Lectures 7/8

- Recursion II: inhomogeneous problems
    - Variating coefficients
    - Generating function.
- Graph Theory
    - Definitions (recall AoK)
    - Subgraphs complement &
      isomorphism

Non-homogeneous problems

Example

$$a_n - a_{n-1} = n \qquad a_0 = 0$$

Example

$$a_n - 3a_{n-1} = 5 \cdot 7^n \qquad a_0 = 1$$

Solution of the homogeneous :

Solution of the non homogeneous

Try $\quad a_n^{(p)} = A\,7^n$

Table 10.2

$f(n)$	$a_n^{(p)}$
c, a constant	A, a constant
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
$n^t,\ t \in \mathbf{Z}^+$	$A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0$
$r^n,\ r \in \mathbf{R}$	Ar^n
$\sin \theta n$	$A \sin \theta n + B \cos \theta n$
$\cos \theta n$	$A \sin \theta n + B \cos \theta n$
$n^t r^n$	$r^n (A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0)$
$r^n \sin \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$
$r^n \cos \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$

But there are limits

$$a_n - 3a_{n-1} = 5\,3^n$$

we cannot set $a_n = A\,3^n$

we try $a_n = An3^n$

Method of undetermined coefficients

First order

$$a_n + C a_{n-1} = k r^n$$

1) if r is not a root of the char equation

$$a_n^{(p)} = A r^n \quad \text{and find } A$$

2) if r is a root

$$a_n^{(p)} = A n r^n \quad \text{and find } A$$

Second order

$$\boxed{a_n - C_1 a_{n-1} - C_0 a_{n-2} = k r^n}$$

1) $\underline{\underline{A r^n}}$ if r is not a root of the chain
2) $\underline{\underline{A n r^n}}$ if r is a single root
3) $\underline{\underline{A n^2 r^n}}$ if r is a double root

$\longrightarrow$ $x^2 - C_1 x - C_0 = 0$

find the roots

Example Find a solution of

$$\begin{cases} a_{n+2} - 4a_{n+1} + 3a_n = -200 = -200(1^n) \\ \left[\begin{array}{l} a_0 = 3000 \\ a_1 = 3300 \end{array}\right. \end{cases}$$

$a \cdot 1^n$

gen

① Find the solution of the hom problem

② Find a solution of the non-hom prob

③ Use the boundary conditions.

$r = 1$ root of the equation

3 & 1

1. $\boxed{x^2 - 4x + 3 = 0}$ (*)

$(x-3)(x-1) = 0$ we have 2 different roots

$$a_n^{(h)} = A_1 3^n + A_2 1^n = A_1 3^n + A_2$$

2. The RHS is of the form $\alpha \cdot 1^n$ & 1 is a root of (*) ⇝ simple root

$$a_n^{(p)} = B \cdot n 1^n = B \cdot n$$

$$B(n+2) - 4Bn + 3B(n-2) = -200$$

$$\cancel{Bn} + 2B - \cancel{4Bn} + \cancel{3Bn} - 6Bn =$$

$$2B(1 - 6n) = -200$$

$$\cancel{2} \; 2B = -200$$

$$B = 100.$$

general solution

$$a_n^{(h)} + (a_n^{(p)}) = A_1 3^n + A_2 + 100n$$

Use the Boundary to get A_1 A_2

$$\begin{cases} A_1 + A_2 = 2000 \\ 3A_1 + A_2 + 100 = 3300 \end{cases}$$

Solve for A_1 & A_2

The method of generating functions

$$a_n - 3a_{n-1} = n \qquad a_0 = 1$$

What does this says in terms of generating series of a_n ?

$$\begin{aligned}
&\rightsquigarrow a_n &&\rightsquigarrow \sum a_n x^n \\
&\rightsquigarrow -3a_{n-1} &&\rightsquigarrow -3\sum a_{n-1} x^n \\
&\quad = n &&\rightsquigarrow \quad = \sum n x^n
\end{aligned}$$

$$a_n x^n - 3a_{n-1} x^n = n x^n$$

$$\underbrace{\sum_{n=1}^{\infty} a_n x^n}_{f(x) - a_0} - 3 \underbrace{\sum_{n=1}^{\infty} a_{n-1} x^n}_{3x f(x)} = \sum_{n=1}^{\infty} n x^n$$

$$= x \sum_1 n x^{n-1} \qquad = x \frac{d}{dx}\left(\sum_1 x^n\right)$$

$$= x \frac{d}{dx} \frac{1}{1-x}$$

$$= x \frac{1}{(1-x)^2}$$

$f(x) - a_0 = 1$

$f(x) - 1$

$3 f(x)$

$$f(x) - 1 - 3x f(x) = \frac{x}{(1-x)^2}$$

$$f(x)\left[1 - 3x\right] = \frac{x}{(1-x^2)} + 1$$

$$f(x) = \frac{1}{1-3x}\left[\frac{x}{1-x^2} + 1\right]$$

$$\sum (3x)^n \left[\sum n x^n + 1\right]$$ product rule.

Example Recover $\binom{n+r-1}{r}$

$a(n,r)$ = # ways to distribute r objects out of n

Two cases

1) We do not distribute the first object

$\longrightarrow a(n-1, r)$

2) We distribute the first object $\rightarrow a(n, r-1)$

$$\sum_{r=1}^{\infty} a(n,r)x^r = \sum_{r=1}^{\infty} a(n-1,r)x^r + a(n,r-1)x^r$$

$$f_n(x) = \sum_{r=0}^{\infty} \underline{a(n,r)}x^r$$

Fixed

$$f_n(x) - \underset{1}{\underset{\shortparallel}{a(n,0)}} = f_{n-1}(x) - \underset{1}{\underset{\shortparallel}{a(n-1,0)}} + x f_n(x)$$

$$f_n(x)(1-x) = f_{n-1}(x)$$

$$f_n(x) = \frac{f_{n-1}(x)}{(1-x)}$$

$$f_0 \equiv 1$$

$$\Updownarrow$$

McLaurin

$$f_n(x) = \frac{1}{(1-x)^n} = \sum_{r \in \mathbb{N}} \binom{n+r-1}{r} x^r$$

$$\sum_{r=0}^{\infty} a(0,r)\, x^r$$

$$= a(0,0) = 1$$

$$f_n = \frac{f_{n-1}}{(1-x)} = \frac{f_{n-2}}{(1-x)^2} = \frac{f_{n-3}}{(1-x)^3}$$

$$= \cdots = \frac{f_0}{(1-x)^n} = \frac{1}{(1-x)^n}$$

$$= \sum \binom{n+r-1}{r} x^r$$

Graph Theory

Definition A directed Graph is a pair

(V, E) · V set of vertices

· $E \subseteq V \times V$ edges

↳ ordered pair of vertices

Example

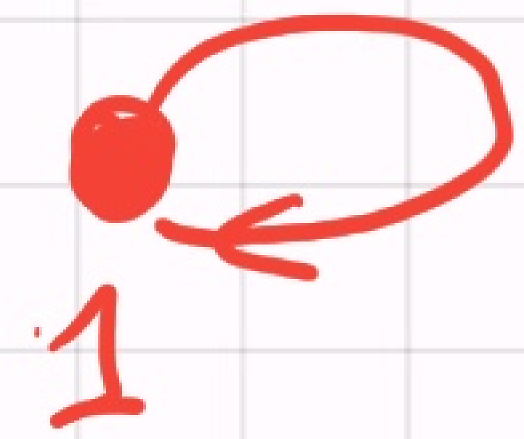
$(\{1,2\}, \{(1,2)\})$

$\underbrace{\{1,2\}}_{V}$ $\underbrace{\{(1,2)\}}_{E}$

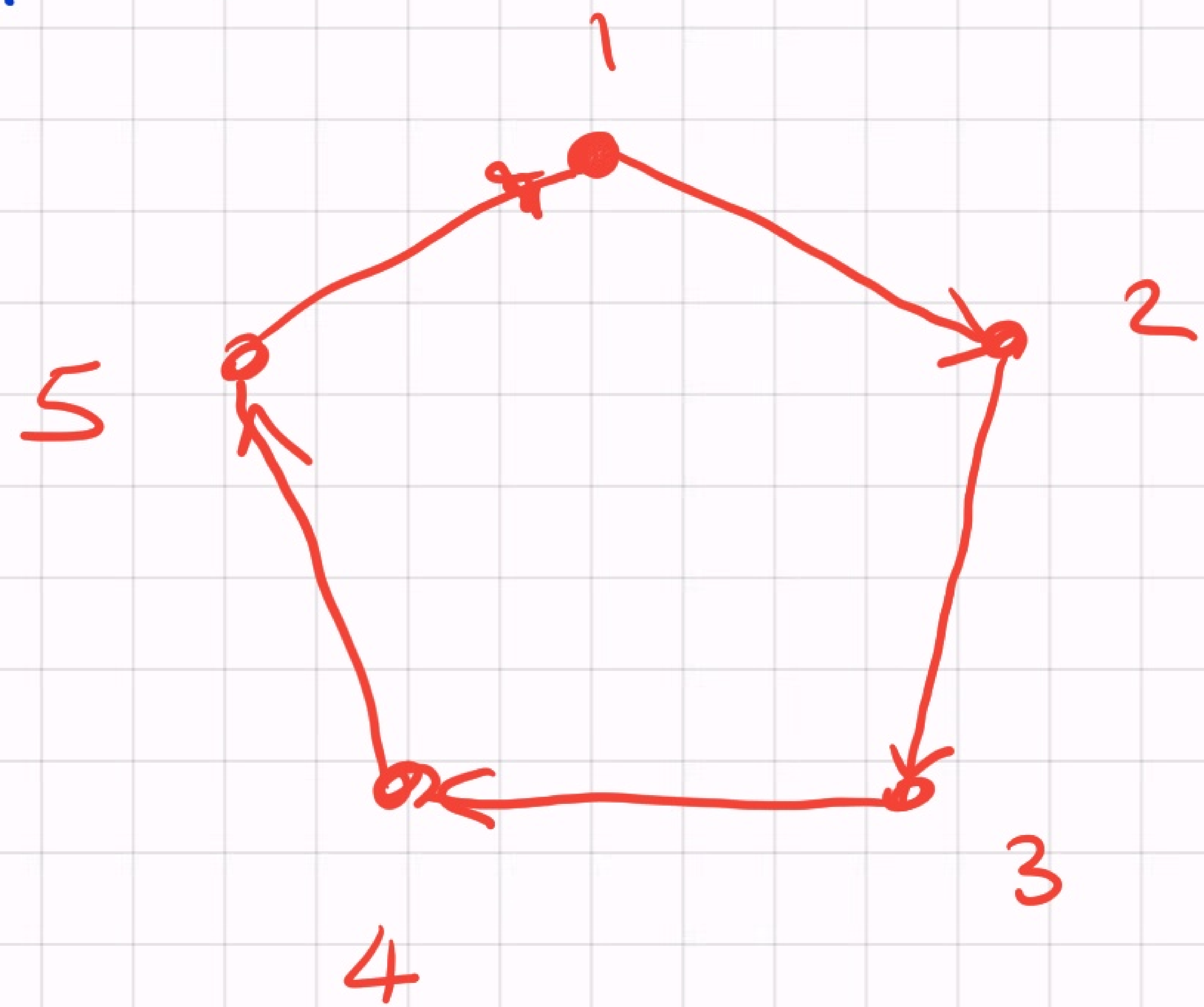
$(\{1,2\}, \{(1,2), (2,1)\})$

$(\{1,2\}, \{\{1,2\} = \{2,1\}\})$

$(\{1\}, \{(1,1)\})$



$(\{1,2,3,4,5\}, \{(1,2), (2,3), (3,4), (4,5), (5,1)\})$



· Definition : An undirected graph is (V, E)

V · set

$E \subseteq \{ A \in \mathcal{P}(V) \mid |A| = 1 \text{ or } 2 \}$

↳ Collection of all subset of V

– Set of parts of V

These are the real graphs

What is different? $\{a_1, a_2\}$

The order of the vertices does not matter

Ex $\left(V = \{1,2\}, \{\{1,2\}, \{2\}\}\right)$

Def Given a directed graph we have two projection

$$s : E \longrightarrow V \qquad \text{source map}$$
$$(v_1, v_2) \longmapsto v_1$$

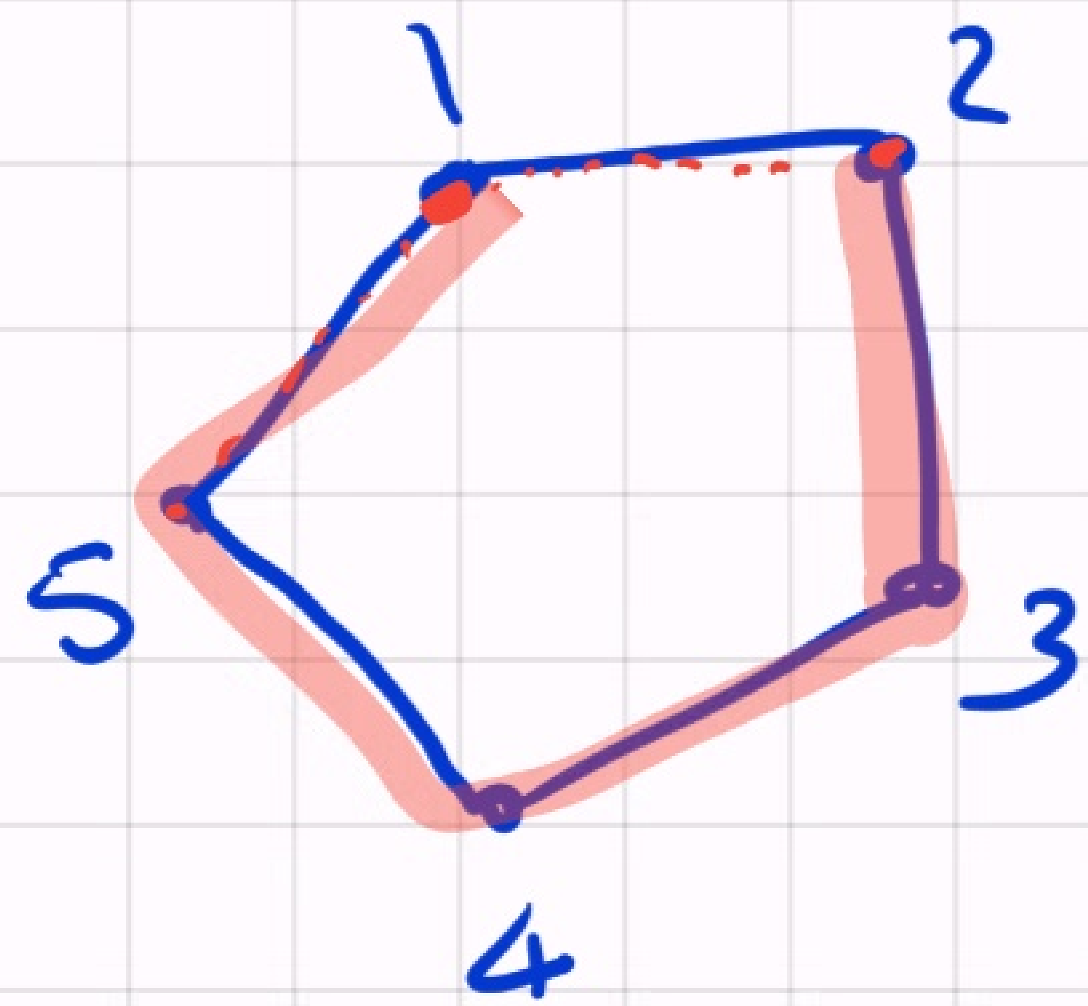
$$r : E \longrightarrow V \qquad \text{range map}$$
$$(v_1, v_2) \longmapsto v_2 \qquad \text{(or target)}$$

Given a graph (either directed or not) we say that two vertices are adjacent if there is an edge connecting them

A loop is an edge of type (v,v) or $\{v\}$

A graph with no loop is said loop free.

Example



loop free

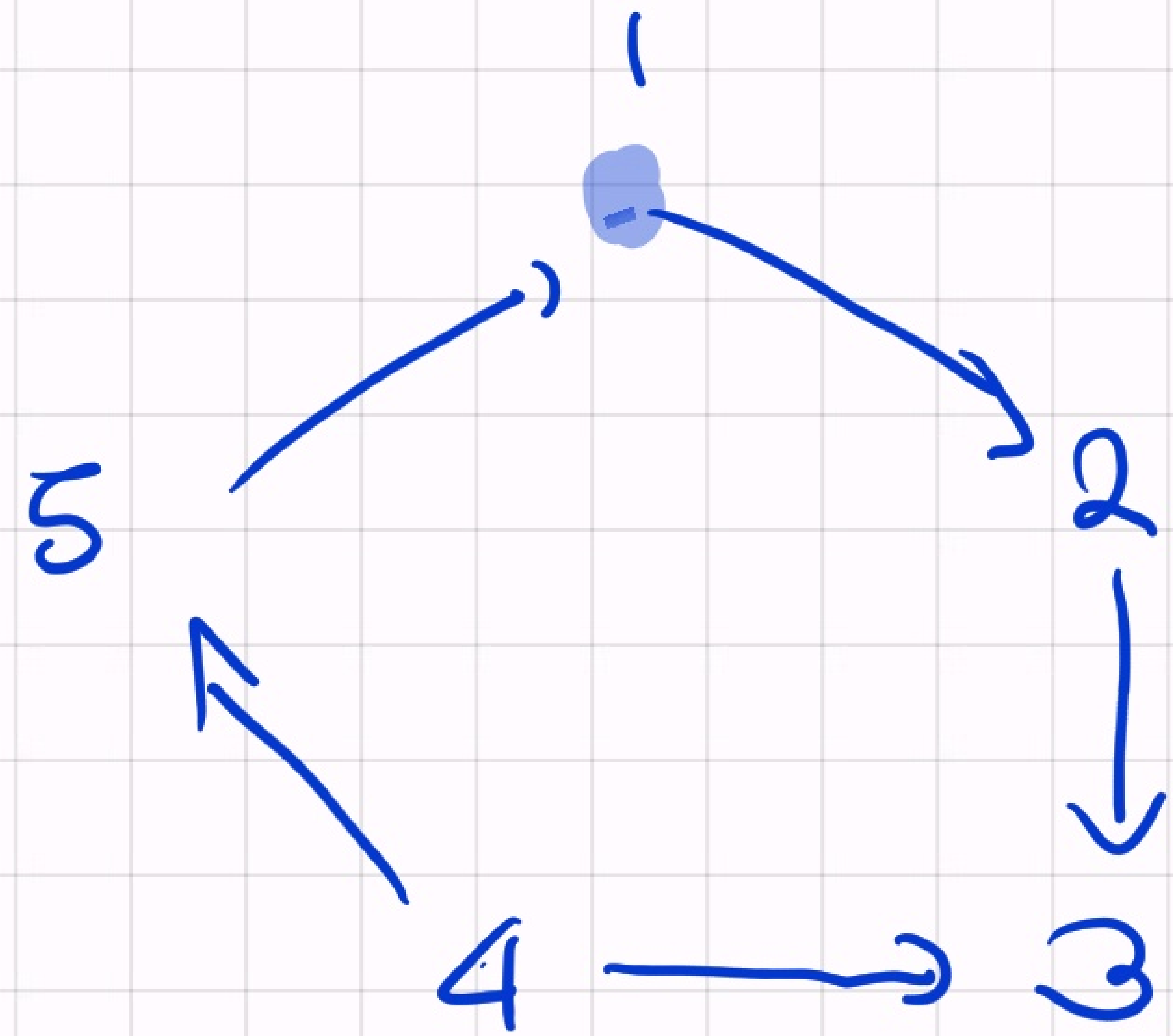
1 is adjacent to 2, 5

and not adjacent to. 3, 4, 1

Walk from 3 to 1

$3 \to 2 \to 3 \to 2 \to 3 \to 4 \to 5 \to 1$

$3 \to 2 \to 1$



there is no walk from 3 to 1

$4 \to 5 \to 1$ walk from 4 to 1

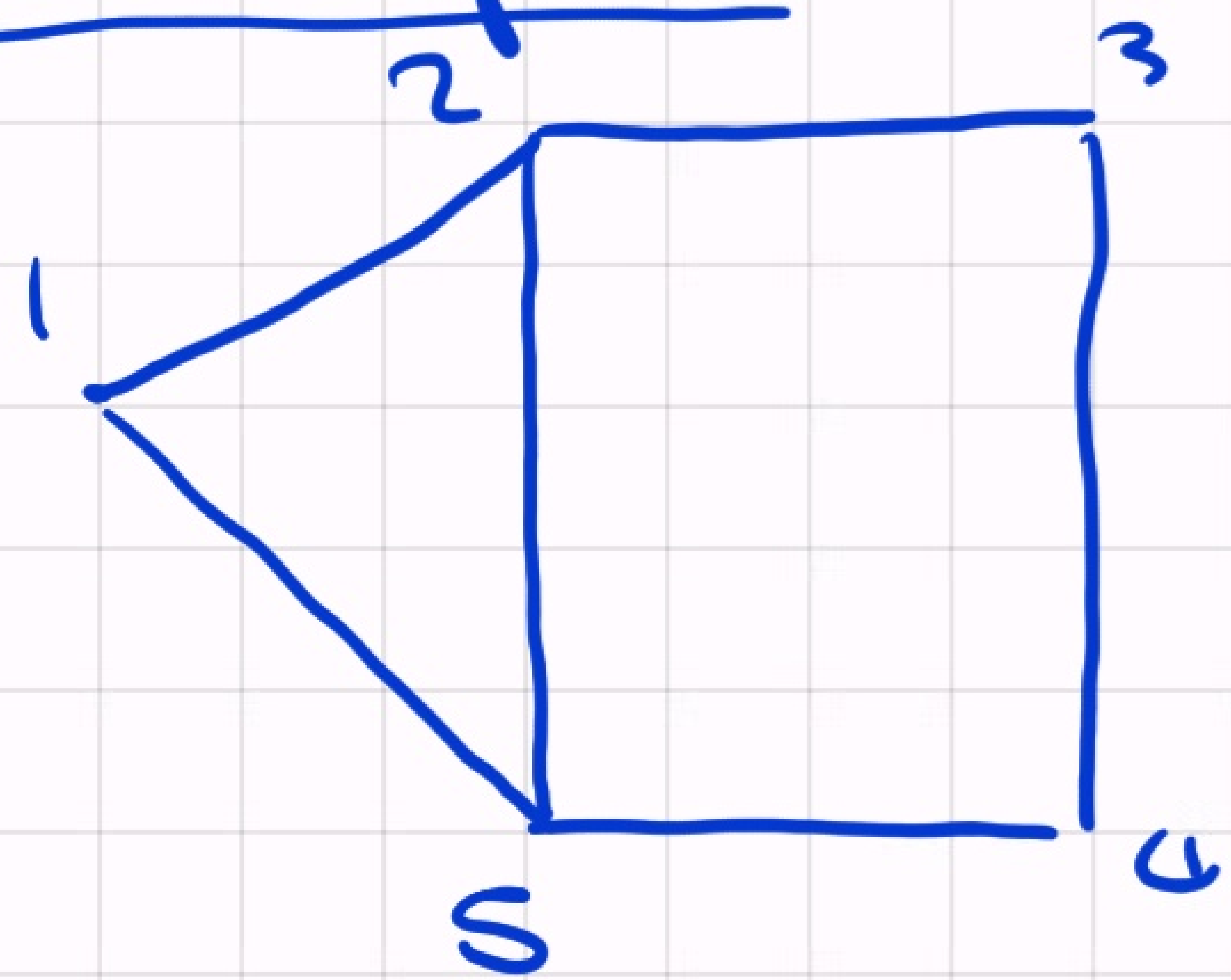
Definition A walk of length n from $x \in V$ to $y \in V$ on a (directed) graph is a n-tuple $(v_1 \_\_\_ v_n) \in V^n$ such that

1) $v_1 = x$, $v_n = y$
2) $\{v_i, v_{i+1}\} \in E$ $\left((v_i\ v_{i+1}) \in E \right)$

If $x = y$ we say that the walk is closed

Notation $v_1 \to v_2 \longrightarrow v_3 \to \cdots \to v_n$

Example



from 1 to 3

1 → 5 → 4 → 3

1 → 2 – 3

~~1 → 2 → 5 → 1 → 2 → 3~~

1 → 2 → 5 → 4 → 3

trail = walk with no repeated edges

circuit = closed trail

path = no repeated vertices.

cycle = closed path

(we have directed analogy)

Prop (V,E) graph $x, y \in V$. Every walk from x to y with minimal length is a path

Proof **Rmk** all walk have non-negative length $S := \{ n \,/\, \text{there is a walk of length } n \text{ from } x \text{ to } y \} \subseteq \mathbb{Z}$

lower bounded (by 0) $\Rightarrow$ the principle of minimum S has a min. d

Let $(v_0 \,\text{----}\, v_d)$ a walk from

x to y of length d. Suppose by contradiction that is not a path, then there is a repeated vertex v. Let v_j be the first time in which v is encountered & let v_{j+k} to be another occurence of v

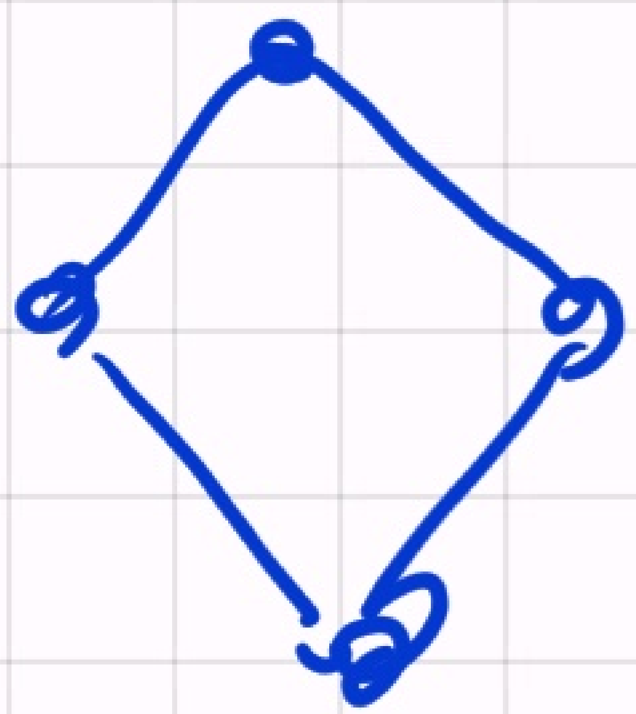
Then $(v_0 \dots v_j, v_{j+k+1}, \dots v_d)$ is another walk from x to y which is SHORTER than $(v_0 \dots v_d)$ ξ

QED.

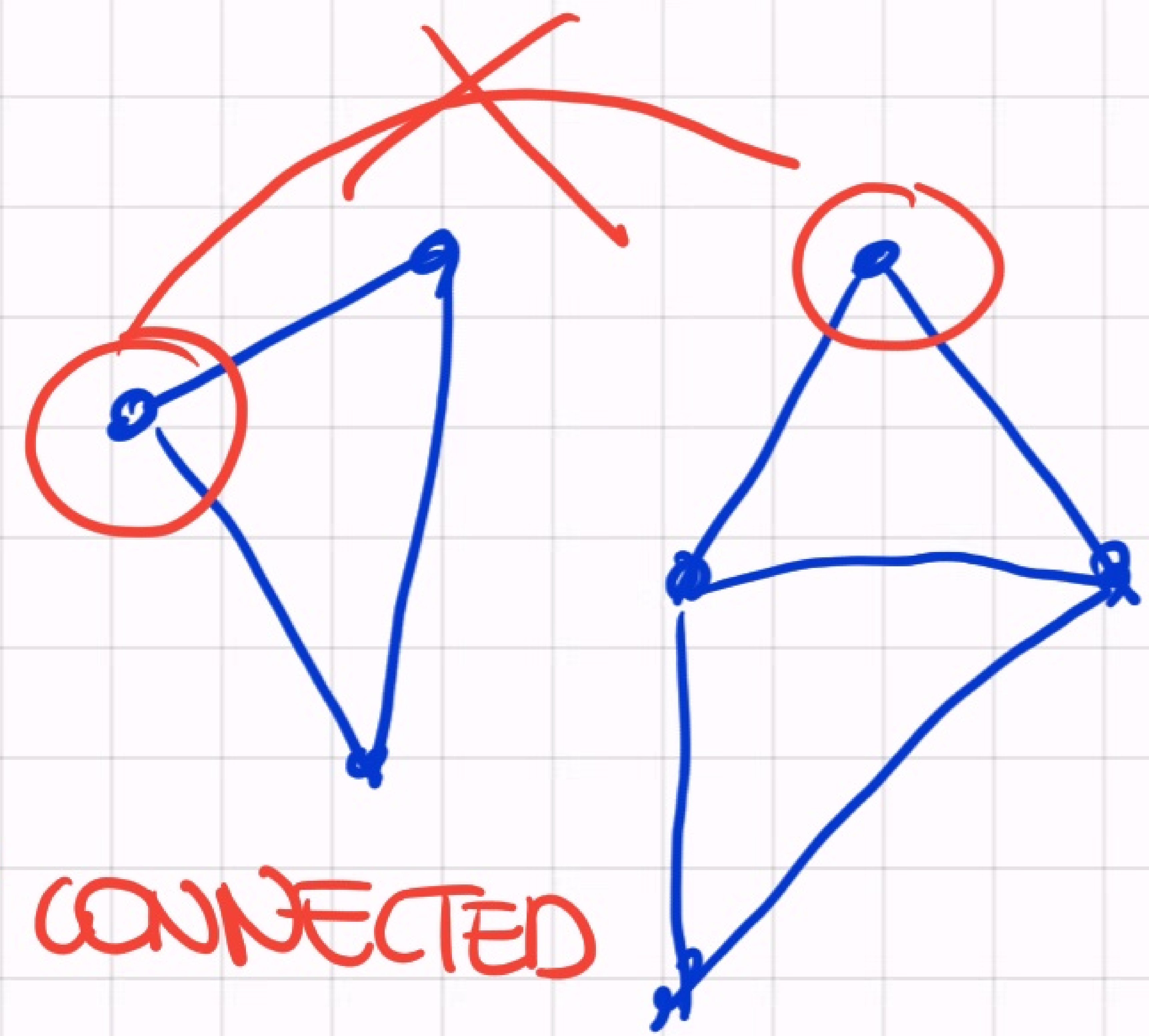
Corollary If there is a walk between x and y then there is a path between x and y

$\rightsquigarrow$ create a path by eliminating the repetition as in the proof before

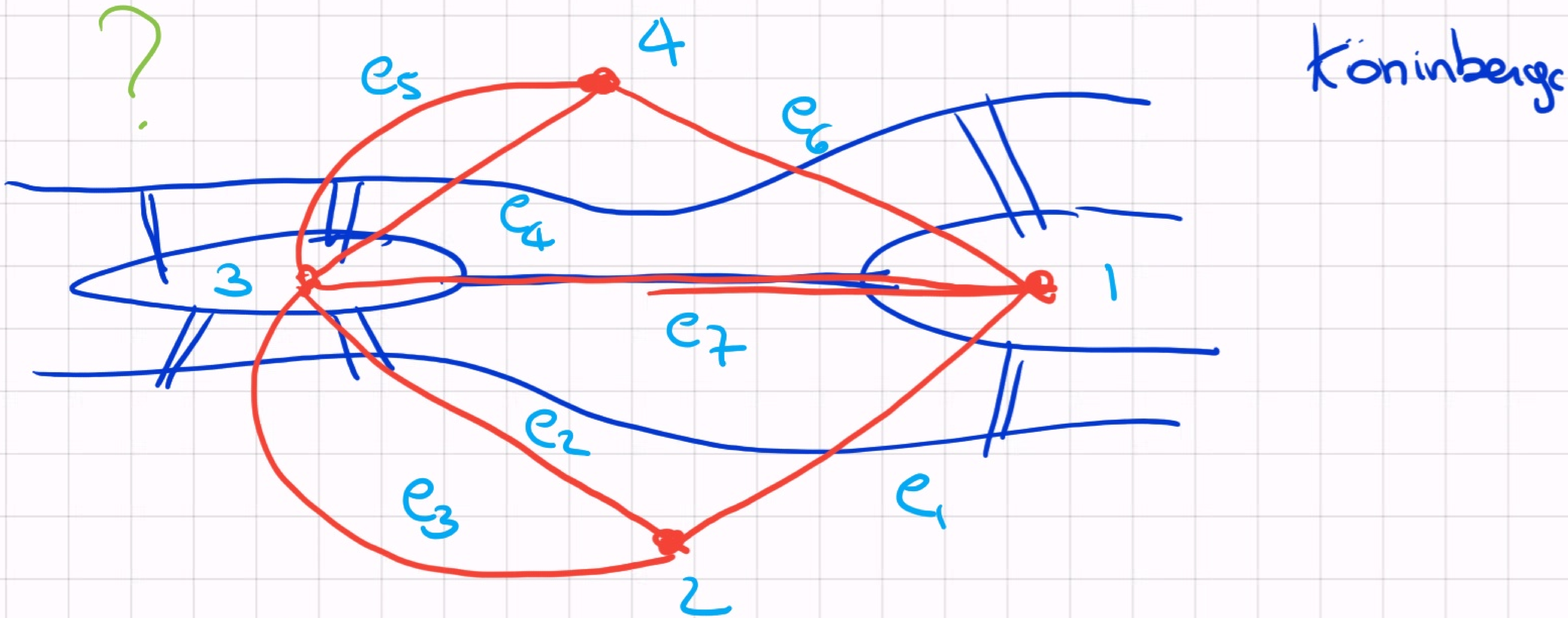
<u>Def</u> A graph is called connected if for every $x \neq y \in V$ there is a path between x and y



CONNECTED



NOT CONNECTED



We have multiple edges

A <u>multigraph</u> is (V, E, f) V, E sets

$$f : E \longrightarrow \{ A \in \mathcal{P}(V) \mid |A| = 1 \text{ or } 2 \}$$

$$V = \{1, 2, 3, 4\}$$

$$E = \{e_1\ e_2\ e_3\ e_4\ e_5\ e_6\ e_7\}$$

$$\rho : E \longrightarrow \{A \subseteq \mathcal{P}(V) \mid |A| = 1 \text{ or } |A| = 2\}$$

$$\begin{aligned}
e_1 &\longmapsto \{1,2\} \\
e_2 &\longmapsto \{2,3\} \\
e_3 &\longmapsto \{2,3\} \\
e_4 &\longmapsto \{3,4\} \\
e_5 &\longmapsto \{3,4\}
\end{aligned}
\qquad
\begin{aligned}
e_6 &\longmapsto \{4,1\} \\
e_7 &\longmapsto \{1,3\}
\end{aligned}$$

Subgraph complement & Isomorphism

• **Def** Given a graph $G = (V, E)$ a subgraph is $G_1 = (V_1\ E_1)$ with

$V_1 \subseteq V$ and $E_1 \subseteq E$

$$\underset{\underset{E_1}{\cup}}{V_1 \times V_1} \hookrightarrow \underline{V \times V}$$

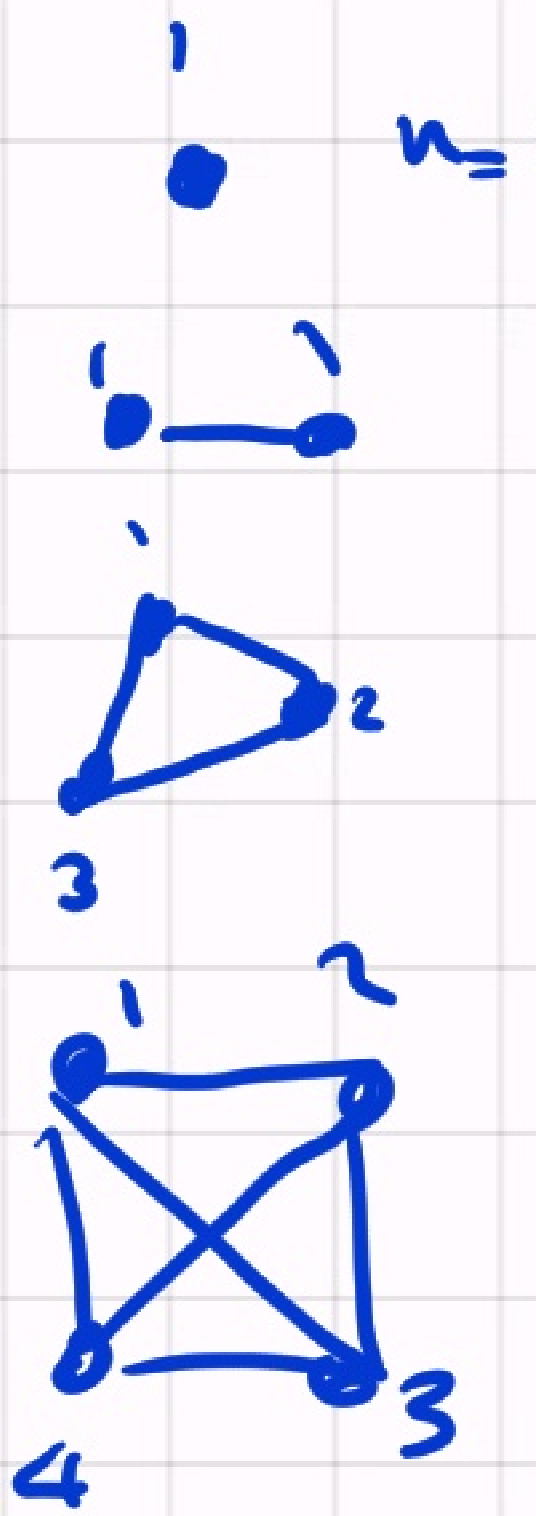
we compare with E here.

If $V_1 = V$ we speak of spanning graph

Example

Subgraphs of 

SPANNING



K_n = complete graph with n vertices

$= (V = \{1 \ldots n\}, \{A \subseteq P(V) \,/\, |A| = 2\})$

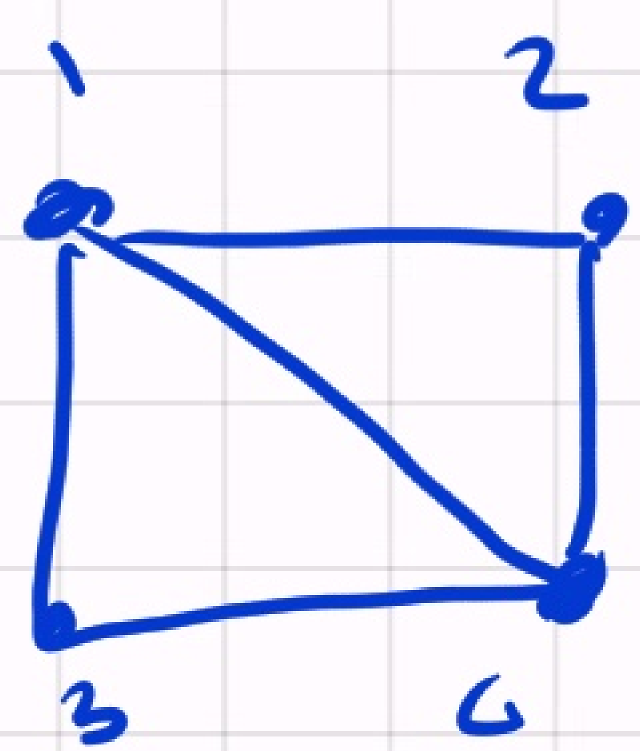
Every (loop free) graph with n vertices is a subgraph of this)

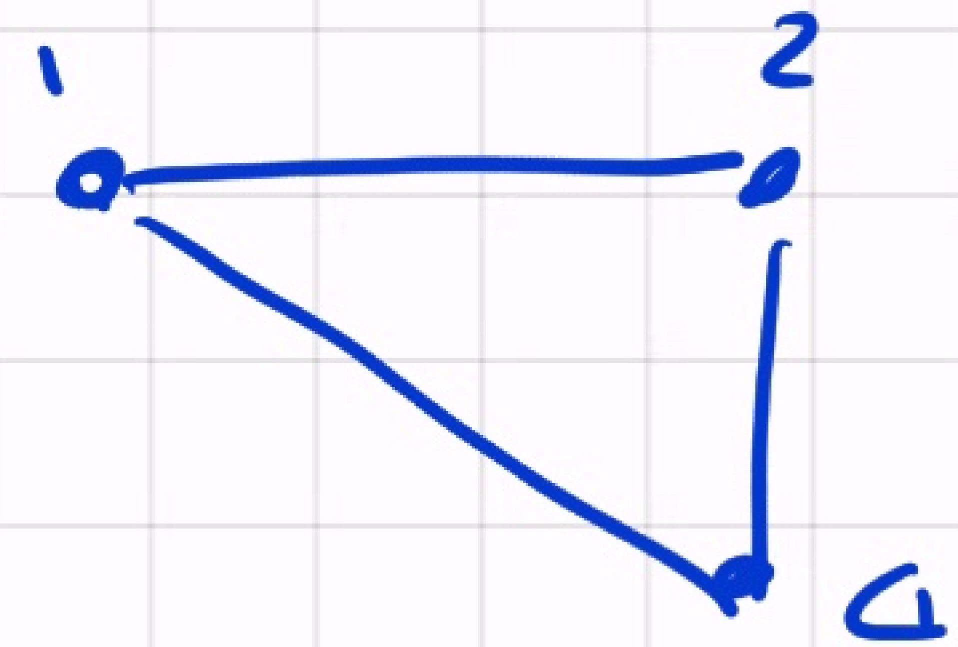
• <u>Def</u> $G = (V, E)$ a graph $U \subseteq V$. The subgraph of G induced by U is

$$\langle U \rangle_G := \langle U, E \cap (U \times U) \rangle$$

Any subgraph of this form is called induced

<u>Example</u> induced graph of $U = \{1, 2, 4\}$

$G =$ 



$G = (V, E)$ graph $v \in V$, $e \in E$

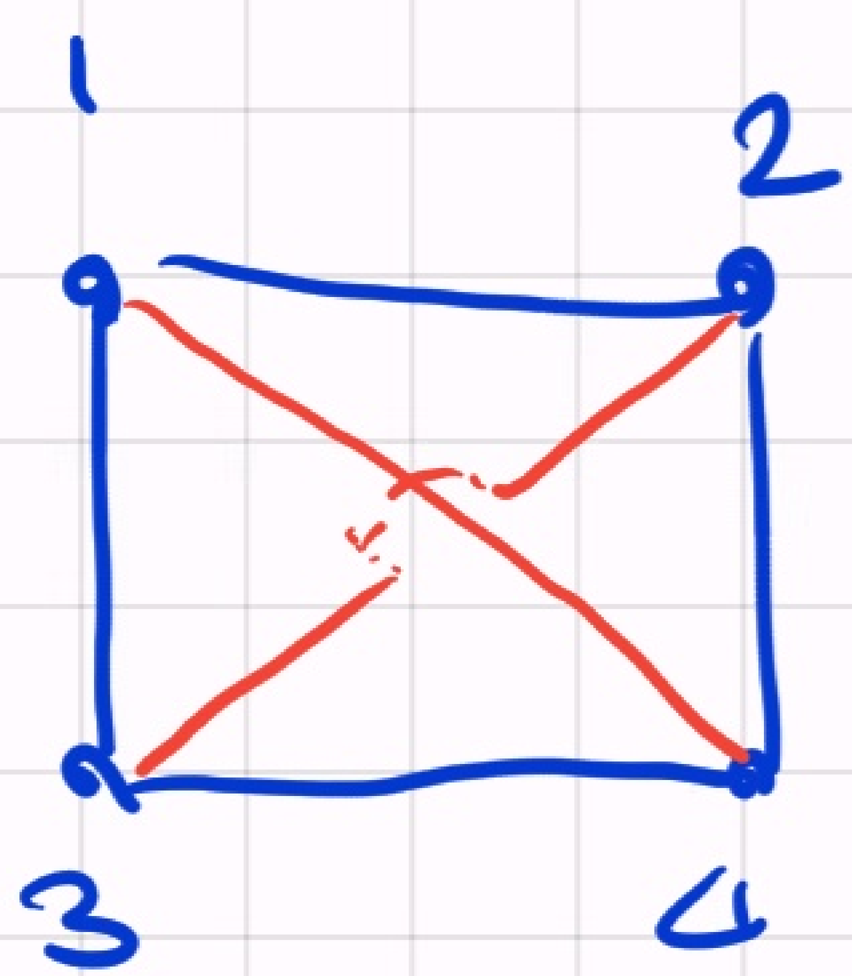
$G - v$ = graph induced by $V \setminus \{v\}$

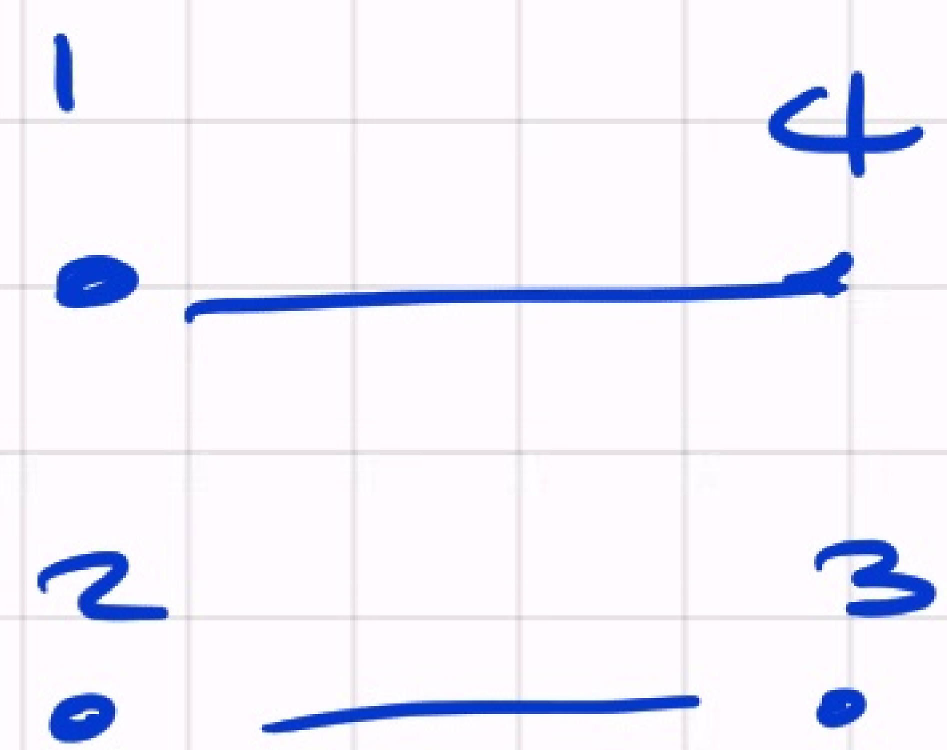
$G - e = (V, E \setminus \{e\})$

If G is loop free the complement of G is $(V, \{A \in \mathcal{P}(V) \mid |A| = 2\} \setminus E)$

<u>Example</u>

G





! can be not connected

Def A graph isomorphism $(V_1, E_1) \to (V_2, E_2)$ is a bijective function

$$f : V_1 \longrightarrow V_2$$

such that $f_1 \times f_2 \,|_{E_1} : E_1 \longrightarrow E_2$

is bijective.

Example $(\{0,1\}^3, E)$ $\{v_1, v_2\} \in E$

iff v_1 and v_2 differ in exactly one coordinate

$$f : V \xrightarrow{\sim} \mathbb{Z}/8\mathbb{Z}$$

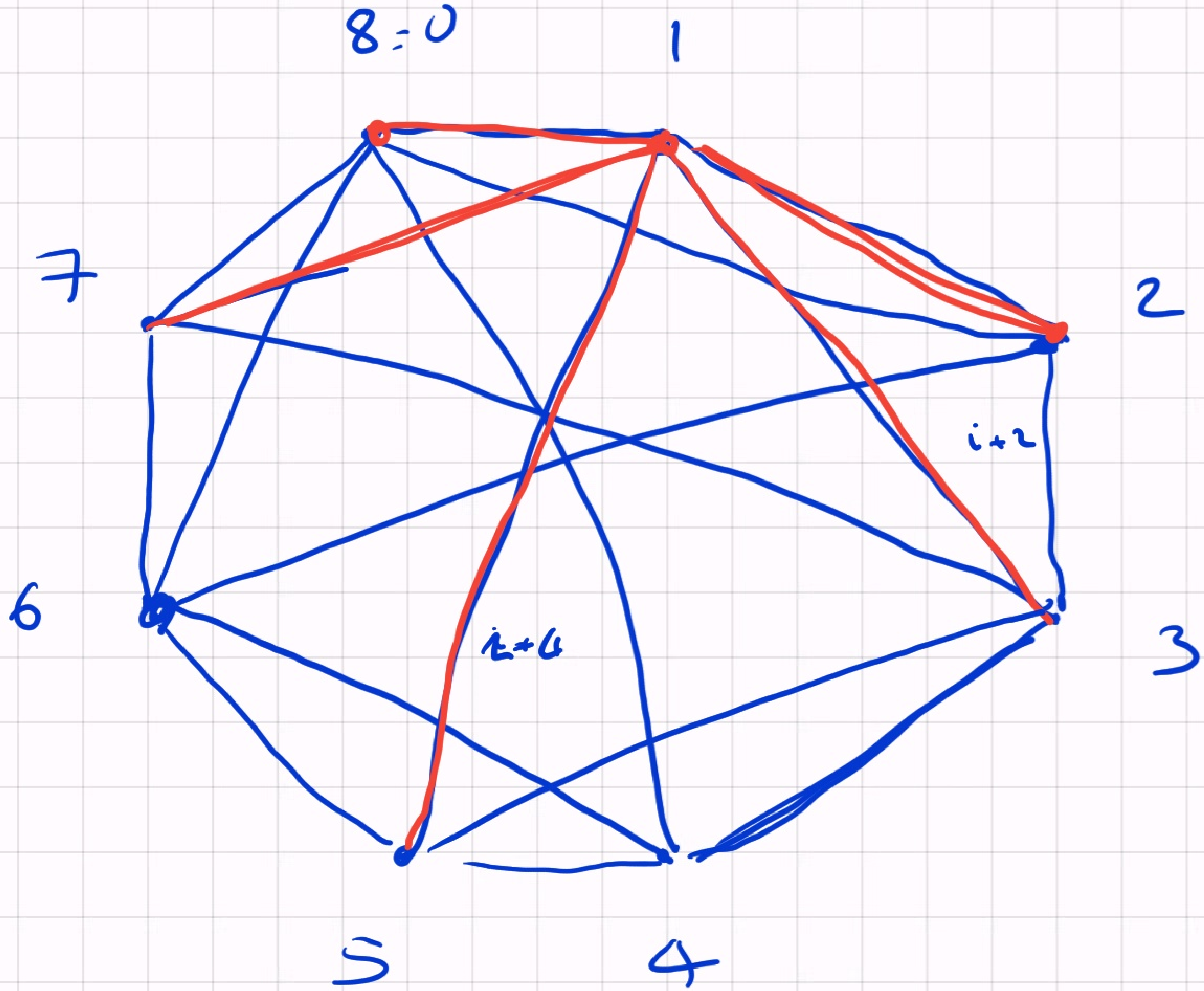
$$(a_1, a_2, a_3) \longmapsto a_1 \cdot 2^2 + a_2 \cdot 2 + a_3$$

Now the vertex i is adjacent to

unit $\quad i+1 \quad$ (units $\neq$)

$i+2$

$i+4$


8 = 0
1
2
3
4
5
6
7
i+2
i+4