x^n - \sum_{n=1}^{\infty} (n-1) \cdot x^n$

$$= 0 + \sum_{n=1}^{\infty} n - (n-1) \cdot x^n = \sum_{n=1}^{\infty} x^n$$

$$= \frac{x}{1-x}$$

$$\rightsquigarrow \boxed{f(x) = \frac{x}{(1-x)^2}}$$

iv) $f(x) = \sum_{n=0}^{\infty} d_n \cdot x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k \right) \cdot x^n$

$$(1-x) \cdot f(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k \right) \cdot x^n - \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} k \right) \cdot x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k - \sum_{k=0}^{n-1} k \right) \cdot x^n$$

$$= \sum_{n=0}^{\infty} n \cdot x^n = \frac{x}{(1-x)^2}$$

$$\rightsquigarrow \boxed{f(x) = \frac{x}{(1-x)^3}}$$

Exercise 4:

If $f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$, i) what is the generating function for the sequence $a_0, a_0+a_1, a_1+a_2, a_2+a_3, \ldots$?

ii) What is the generating function for

$a_0, a_0+a_1, a_0+a_1+a_2, a_1+a_2+a_3, a_2+a_3+a_4, \ldots$?

iii) What is the generating function for

$\frac{a_0}{4}, \frac{a_0}{2}+\frac{a_1}{4}, \frac{a_0}{4}+\frac{a_1}{2}+\frac{a_2}{4}, \frac{a_1}{4}+\frac{a_2}{2}+\frac{a_3}{4}, \ldots$?

Sol:

i) Let $g_1(x) = a_0 + (a_0+a_1)x + (a_1+a_2)x^2 + \ldots$

We see that $g_1(x) - f(x) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots$

$= x \cdot f(x)$.

Therefore, $g_1(x) = x \cdot f(x) + f(x) = (1+x) f(x)$.

ii) Let $g_2(x) = a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + (a_1+a_2+a_3)x^3 + \ldots$

Again, we see that

$$g_2(x) - f(x) = a_0 x + (a_0 + a_1)x^2 + (a_1 + a_2)x^3 + \dots$$

$$= x \cdot g_1(x)$$

$$\rightsquigarrow g_2(x) = x \cdot (1+x) \cdot f(x) + f(x)$$

$$= (1 + x + x^2) \cdot f(x) .$$

iii) Let $g_3(x) = \frac{a_0}{4} + \left(\frac{a_0}{2} + \frac{a_1}{4}\right)x + \left(\frac{a_0}{4} + \frac{a_1}{2} + \frac{a_2}{4}\right)x^2 + \dots$

We compute $g_3(x) - \frac{f(x)}{4} =$

$$\frac{a_0}{2} x + \left(\frac{a_0}{4} + \frac{a_1}{2}\right) x^2 + \left(\frac{a_1}{4} + \frac{a_2}{2}\right) x^3 + \dots$$

$$g_3(x) - \frac{f(x)}{4} - \frac{x}{2} \cdot f(x) =$$

$$\frac{a_0}{4} x^2 + \frac{a_1}{4} x^3 + \frac{a_2}{4} x^4 + \dots$$

$$= \frac{x^2}{4} f(x) \rightsquigarrow g_3(x) = \left(\frac{1}{4} + \frac{x}{2} + \frac{x^2}{4}\right) \cdot f(x)$$