10.3 **Exercise 2:**

Use a recurrence relation to derive the formula for $\sum_{i=0}^{n} i^2$

**Solution:** Let $a_n = \sum_{i=0}^{n} i^2$.

Then $a_n = n^2 + a_{n-1}$

$a_{n-1} = (n-1)^2 + a_{n-2}$

$a_{n-2} = (n-2)^2 + a_{n-3}$

Define $b_n = a_{n+1} - a_n$

$$b_n = a_{n+1} - a_n = (n+1)^2 + a_n - (n^2 + a_{n-1})$$

$$= 2n+1 + a_n - a_{n-1} = (2n+1) + b_{n-1}$$

Define $c_n = b_{n+1} - b_n$

$$c_n = b_{n+1} - b_n = (2n+3) + b_n - ((2n+1) + b_{n-1})$$

$$= 2 + b_n - b_{n-1} = 2 + c_{n-1}$$

$$d_n = c_{n+1} - c_n \rightsquigarrow d_n = d_{n-1}$$

so $d_n$ is constant:

$$0 = d_{n+1} - d_n = c_{n+2} - c_{n+1} - c_{n+1} + c_n$$

$$= b_{n+3} - \underbrace{b_{n+2} - 2b_{n+2}} + \underbrace{2b_{n+1} + b_{n+1}} - b_n$$

$$= a_{n+4} - a_{n+3} - 3a_{n+3} + 3a_{n+2} + 3a_{n+2} - 3a_{n+1}$$

$$- a_{n+1} + a_n$$

$$= a_{n+4} - 4a_{n+3} + 6a_{n+2} - 4a_{n+1} + a_n = 0$$

The characteristic polynomial is

$(x-1)^4$, so the solution is of

the form

$$a_n = c_0 \cdot 1^n + c_1 \cdot n \cdot 1^n + c_2 \cdot n^2 \cdot 1^n + c_3 \cdot n^3 \cdot 1^n$$

$a_0 = 0$

$a_1 = 1$

$a_2 = 5$

$a_3 = 14$

$$\begin{cases} c_0 = 0 \\ c_0 + c_1 + c_2 + c_3 = 1 \\ c_0 + 2c_1 + 4c_2 + 8c_3 = 5 \\ c_0 + 3c_1 + 9c_2 + 27c_3 = 14 \end{cases}$$

$$\begin{cases} c_0 = 0 \\ c_1 + c_2 + c_3 = 1 \\ 2c_2 + 6c_3 = 3 \\ 6c_2 + 24c_3 = 11 \end{cases}$$

$$\begin{cases} c_0 = 0 \\ c_1 + c_2 + c_3 = 1 \\ 2c_2 + 6c_3 = 3 \\ 6c_3 = 2 \end{cases}$$

$$c_3 = \frac{1}{3}, \quad c_2 = \frac{1}{2}, \quad c_1 = \frac{1}{6}, \quad c_0 = 0$$

$$\boxed{\begin{aligned} a_n &= \frac{n}{6} + \frac{1}{2} \cdot n^2 + \frac{1}{3} \cdot n^3 \\ &= \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6} \end{aligned}}$$

## Exercise 8:

Determine the number of $n$-digit quaternary $(0,1,2,3)$ sequences in which there is never a 3 anywhere to the right of a 0.

$a_1 = 4$

$a_2 = 15$ (all except "03")

Let $b_n$ be the number of sequences of n-digits not containing 0, and $a_n$ be the number of valid n-digit sequences.

$$a_{n+1} = 4 \cdot b_n + 3 \cdot (a_n - b_n)$$

add any digit to the right of a sequence not containing 0.

add a digit different than 3 to the right of a valid sequence containing 0.

$$a_{n+1} = 3a_n + b_n = 3a_n + 3^n$$

$b_n = 3^n$

$$a_{n+1} - 3a_n = 3^n$$

We solve this recurrence relation with the technique of generating functions:

Consider $f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$.

The recurrence relation gives

$$\sum_{n=0}^{\infty} (a_{n+1} - 3a_n) \cdot x^n = \sum_{n=0}^{\infty} 3^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} a_{n+1} x^n - 3 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (3x)^n$$

$$= \frac{f(x) - a_0}{x} - 3f(x) = \frac{1}{1-3x}$$

$$= f(x)(1-3x) - a_0 = \frac{x}{1-3x}$$

$$\rightsquigarrow f(x) = \frac{x}{(1-3x)^2} + \frac{a_0}{1-3x}$$

$$= x \sum_{n=0}^{\infty} \binom{1+n}{n} \cdot (3x)^n + \sum_{n=0}^{\infty} a_0 \cdot (3x)^n$$

$$= x \sum_{n=0}^{\infty} (n+1+a_0) \cdot 3^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} 3^n (n+1+a_0) \cdot x^{n+1}$$

$$= \sum_{n=1}^{\infty} 3^{n-1} \cdot (n+a_0) \cdot x^n$$

So this means that $a_n = (n+a_0) \cdot 3^{n-1}$

$$a_1 = 4 = (1+a_0) \cdot 3^0 \rightsquigarrow a_0 = 3$$

$$a_2 = (2+3) \cdot 3^1 = 15$$

$$\rightsquigarrow a_n = (n+3) \cdot 3^{n-1}$$

## 10.4: Exercise 1:

Solve the following recurrence relations by the method of generating functions:

a) $a_{n+1} - a_n = 3^n \quad n \geq 0 \quad a_0 = 1$

b) $a_{n+1} - a_n = n^2 \, , \; n \geq 0 \, , \; a_0 = 1$

c) $a_{n+2} - 3a_{n+1} + 2a_n = 0 \, , \; n \geq 0 \, , \; a_0 = 1, a_1 = 6$

d) $a_{n+2} - 2a_{n+1} + a_n = 2^n \, , \; n \geq 0 \, , \; a_0 = 1, a_1 = 2$

Solution:

a) Let $f_1(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$.

The relation gives

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) \cdot x^n = \sum_{n=0}^{\infty} 3^n \cdot x^n ,$$

so

$$\frac{f_1(x) - a_0}{x} - f_1(x) = \frac{1}{1-3x}$$

$$\rightsquigarrow \quad f_1(x)(1-x) - a_0 = \frac{x}{1-3x}$$

$$\rightsquigarrow \quad f_1(x) = \frac{x}{(1-3x)(1-x)} + \frac{1}{1-x}$$

$$\frac{x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$$A(1-x) + B(1-3x) = x$$

$$\begin{cases} A + B = 0 \\ -A - 3B = 1 \end{cases} \qquad \begin{array}{l} B = -A \\ 2A = 1 \rightsquigarrow A = \frac{1}{2} \end{array}$$

$$f_1(x) = \frac{1}{2} \cdot \frac{1}{1-3x} - \frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{1-x}$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{\infty} 3^n \cdot x^n + \frac{1}{2} \sum_{n=0}^{\infty} x^n$$

$$\rightsquigarrow \boxed{a_n = \frac{1}{2} \cdot 3^n + \frac{1}{2}}$$

b) Let $f_2(x) := \sum_{n=0}^{\infty} a_n \cdot x^n$

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) \cdot x^n = \sum_{n=0}^{\infty} n^2 \cdot x^n$$

$$\leadsto \frac{f_2(x) - a_0}{x} - f_2(x) = \sum_{n=0}^{\infty} n^2 \cdot x^n$$

recall: $\sum_{n=0}^{\infty} \binom{i-1+n}{n} \cdot x^n = \frac{1}{(1-x)^i}$

$$i = 3 \leadsto \sum_{n=0}^{\infty} \binom{n+2}{n} \cdot x^n = \frac{1}{(1-x)^3}$$

$$\parallel$$

$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} \cdot x^n$$

$$\sum_{n=0}^{\infty} n^2 \cdot x^n = \sum_{n=0}^{\infty} 2 \cdot \binom{n+2}{n} \cdot x^n - \sum_{n=0}^{\infty} (3n+2) \cdot x^n$$

$$= 2 \cdot \sum_{n=0}^{\infty} \binom{n+2}{n} \cdot x^n - 3 \cdot \sum_{n=0}^{\infty} \binom{n+1}{n} \cdot x^n + \sum_{n=0}^{\infty} x^n$$

$$= \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x}$$

$$F_2(x)(1-x) - a_0 = \frac{2x}{(1-x)^3} - \frac{3x}{(1-x)^2} + \frac{x}{1-x}$$

$$F_2(x) = \frac{2x}{(1-x)^4} - \frac{3x}{(1-x)^3} + \frac{x}{(1-x)^2} + \frac{1}{1-x}$$

$$= 2x \sum_{n=0}^{\infty} \binom{n+3}{n} \cdot x^n - 3x \sum_{n=0}^{\infty} \binom{n+2}{n} x^n$$

$$+ x \sum_{n=0}^{\infty} (n+1) \cdot x^n + \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} \left( 1 + n - 3\binom{n+1}{n-1} + 2\binom{n+2}{n-1} \right) x^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\left( 1 + n - 3\frac{(n+1)n}{2} + 2\frac{(n+2)(n+1)\cdot n}{6} \right)}_{= a_n} \cdot x^n$$

So $\boxed{\begin{aligned} a_n &= \frac{n^3+3n^2+2n}{3} - \frac{3}{2}\cdot(n^2+n) + n + 1 \\ &= \frac{2n^3-3n^2+n+6}{6} \end{aligned}}$

c) Let $f_3(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$

We get $\sum_{n=0}^{\infty} (a_{n+2} - 3a_{n+1} + 2a_n)x^n = 0$

$$\rightsquigarrow \frac{f_3(x) - a_0 - a_1 x}{x^2} - 3 \cdot \frac{f_3(x) - a_0}{x} + 2 f_3(x) = 0$$

$$\rightsquigarrow f_3(x)\left(1 - 3x + 2x^2\right) = a_0 + a_1 x - 3a_0 x$$

$$\rightsquigarrow f_3(x) = \frac{1+3x}{(1-x)(1-2x} = \frac{A}{1-x} + \frac{B}{1-2x}$$

$$A(1-2x)+B(1-x)=1+3x$$

$$\begin{cases} A+B = 1 \\ -2A-B = 3 \end{cases} \qquad \begin{matrix} B = 1-A \\ -A-1=3 \end{matrix} \rightsquigarrow \begin{matrix} B=5 \\ A=-4 \end{matrix}$$

$$f_3(x) = \frac{-4}{1-x} + \frac{5}{1-2x} = -4\sum_{n=0}^{\infty} x^n + 5\sum_{n=0}^{\infty} 2^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} (5\cdot 2^n - 4)x^n$$

$$\rightsquigarrow \boxed{a_n = 5\cdot 2^n - 4}$$

d) Let $f_4(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$

$$\rightsquigarrow \sum_{n=0}^{\infty} (a_{n+2} - 2a_{n+1} + a_n)x^n = \sum_{n=0}^{\infty} 2^n \cdot x^n$$

$$\rightsquigarrow \frac{f_4(x) - a_0 - a_1 x}{x^2} - 2\cdot \frac{f_4(x) - a_0}{x} + f_4(x) = \frac{1}{1-2x}$$

$$\rightsquigarrow f_4(x)\left(1-2x+x^2\right) = \frac{x^2}{1-2x} + a_0 + a_1 x - 2a_0 x$$

$$\rightsquigarrow f_4(x) = \frac{x^2}{(1-2x)(1-x)^2} + \frac{1}{(1-x)^2}$$

$$= \frac{x^2 + 1 - 2x}{(1-2x)(1-x)^2} = \frac{1}{1-2x}$$

$$= \sum_{n=0}^{\infty} 2^n \cdot x^n$$

$$\rightsquigarrow \boxed{a_n = 2^n}$$