10.1 Exercise 2:

Find the solution for the following recurrence relations:

a) $a_{n+1} - 1.5\, a_n = 0 \quad n \geq 0$

b) $4a_n - 5a_{n-1} = 0, \; n \geq 1$

c) $3a_{n+1} - 4a_n = 0 \quad , n \geq 0 \quad a_1 = 5$

d) $2a_n - 3a_{n-1} = 0 \quad , n \geq 1, \; a_4 = 81$

Solution:

a) $a_{n+1} = 1.5\, a_n = 1.5^2 \cdot a_{n-1} =$

$\quad \ldots = 1.5^{n+1} a_0$

$$\boxed{a_{n+1} = 1.5^{n+1} \cdot a_0}$$

b) $$\boxed{a_n = \frac{5}{4} a_{n-1} = \ldots = \left(\frac{5}{4}\right)^n \cdot a_0}$$

c) $3a_{n+1} = 4a_n \rightsquigarrow a_{n+1} = \left(\frac{4}{3}\right)^{n+1} \cdot a_0$

$5 = a_1 = \frac{4}{3} \cdot a_0 \rightsquigarrow a_0 = \frac{15}{4}$

$$\rightsquigarrow \boxed{a_n = \left(\frac{4}{3}\right)^n \cdot \frac{15}{4}}$$

d) $2a_n = 3a_{n-1} \rightsquigarrow a_n = \left(\frac{3}{2}\right)^n \cdot a_0$

$81 = a_4 = \left(\frac{3}{2}\right)^4 \cdot a_0 \longrightarrow a_0 = 81 \cdot \frac{16}{81} = 16$

$$\longrightarrow \boxed{a_n = \left(\frac{3}{2}\right)^n \cdot 16}$$

10.2 Exercise 4:

Find and solve a recurrence relation for the number of ways to park motorcycle and cars in a row of n spaces, if a motorcycle takes 1 space, a car takes 2 spaces, and we want to use all the n spaces.

Solution:

Let a_n be the number of ways for n spaces. We need to: - find the recurrence relation
- find the initial values.

- recurrence relation: the last vehicle in the row can be either a motorcycle or a car. In the first case we have a_{n-1} ways to fill up the remaining spaces. In the second case we have a_{n-2} ways. Therefore,

$$a_n = a_{n-1} + a_{n-2} .$$ *

- initial values: $n=0$: there is one way: don't park any vehicle. $\rightsquigarrow a_0 = 1$

$n=1$: there is one way: place a motorcycle: $\rightsquigarrow a_1 = 1$

characteristic polynomial: $x^2 - x - 1 = 0$ *

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$$

$$a_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{cases} 1 = a_0 = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 \\ 1 = a_1 = C_1 \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \left(\frac{1-\sqrt{5}}{2}\right) \end{cases}$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$$C_2 = 1 - C_1$$

$$C_1 \left(\frac{1+\sqrt{5}}{2}\right) + (1 - C_1)\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$C_1\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right)=1-\frac{1-\sqrt{5}}{2}$$

$$C_1\cdot\sqrt{5}=\frac{1+\sqrt{5}}{2}$$

$$C_1=\frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$C_2=1-C_1=\frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$a_n=\frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

<u>Remark</u>: a_n is the sequence of Fibonacci numbers.

<u>Remark</u>: Exercise 9 is exactly the same exercise.

Exercise 11:

a) For $n \geq 1$ let a_n count the number of binary strings of length n, where there are no consecutive 1's. Find and solve a recurrence relation for a_n.

b) Let b_n count the number of strings where there are no consecutive 1's, and furthermore the first and last bit are not both 1.

Solution:

a) We distinguish two cases. If the last bit is 0, the remaining $n-1$ bits can be any valid sequence, so there are a_{n-1} of them. If the last bit is 1, then the previous bit has to be 0, and so there are a_{n-2} valid sequences.

So the recurrence relation is $a_n = a_{n-1} + a_{n-2}$

. _ _ . 0

. _ _ . 01

we see that $a_1 = 2$ $\quad 0; 1$ and

$a_2 = 3$ $\quad 00; 01; 10$

We already know that $a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

$$\begin{cases} 2 = a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) \\ 3 = a_2 = C_1\left(\frac{3+\sqrt{5}}{2}\right) + C_2\left(\frac{3-\sqrt{5}}{2}\right) \end{cases}$$

we multiply the first equation by $\frac{1+\sqrt{5}}{2}$:

$$2\left(\frac{1+\sqrt{5}}{2}\right) = C_1\left(\frac{3+\sqrt{5}}{2}\right) + C_2(-1)$$

$$3 - (1+\sqrt{5}) = C_2\left(\frac{3-\sqrt{5}}{2} + 1\right)$$

$$C_2 = (2-\sqrt{5}) \cdot \frac{2}{5-\sqrt{5}} = 2 \cdot \frac{2-\sqrt{5}}{5-\sqrt{5}}$$

$$c_1 = \frac{2 - \frac{1-\sqrt{5}}{2} \cdot c_2}{\frac{1+\sqrt{5}}{2}} \qquad \text{(by the } 1^{st} \text{ eqn)}$$

$$= \frac{2 + \frac{2-\sqrt{5}}{\sqrt{5}}}{\frac{1+\sqrt{5}}{2}}$$

$$= \frac{\frac{\sqrt{5}+2}{\sqrt{5}}}{\frac{1+\sqrt{5}}{2}} \qquad \text{(} c_1 \text{ and } c_2 \text{ could be simplified further)}$$

$$\boxed{a_n = \frac{2(2+\sqrt{5})}{5+\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{2(2-\sqrt{5})}{5-\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n}$$

b) If the last digit is 0, there are a_{n-1} ways to fill up the remaining digits (and <u>not</u> b_{n-1}, because the sequence of length $n-1$ <u>can</u> have a 1 both at the beginning and at the end.

If the last digit is 1, then the previous and the first digit both have to be 0, so there are a_{n-3} valid sequences.

So, we get $b_n = a_{n-1} + a_{n-3}$, and since we already know a_n we can use it to compute b_n.

An equivalent formulation for a_n is

$$a_n = \frac{1}{\sqrt{5}}\left[\frac{3+\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{3-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$\text{so } b_n = a_{n-1} + a_{n-3} = \frac{1}{\sqrt{5}}\left[\frac{3+\sqrt{5}}{2}\left(1+\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-3}\right.$$

$$\left. - \frac{3-\sqrt{5}}{2}\left(1+\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-3}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{5+\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{5-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

$$\boxed{= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n}$$

Remark: In particular this formula gives $b_1 = 1$, which is correct since in the sequence "1", the first and last digit are both 1.

Exercise 14:

An alphabet Σ consists of seven numeric characters and k alphabetic characters. For $n \geq 0$, a_n counts the number of strings (in Σ^*) of length n that contain no consecutive (identical or distinct) alphabetic characters. If $a_{n+2} = 7a_{n+1} + 63a_n$, $n \geq 0$, what is the value of k?

Solution:

We want to express the recurrence relation as a function of k:

Consider a string of length $n+2$.

i) If the last element of the string is a numeric character, then we can complete any valid string of length $n+1$ by that character, and there are 7 numeric characters.

(ii) If the last element is an alphabetic character, then the previous element has to be a numerical character, and any valid string of length n can be completed by such a pair. How many such pairs are there?
We know that there are k alphabetic characters and 7 numerical characters, so there are $7k$ such pairs.
Therefore the recurrence relation is;

$$a_{n+2} = 7 \cdot a_{n+1} + 7k \cdot a_n$$

Compare with the given recurrence relation, we see that $7k = 63$, so $\boxed{k = 9}$