Mm5023 lecture 7

Recursion II:
the non homogeneous problem

Plan

- Method of moving coefficients (educated guess)
- Method of generating functions

The problem:

find a closed formula for a sequence

$$(a_n)_{n \in \mathbb{N}}$$

satisfying

$$(*) \quad c_k a_{n+k} + c_{k-1} a_{n+k-1} + \quad + c_0 a_n = f(n)$$

We know that

a particular solution of (*)

$$a_n = \underbrace{a_n^{(h)}}_{\text{all solutions of hom.}} + \boxed{a_n^{(p)}}$$

$a_n^{(h)}$ solution of the hom. relat.

<u>Today</u> find $a_n^{(p)}$ a particular solution of the non hom problem

Example:

$a_n - a_{n-1} = 3n^2 \qquad a_0 = 7$

$a_n = a_{n-1} \quad f(n)$

$a_1 = a_0 + f(1)$

$a_2 = a_0 + f(1) + f(2)$

$a_3 = \underbrace{a_0 + f(1) + f(2)}_{a_2} + f(3)$

$a_n = a_0 + \sum_{k=0}^{n} f(k)$

Our CASE $f(n) = 3n^2$

$= 7 + 3\sum_{k=0}^{n} k^2 \overset{\text{formula}}{=} 7 + \frac{3}{2} n(n+1)(2n+1)$

Example

$$a_n - 2a_{n-1} = 3^n$$

Want to find $a_n^{(p)}$

Guess: $a_n^{(p)} = A \cdot 3^n$

↳ leave some margin.

Plug this into my problem and find A such that this is indeed a solution.

$$\boxed{A \cdot 3^n - 2A3^{n-1} = 3^n}$$

$$A\cancel{3^{n-1}}(3-2) = 3^{\cancel{n}}$$

$$A = 3$$

$$a_n^{(p)} = 3 \cdot 3^n = 3^{n+1}$$ this solve the problem.

$$\sum_{i=0}^{k} c_i a_{n+i} = f(n)$$

The method of varning coefficients

$\sum c_i \cdot a_{n+i} = f_j$

first reduction : if

$$f = \sum \alpha_i f_j \qquad \alpha_i \in \mathbb{R}$$

then it is enough to give a solution to $\sum c_i a_{n+i} = f_j$

We solve $\sum c_i a_{n+i} = f_j(n)$ $\qquad a_n^{(p),j}$

The solution $a_n^{(p)} = \sum_j \alpha_j a_n^{(p),j}$

We need to deal only the case in which f is an elementary functions

Strategy: you look at $f(n)$ and, based on its shape you formulate an educated guess about the shape of $a_n^{(p)}$ leaving margin for adjustment.

The educated guess:

$$\sum c_i a_{n+i} = f_j$$

Table 10.2

f_j	$a_n^{(p)}$
c, a constant	A, a constant
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
$n^t, t \in \mathbf{Z}^+$	$A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0$
$r^n, r \in \mathbf{R}$	$A r^n$
$\sin \theta n$	$A \sin \theta n + B \cos \theta n$
$\cos \theta n$	$A \sin \theta n + B \cos \theta n$
$n^t r^n$	$r^n (A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0)$
$r^n \sin \theta n$	$A r^n \sin \theta n + B r^n \cos \theta n$
$r^n \cos \theta n$	$A r^n \sin \theta n + B r^n \cos \theta n$

for diff equation you have the same table $x \leftrightarrow n$

Example

$a_{n+1} - 3a_n = 5 \cdot 7^n \qquad a_0 = 7$

$$a_n^{(p)} = A 7^n$$

$$A 7^{n+1} - 3A7^n = 5 \cdot 7^n$$

$$\cancel{7^n} A(1-3) = 5 \cancel{7^n}$$

$$A = -\frac{5}{2} \qquad a_n^{(p)} = -\frac{5}{2} 7^n$$

We want a solution with the right initial condition
the homogeneous problem is

$$a_{n+1} - 3a_n = 0 \qquad \lambda - 3 = 0 \qquad \lambda = 3$$

$$a_n^{(h)} = A \cdot 3^n$$

$$a_n = A \cdot 3^n - \frac{5}{2} 7^n \qquad \text{impose } a_0 = 7$$

$$7 = A \cdot 3^0 - \frac{5}{2} 7^0 \qquad A = 7 + \frac{5}{2} = \frac{19}{2}$$

Solution

$$\boxed{a_n = \frac{19\,3^n}{2} - \frac{5\,7^n}{2}}$$

Example

$a_{n+1} - 3a_n = 5\,3^n \qquad a_0 = 5$

$f(n) = 5\,3^n \implies$ Guess $\boxed{a_n^{(P)} = A\cdot 3^n}$

plug in

$A\cdot 3^{n+1} - 3A3^n = 5\,3^n$

$3^n A(3-3) = 5\,3^n$

$0 = 5\,3^n$!! $\frown$ Impossible to find A.

Why we do not get a result?

the general solution of the homogeneous problem would be.

$\lambda - 3 = 0 \qquad \lambda = 3$

$\boxed{a_n^{(h)} = A3^n}$

$a_n = A \cdot 3^n$ is the general solution of the homog. problem. So it makes sense that we cannot find A such that $A \cdot 3^n$ solves the non hom. problem.

$\Longrightarrow$ SOLVE THE HOMOGENEOUS PROBLEM FIRST.

$\Rightarrow$ you know if you are going to have problem with the educated guess

$\Rightarrow$ If you have problems you need to fix the guess.

What is the problem?

Rule of thumb

If the guessed solution is a solution of the homogeneous equation you multiply it by the smallest power of n such that the guess is not anymore a solution of the homogeneous.

→ this is your new guess !

$a_{n+1} - 3a_n = 5\,3^n$

$x - 3 = 0$

$$a_n^{(h)} = A \cdot 3^n$$

3 is a root with multiplicity 1

$$a_n^{(p)} = B(n \cdot 3^n)$$

is not a solution of the homo. problem

$$B(n+1) \cdot 3^{n+1} - 3Bn3^n = 5\,3^n$$

$$3^{n+1} B(\cancel{n}+1-\cancel{n}) = 5\,3^n$$

$$B = \frac{5}{3}$$

$$\Rightarrow \boxed{a_n^{(p)} = \frac{5}{3}3^n = 5\,3^{n-1}}$$

$$a_n = A \cdot 3^n + 5 \cdot 3^{n-1}$$

find A to fit initial conditions

Example: First order $a_{n+1} = d\,a_n \qquad a_0 = A$

Second order

There are problems in two instances

1) $a\, a_{n+2} + b\, a_{n+1} + c\, a_n = \alpha\, r^n$

with $r \in \mathbb{R}$ a root of the char equation

2) $a a_{n+2} + b a_{n+1} + c a_n =$

$$r^n(\alpha \cos n\theta + \beta \sin n\theta)$$

with

$$r^n(\cos\theta + i \sin\theta)$$

a root of the char equation

Example

$$\begin{cases} a_{n+2} - 4a_{n+1} + 3a_n = \boxed{200} \\ a_0 = 3000 \\ a_1 = 3300 \end{cases}$$

$\hookrightarrow$ costant

$200 \cdot 1^n$

Homogeneous

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$$a_n^{(h)} = A3^n + B1^n = A3^n + B$$

Non homogeneous

$$a_n^{(p)} = \cancel{\alpha = \alpha \cdot 1^n}$$

$$a_n^{(p)} = \alpha n$$

$$\alpha(n+2) - 4\alpha(n+1) + 3\alpha n = 200$$

$$\alpha(\cancel{n} + 2 - \cancel{4n} - 4 + \cancel{3n}) = 200$$

$$\alpha = \frac{200}{-2} = -100$$

$$a_n^{(p)} = -100n$$

$$a_n = A3^n + B - 100n$$

Find A & B that satisfy the initial condition.

Example

$$a_{n+2} - 4a_{n+1} + 4a_n = 2^n$$

Solve the homogeneous problem.

$$x^2 - 4\lambda + 4 = 0 \iff (\lambda - 2)^2 = 0$$

$$a_n^{(h)} = A \cdot 2^n + Bn 2^n$$

Specific solution of the non homogeneous problem

$$a_n^{(p)} = \alpha \, n^2 \, 2^n \qquad \text{Find } \alpha$$

$$\alpha(n+2)^2 2^{n+2} - 4\alpha(n+1)^2 2^{n+1} + 4\alpha n^2 \cdot 2^n = 2^n$$

$$\alpha(n^2+4n+4) \cdot 4 - 8\alpha(n^2+2n+1) + 4\alpha n^2 = 1$$

$$\alpha\left(\cancel{4n^2} + \cancel{16n} + 16 - \cancel{8n^2} - \cancel{16n} - 8 + \cancel{4n^2}\right) = 1$$

$$\alpha = \frac{1}{8}$$

$$a_n^{(p)} = \frac{1}{8} n^2 \cdot 2^n = n^2 2^{n-3}$$

The method of generating functions

$a_n - 3a_{n+1} = n \qquad a_0 = 1$

$n \geq 1$

$a_{n-1}x^n - 3a_n x^n = n x^n$

$(\star) \quad \sum_{n=1}^{\infty} a_{n-1} x^n - 3\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n x^n + 0x^0$

suppose that $\sum_{n=0}^{\infty} a_n x^n \longrightarrow f(x)$ absolutely somewhere.

$(\star) \quad x f(x) - 3\left(f(x) - a_0 x^0\right) = \sum_{n=0}^{\infty} n x^n$

$f(x)(x-3) + 3\cdot 1 = \frac{x}{(1-x)^2}$

$\sum n x^n = x \sum_n n x^{n-1} = x \frac{d}{dx} \frac{1}{1-x} = x \cdot -(-1)\frac{1}{(1-x)^2}$

$$f(x) = \frac{1}{(x-3)}\left(\frac{x}{(1-x)^2} - 3\right)$$

We want a_n

$$f(x) = \frac{x - 3(1-x)^2}{(x-3)(1-x)^2}$$

$$= \frac{x - 3(1-2x+x^2)}{(x-3)(1-x)^2} = \frac{-3x^2+7x-3}{(x-3)(1-x)^2}$$

$$= \frac{A}{(x-3)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

(like in solving the integrals) find A B C

Method of partial fractions.

$$\frac{A}{(x-3)} = -\frac{A}{3-x} = -\frac{A}{3}\,\frac{1}{1-\frac{x}{3}} = -\frac{A}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

$$\frac{B}{1-x} = B\sum_{n=0}^{\infty} x^n$$

$$\frac{C}{(1-x)^2} = C\sum_{n=1}^{\infty} n x^{n-1}$$

$$a_n = -\frac{A}{3}\left(\frac{1}{3}\right)^n + B + C(n+1)$$

$$\begin{cases} a_0 x - 3a_1 x = 1x \\ a_1 x^2 - 3a_2 x^2 = 2x^2 \\ a_2 x^3 - 3a_3 x^3 = 3x^3 \\ \vdots \\ a_{n-1} x^n - 3a_n x^n = n x^n \\ \vdots \end{cases}$$

$f(x)$ gf of $\sum a_n x^n$ (if it exists)

$$-3\Big(\overbrace{\sum_{n=0}^{\infty} a_n x^n}^{f(x)} - a_0 x^0\Big) = -3\big(f(x) - a_0\big)$$

$$\underbrace{\sum_{n=1}^{\infty} a_{n-1} x^n} - 3\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n x^n$$

$$\sum_{n=0}^{\infty} a_n x^{n+1} = x \sum_{n=0}^{\infty} a_n x^n = x f(x) \qquad \sum_{n=0}^{\infty} n x^n$$

Example

$m-1$ divison.

$a(n,r)$ = # of ways to choose r objects out of n types allowing repetitions

$$\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$$

$\{ b_1 \ldots b_n \}$

1) b_1 is selected

$a(n, r-1)$

2) b_1 is not selected. $a(n-1, r)$

RULE OF SUM

$$\underset{\substack{\downarrow \\ \text{Types}}}{a(n,r)} = a(n, r-1) + a(n-1, r)$$

$$f_n(x) = \sum_{r \geq 0} a(n,r)\, x^r$$

gen functions

$a(n\ r)$ n fixed.

$$f_n(x) - \underbrace{a(n,0)} = \sum_{r \geq 1} a(n,r)\, x^r$$

$$f_n(x) - \underbrace{a(n,0)}_{1} = \sum_{r\geq 1} \big(a(n-1,r) + a(n,r-1)\big)x^r$$

$$= \sum_{r\geq 1} a(n-1,r)x^r + \sum_{r\geq 1} a(n,r-1)x^r$$

$$= f_{n-1}(x) - \underbrace{a(n-1,0)}_{1} + x \underbrace{\sum_{r\geq 1} a(n,r-1)x^{r-1}}_{\sum_{r\geq 0} a(n,r)x^r \,=\, f_n(x)}$$

$$f_n(x) - \cancel{1} = f_{n-1}(x) - \cancel{1} + x f_n(x)$$

$$(1-x) f_n(x) = f_{n-1}(x)$$

$$f_n(x) = \frac{1}{1-x} f_{n-1}(x) = \left(\frac{1}{1-x}\right)^2 f_{n-2}(x) = \left(\frac{1}{1-x}\right)^3 f_{n-3}(x)$$

$$= \quad \cdots \quad = \left(\frac{1}{1-x}\right)^n f_0(x) \quad = \left(\frac{1}{1-x}\right)^n$$

$$f_0(x) = \sum_{r \geq 0} a(0,r)\, x^r = 1 + 0x + 0x^2 + \cdots$$

$$f_n(x) = \sum a(n,r)\, x^r = \left(\frac{1}{1-x}\right)^n$$

coeff deg r $\quad \binom{n+r-1}{n-1}$

$$a_n - a_{n-1} = 3n^2 \qquad a_0 = 7$$

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 3n^2 x^n + 3 \cdot 0 \cdot x^0$$

$$\left(f(x) - a_0\right) - x \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=0}^{\infty} 3n^2 x^n$$

$$f(x) - 7 - x f(x) = 3 \frac{(1+x)x}{(1-x)^3}$$

$$(1-x) f(x) = \frac{+7(1-x)^3 + 3(1+x)x}{(1-x)^3}$$

$$f(x) = \frac{+7(1-x)^3 + 3(1+x)x}{(1-x)^4} \qquad \frac{1}{1-x} \text{ gf of } n^2$$

$$\frac{7}{1-x} = b_n = 7$$

$$\text{gf } \boxed{\sum_{k=0}^{n} k^2}$$

$$a_{n+2} - 2a_{n+1} + a_n = 0 \qquad a_0 = a_1 = 1$$

$$(x-1)^2$$

$$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$A + Bn$

$B = 0$

$A = 1$

$$\left(f(x) - a_0 x^0 - a_1 x\right) - 2x\left(f(x) - a_0 x^0\right) + x^2 f(x)$$

$$f(x) - 1 - x - 2x f(x) + 2x + x^2 f(x) = 0$$

$$f(x)\left(1 - 2x + x^2\right) = 1 - x$$

$$f(x) = \frac{1-x}{1-2x+x^2} = \frac{\cancel{1-x}}{(1-x)^{\cancel{2}}} = \frac{1}{1-x}$$

$a_n = 1 \quad \forall n.$

$$\sum x^n$$

Denote by $f_n(x)$ the generating function of $a(n,r)$

$$f_n(x) = \sum_{r=0}^{\infty} a(n,r)x^r$$

$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$$= \left(\frac{1}{1-x}\right)^n$$