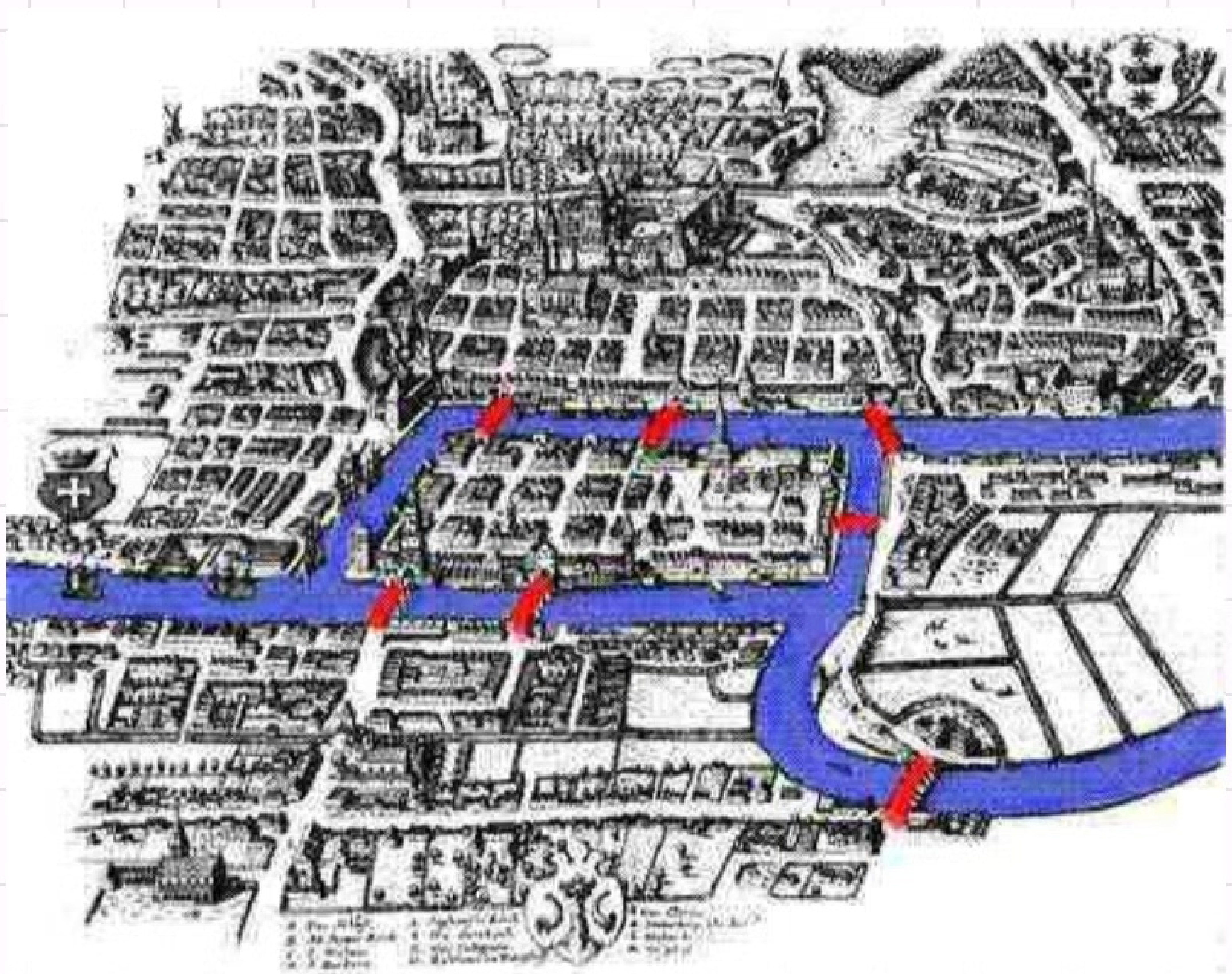
Mm5023 lecture 8

Graphs I



Plan

- Basic definitions
- Walks paths trails circuit
- Connected graph
- Subgraph
- Isomorphism

definiția

Definition a directed graph is a pair (V, E)

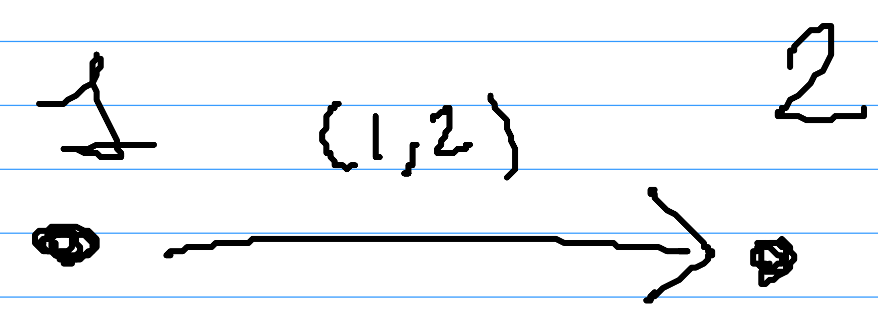
V set (the elements are vertices)

E set (______ edges)

$E \subseteq V \times V$

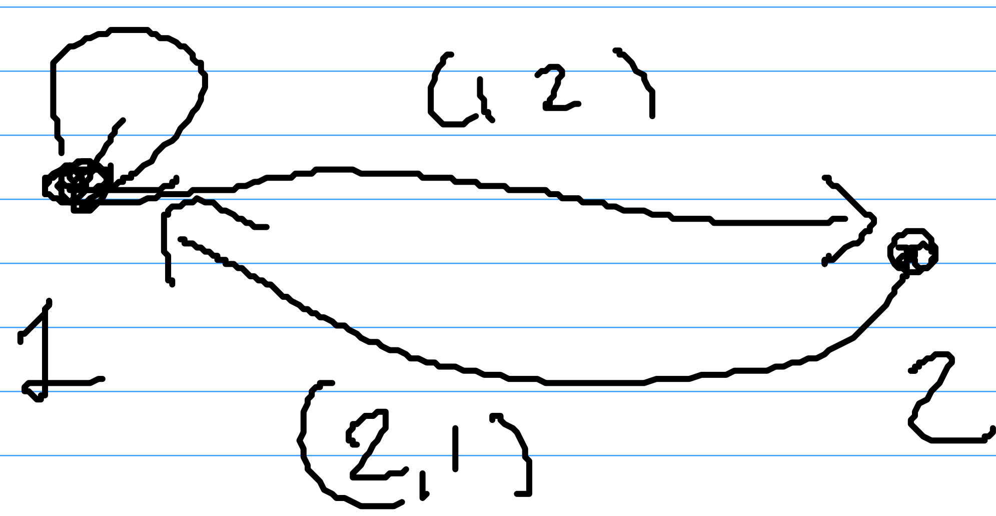
Example

$V = \{1, 2\}$ $E = \{(1,2)\}$

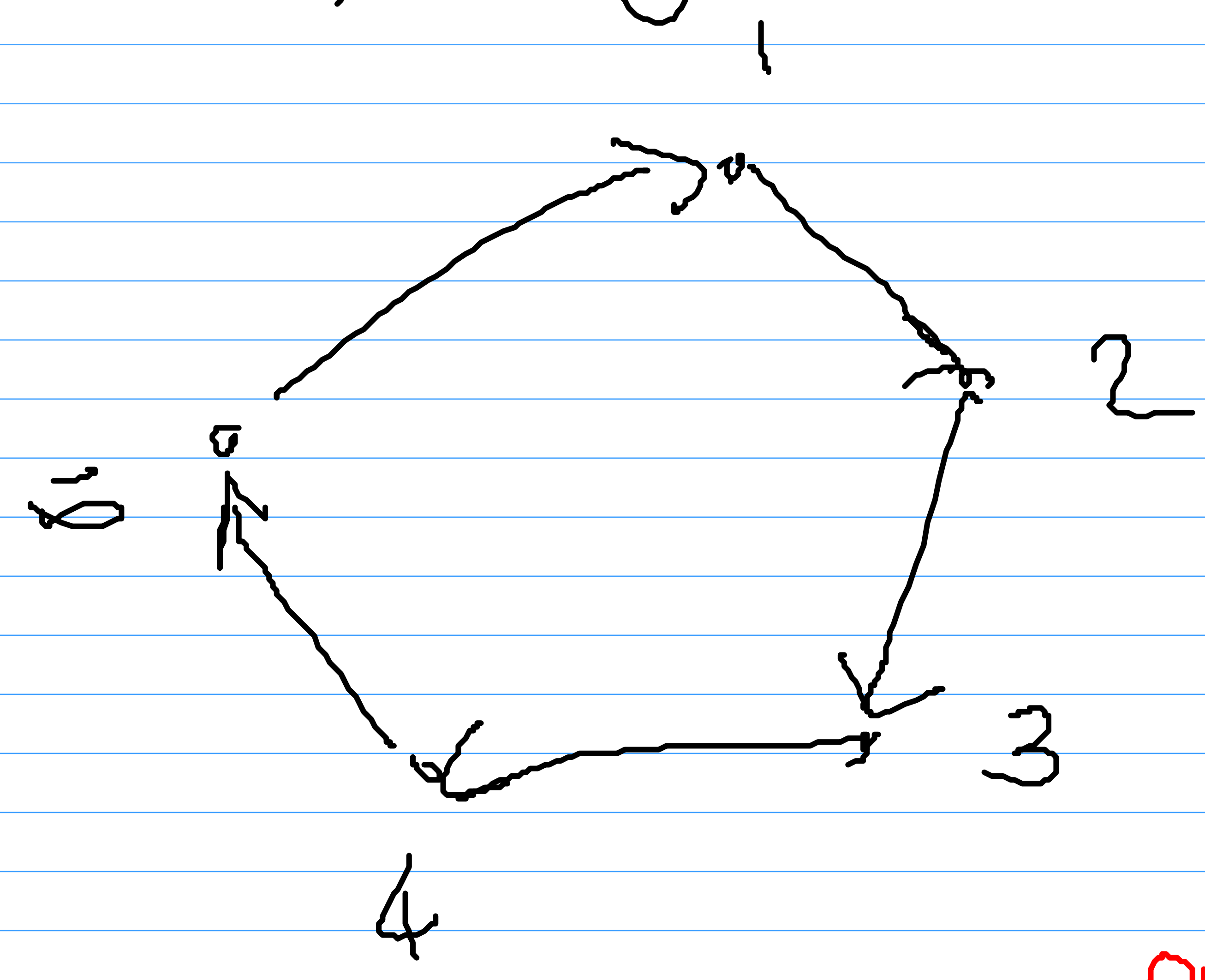


$V = \{1, 2\}$ $E = \{(1,2)\ (1,1)\ (2,1)\}$

loop. →



$V = \{1, 2, 3, 4, 5\}$, $E = \{(i\ i+1)\ i = 1 \dots 4\} \cup \{(5\ 1)\}$



graph = undirected & simple

Def a (undirected simple) graph is (V, E)

V set

$E \subseteq \{A \in \mathcal{P}(V) \mid |A| = 1 \text{ a } 2\}$

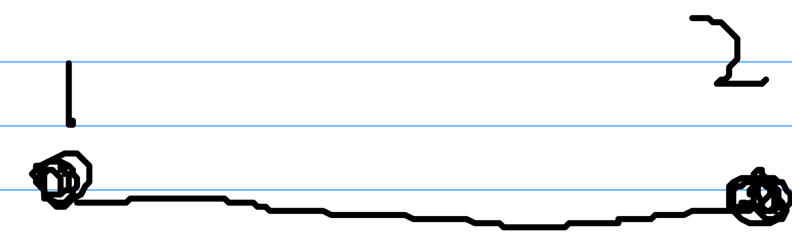
$\mathcal{P}(V)$: Powers set of V

Directed
$e = (1, 2)$

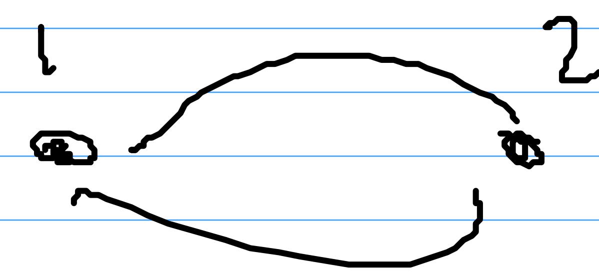
Undirected
$e = \{1, 2\}$

Examples

$V = \{1, 2\}$ $\quad E = \{\{1,2\}\}$



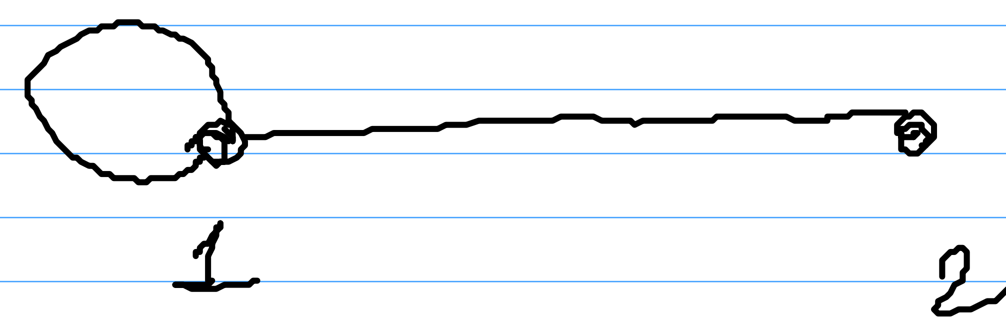
Rmk

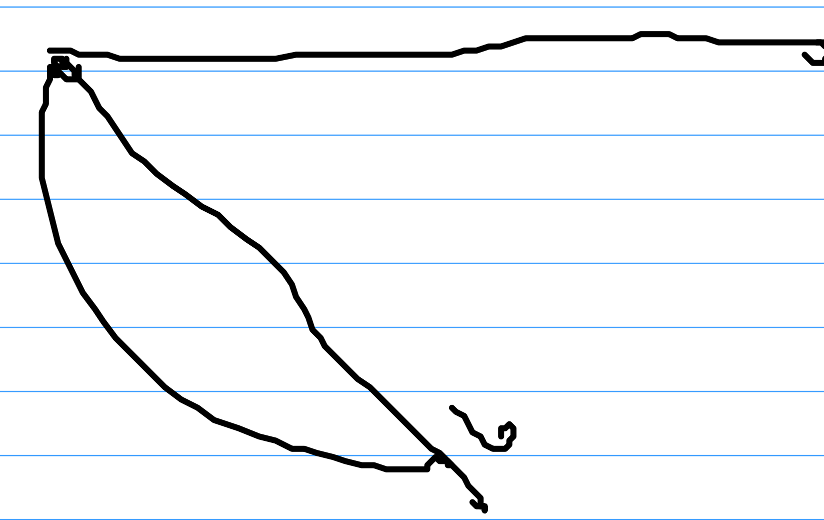


not a (simple) graph

$V = \{1\ 2\}$ $\quad E = \{\{1\}, \{1,2\}\}$

loop:



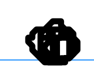


K_n the complete (loop free) graph on n vertices

$$V(K_n) = \{1 \ldots n\}$$

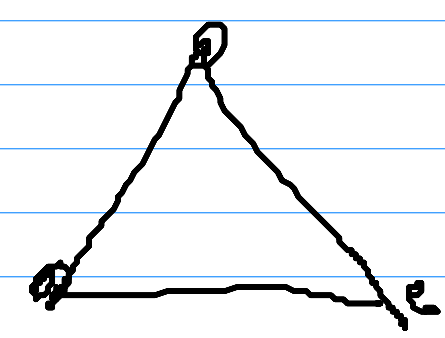
$$E(K_n) = \{\{i,j\} \mid i \neq j\}$$

$n = 1$

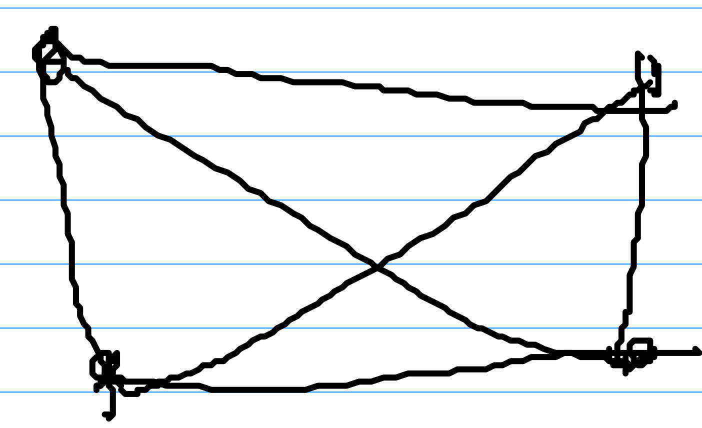


$n = 2$

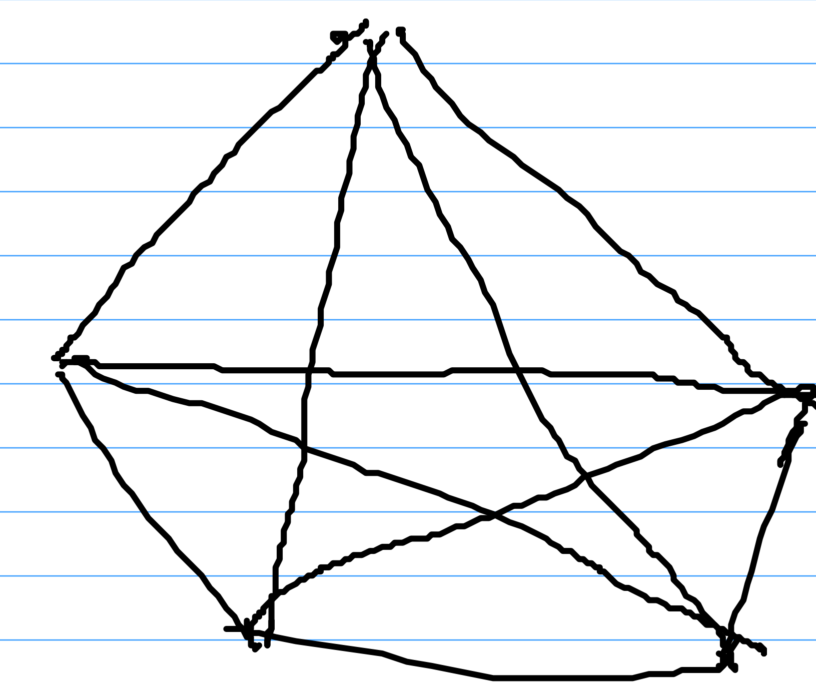
$n = 3$



$n = 4$



$n = 5$



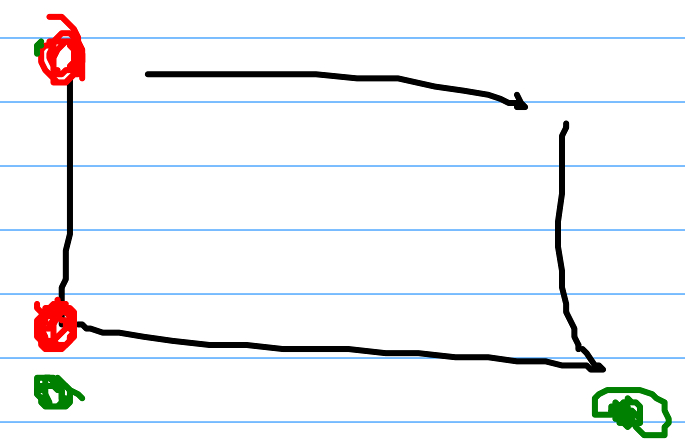
C_n Cyclic graph on n vertices ($n \geq 3$)

$V = \{1 \dots n\}$

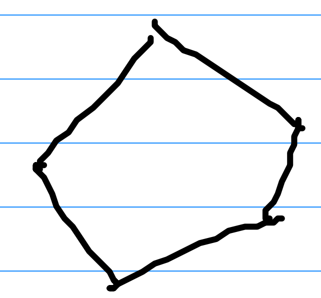
$E = \{ \{i, i+1\} \ i = 1 \dots n-1 \} \cup \{\{n, 1\}\}$

$n = 3 = K_3$

$n = 4$



$n = 5$



Given a directed graph there are two functions

$$s : E \longrightarrow V \qquad \text{SOURCE}$$
$$(i,j) \longmapsto i$$

$$r : E \longrightarrow V \qquad \text{RANGE}$$
$$(i,j) \longmapsto j$$

We say that two vertices in a (directed or und) graph are ADJACENT if there is an edge connecting them

$v \sim w$

DIRECTED	UNDIRECTED
$v \sim w$ iff	$v \sim w$ iff
(v,w) or $(w,v) \in E$	$\{v,w\} \in E$

IS NON AN EQUIVALENCE RELATION

Loops:

DIRECTED	UNDIRECTED
(V, E) directed grap $e \in E$ is said to be a loop if $e = (i, i)$ for $i \in V$	(V, E) undirected graph $e \in E$ is called a loop if $\lvert e \rvert = 1$

A graph (directed / undirected) is called loop-free if it has no loops.

Def a (undirected) multigrap is (V, E, p)

V set

E set

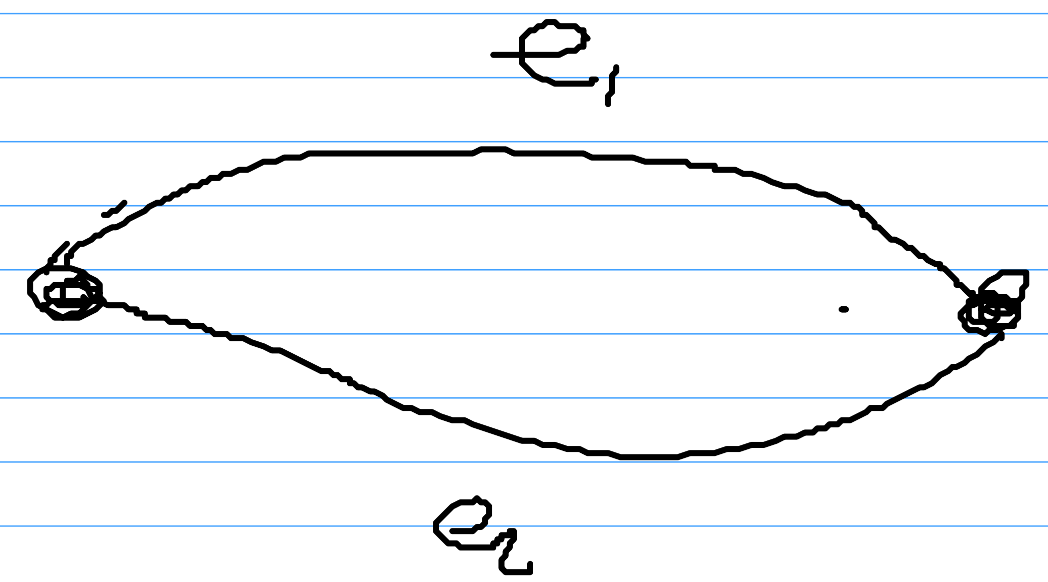
$p : E \longrightarrow \mathcal{P}(V)$

$\operatorname{Im} p \subseteq \{ A \in \mathcal{P}(V) \mid |A| = 1, 2 \}$

Example

$V = \{1, 2\}$ $\quad E = \{e_1 \; e_2\}$

$$p(e_1) = \{1, 2\}$$
$$\shortparallel$$
$$p(e_2)$$



Definitions

(V, E) undirected not necessarily simple graph.

A walk of length n from x to y $\quad x, y \in V$

is $(v_1, e_1, v_2, e_2 \dots e_n\ v_{n+1})$

$v_i \in V \qquad e_i \in E$

$p(e_i) = \{v_i, v_{i+1}\} \qquad v_1 = x \qquad v_{n+1} = y$

(E is simple $\quad p =$ inclusion

$E \subseteq \{A \in \mathcal{P}(V) \mid |A| = 1, 2\}$)

$\hookrightarrow$ ignore the edges

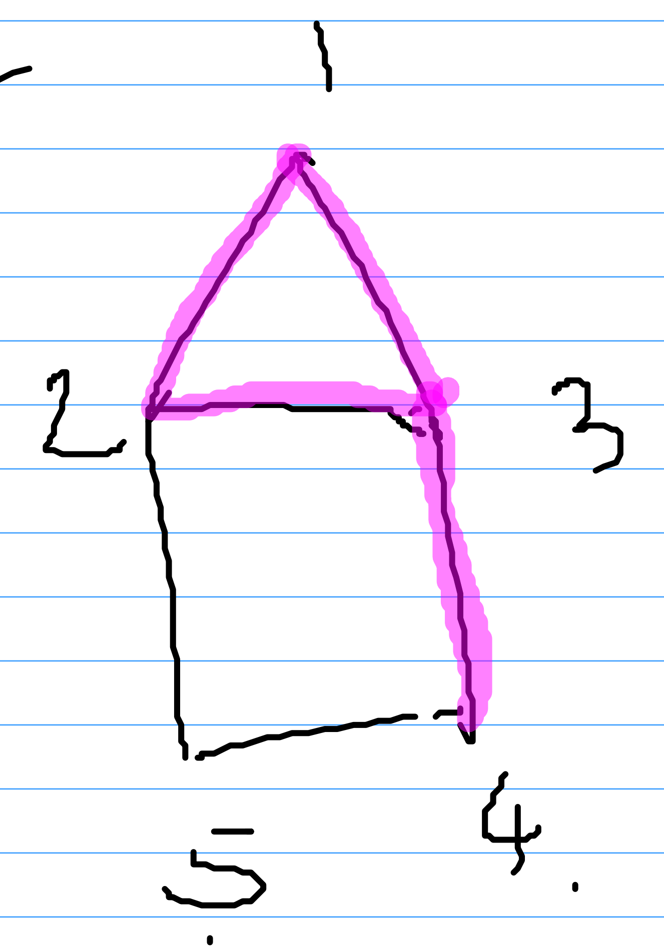
"walk" $(v_1 \dots v_{n+1})$

$v_1 = x \qquad v_{n+1} = y$

Ex modify for directed graph.

If $x = y$ we say that the walk is CLOSED

Example



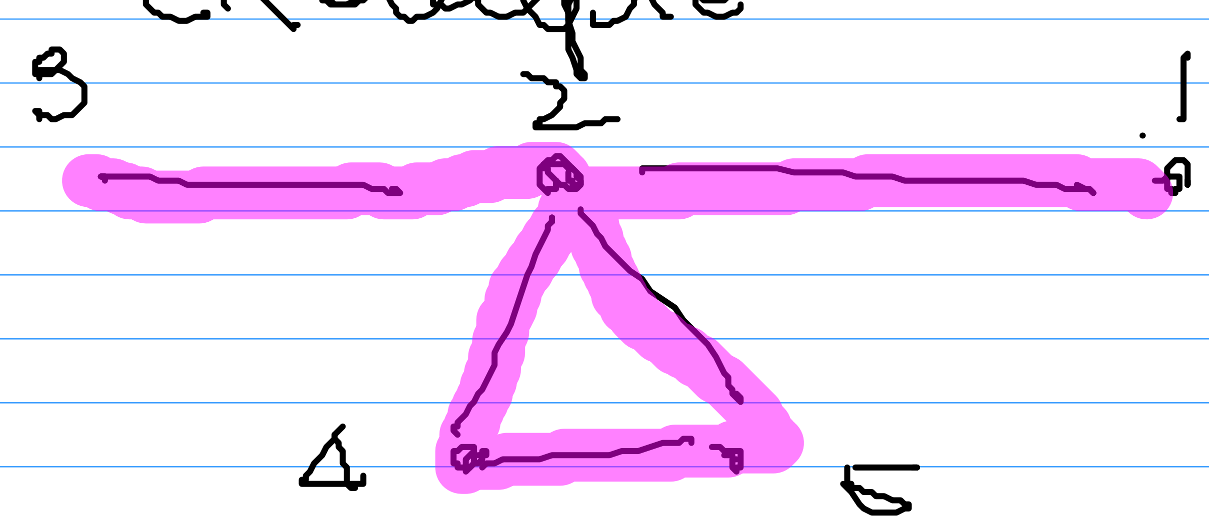
(1, 2, 3, 4, 3, 1)

<u>Definition</u> G graph (not necessarily simple)

a walk in G is a

1) TRAIL iff no edge is repeated
2) CIRCUIT iff it is a closed trail
3) A PATH iff not vertex is Repeated
4) A CYCLE iff it is a closed path.
(starting one does not count)

Example



(1 2 4 5 2 3)

TRAIL but no PATH

Plan Graph theory

- Euler circuit / trail.
- Hamilton path / cycle.
- Planar graphs
- Coloring.

we know everything
we have necessary
& sufficient
conditions
for the existence

An Euler circuit / trail is a circuit / trail that "walks" every edges

An Hamilton path / cycle is a path / cycle that "visits" every vertices

we have conditions
still open problems.

Prop (V, E) graph. Any walk from x to y which has minimal length is a path.

Rmk such walk $\exists$ by well ordering of the natural numbers

Proof $(v_1 \overset{e_1 \,\ldots\, e_{n-1}}{\ldots} v_{n+1})$ walk from v_1 to v_{n+1}

work for multig. Suppose n minimal. If it is not a path $\exists\ w \in V$ such that $w = v_i$

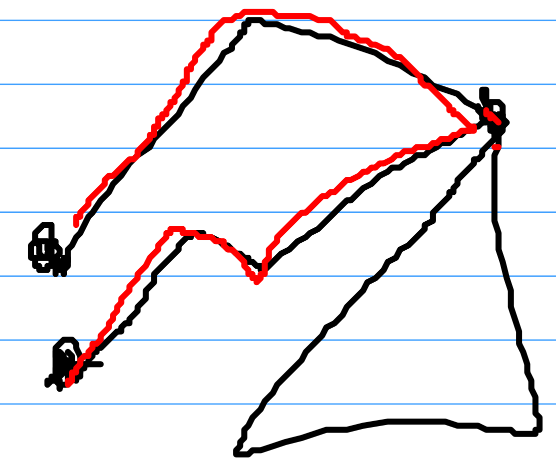
$w = v_j$ for $i \neq j$

up to reordering we can assume $i < j$

$(v_1 \overset{e}{\ldots} v_i\ v_{j+1} \ldots v_{n+1})$ walk from v_1 v_{n+1}

of length $n - j + i < n$ contradicting minimality

$j - i > 0$



<u>Corollary</u> if $\exists$ walk / trail between x, y there is a path connecting the

A Graph (not nec simple) is called connected if there is a path between any two vertices.

Well ordering of natural numbers
$\iff$ Induction

Every subset of the natural numbers which is not empty has a min.

Every subset of $\mathbb{Z}$ which is not empty & bounded below (above) has a min (max)

If there is a walk from x to y

$\Rightarrow$ then there is a walk of minimal length

$$\ell \quad W(x,y) \underset{\neq \emptyset}{=} \{\text{walk from } x \to y\} \longrightarrow \mathbb{N}$$

$$w \longmapsto \text{length of } w$$

$\ell(W(x,y)) \neq \emptyset$ $\qquad$ $\ell(W(x,y))$
$\hookrightarrow$ it has a min. $\qquad$ $\Downarrow$ m. $\qquad$ $\exists\, \bar{w} \in W(x,y)$ $\ell(\bar{w}) = m$.

Example

(a, b, e, f, g, e, d, c)

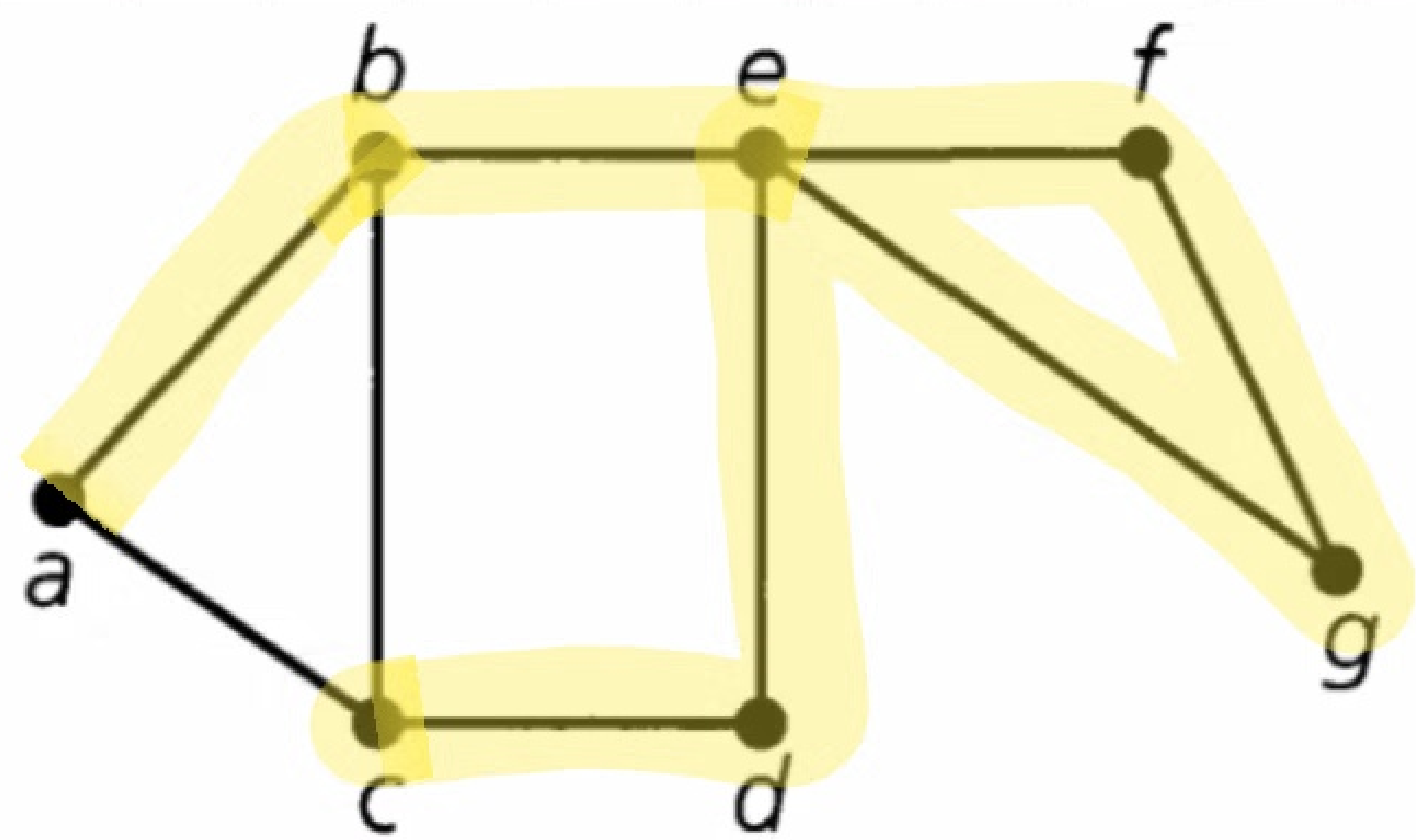


Figure 11.7

Example

2. For the graph in Fig. 11.7, determine (a) a walk from b to d that is not a trail; (b) a b-d trail that is not a path; (c) a path from b to d; (d) a closed walk from b to b that is not a circuit; (e) a circuit from b to b that is not a cycle; and (f) a cycle from b to b.

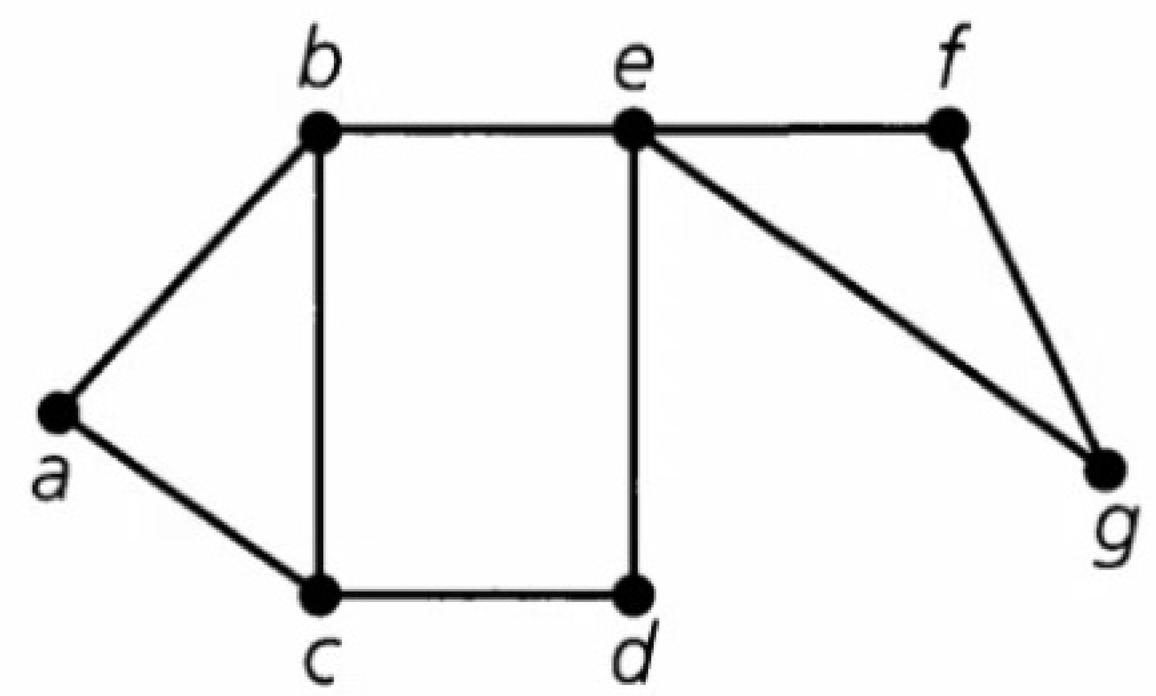


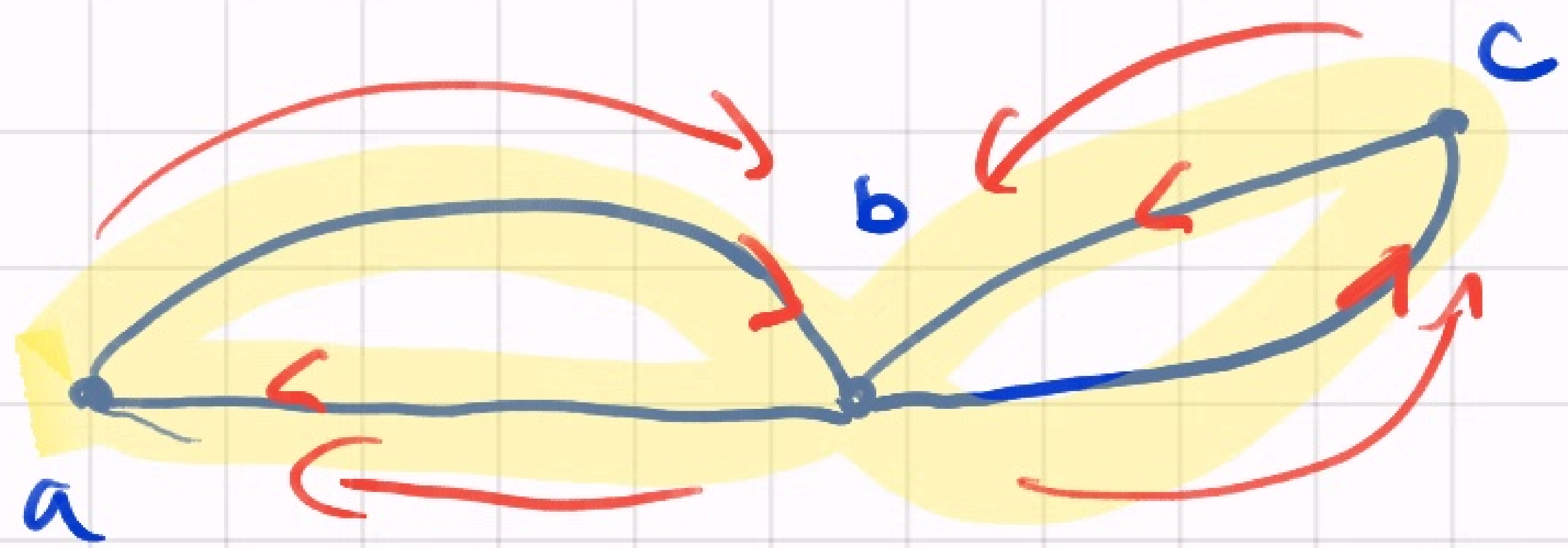
Figure 11.7

3. For the graph in Fig. 11.7, how many paths are there from b to f?

Path vs trail

<u>Graph</u> there is only one edge connecting two vertices

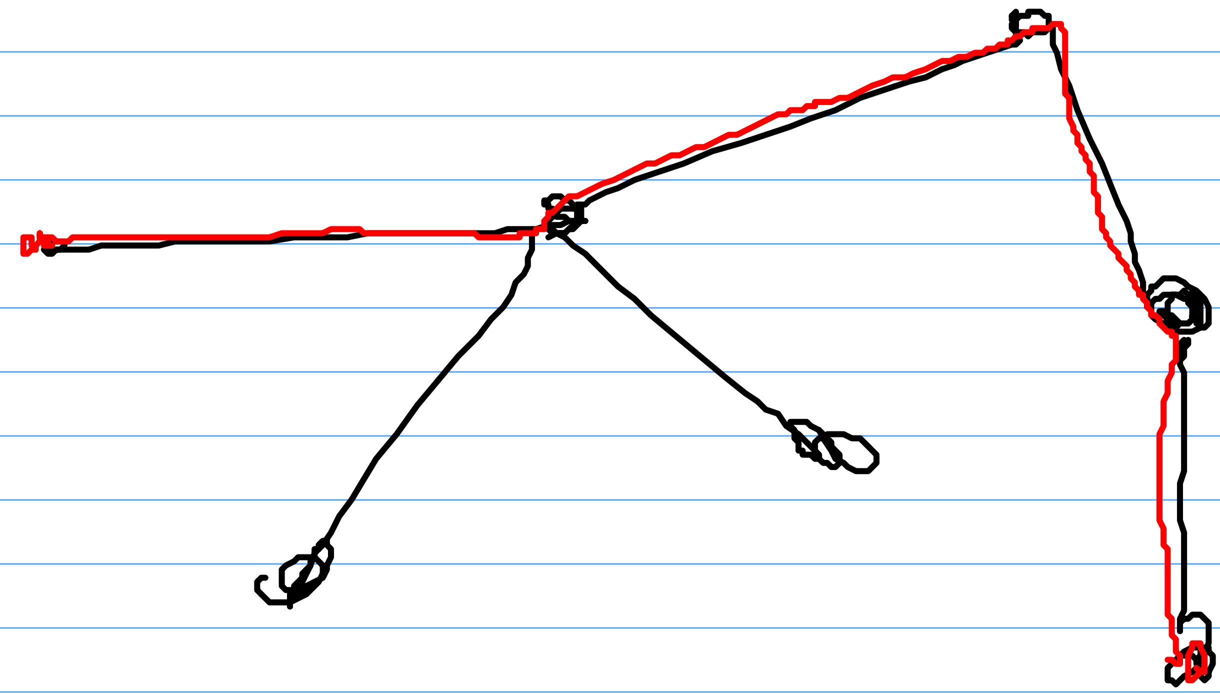
if an edge is repeated vertices are repeated



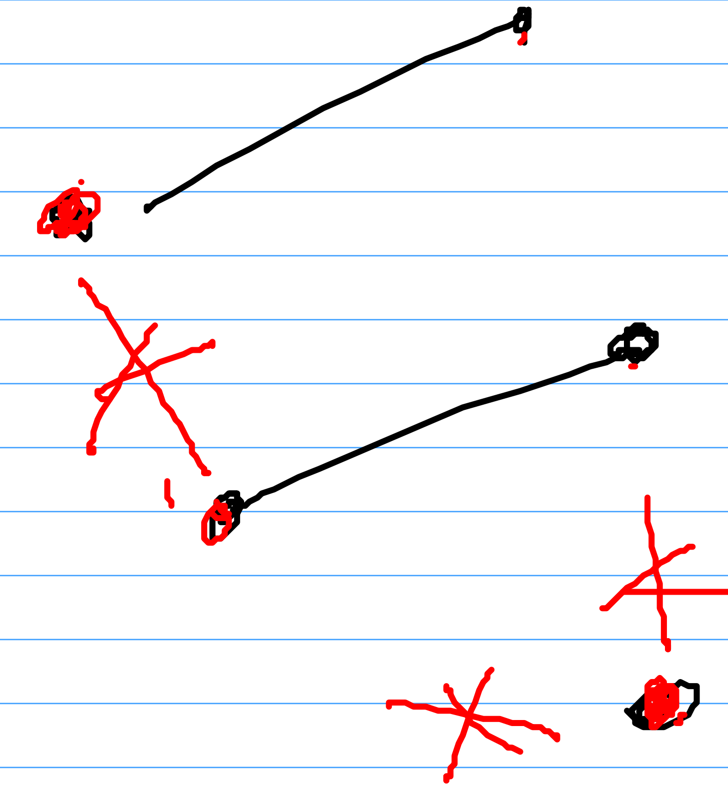
no repeated edges

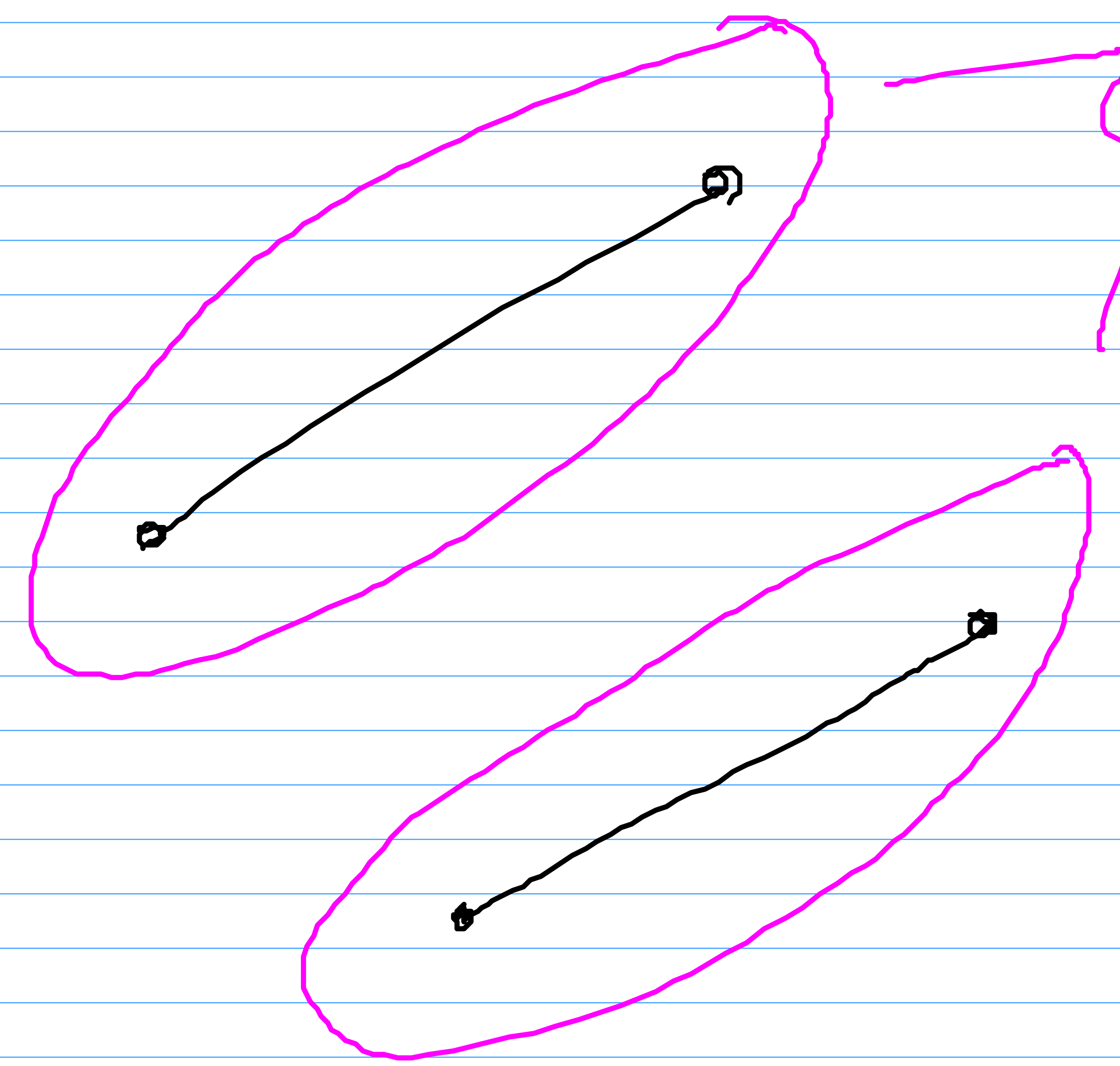
by I have no repeated vertex

Example



Connected





connected
no bigger
connected
subgraph

they are
connected
component.

<u>Def</u> a subgraph in a multigraph $(V, E, p) = G$

$\left(\text{simple } p = \hookrightarrow \right)$

is $G_1 = (V', E', p')$

$V' \subseteq V$

$E' \subseteq E$

$p' = p|_{E'}$ $\quad$ $i \circ p = p|_{E'}$

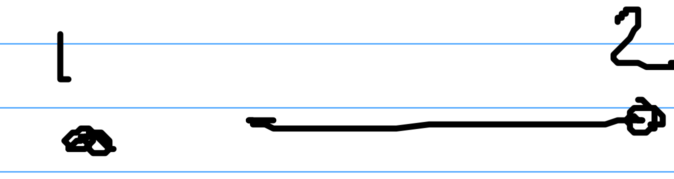
$$\begin{array}{ccccc} p: & E & \longrightarrow & \mathcal{P}(V) & \\ & \uparrow & & \cup\!\!\mid \; i & V' \subseteq V \\ p': & E' & \longrightarrow & \mathcal{P}(V') & \end{array}$$

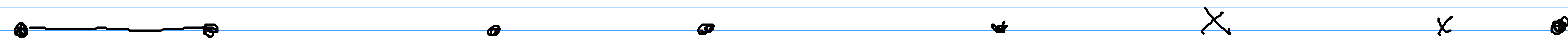
$G = (V, E)$ directed $\quad V' \subseteq V \quad E' \subseteq E$

$E' \subseteq V' \times V'$

$\cap\!\mid$

$E \subseteq V \times V$

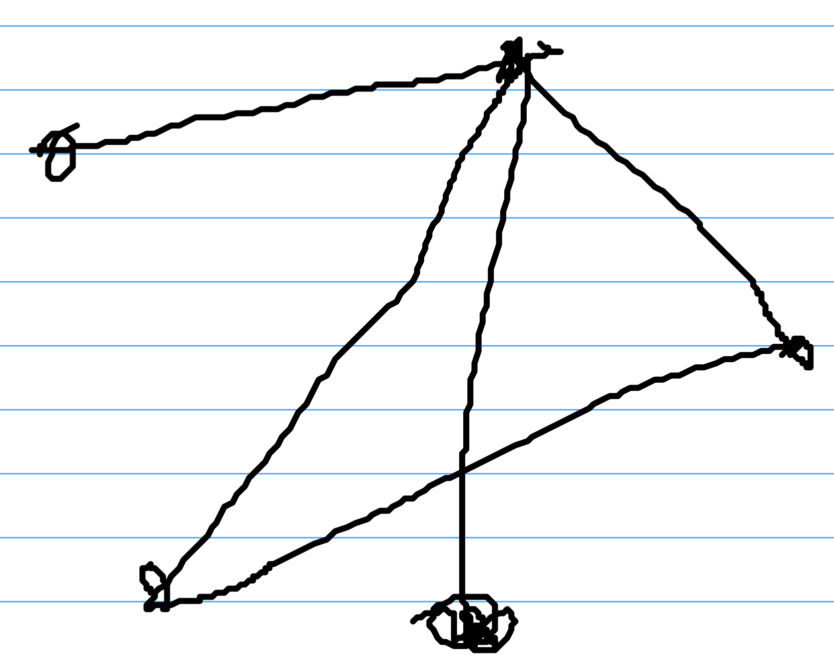
Example the subgraph of 

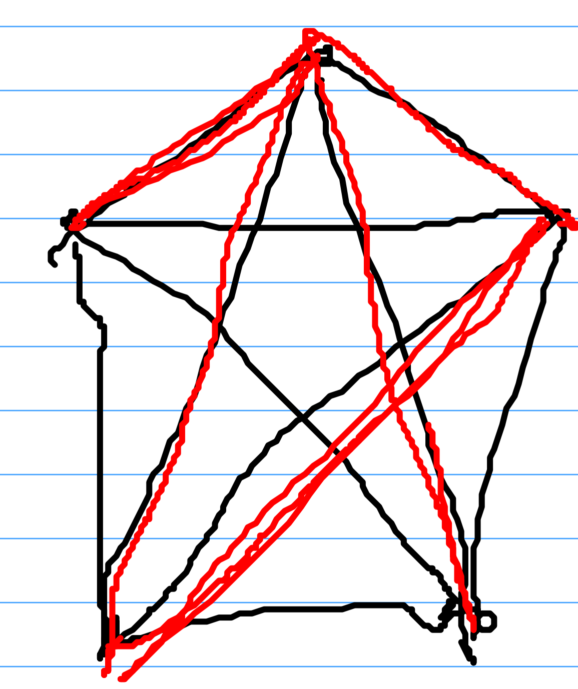


We say that a subgraph is SPANNING if $V' = V$

Rmk Any loop-free graph with n vertices is a spanning subgraph of K_n

Example





Spanning subgraph

We say that a subgraph $H=(V',E')$ of a graph $G=(V,E)$ is <u>spanning</u> if

$$V=V'$$

$\boxed{\text{"Spanning Trees"}}$

Connected components

<u>Def</u> given a $^{\text{multi}}$graph $G=(V,E)$ a connected component is a subgraph $C=(V',E')$ which is

① <u>connected</u>

② maximal among the connected subgraph of G
if C' subgraph C' connected C subgraph of C' then $C=C'$

Operations on graphs

We are going to define 3 operation

1) complement

2) Removing an edge. special case of induced graph

3) Removing a vertex. (induced)

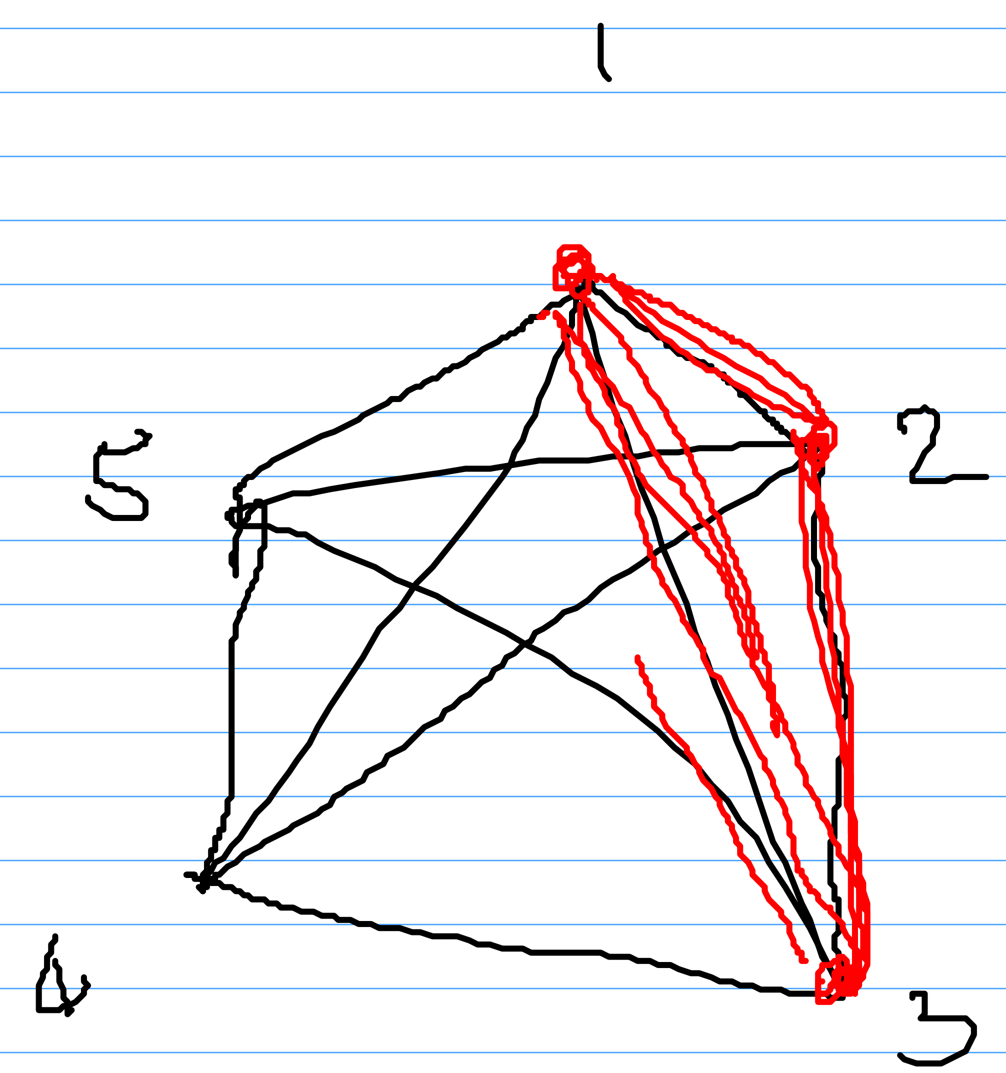
(4) Collapsing an edge → coloring)

→ Lecture Graph III.

$G=(V,E)$ graph $\qquad \emptyset \neq U \subset V$

The subgraph induced by U

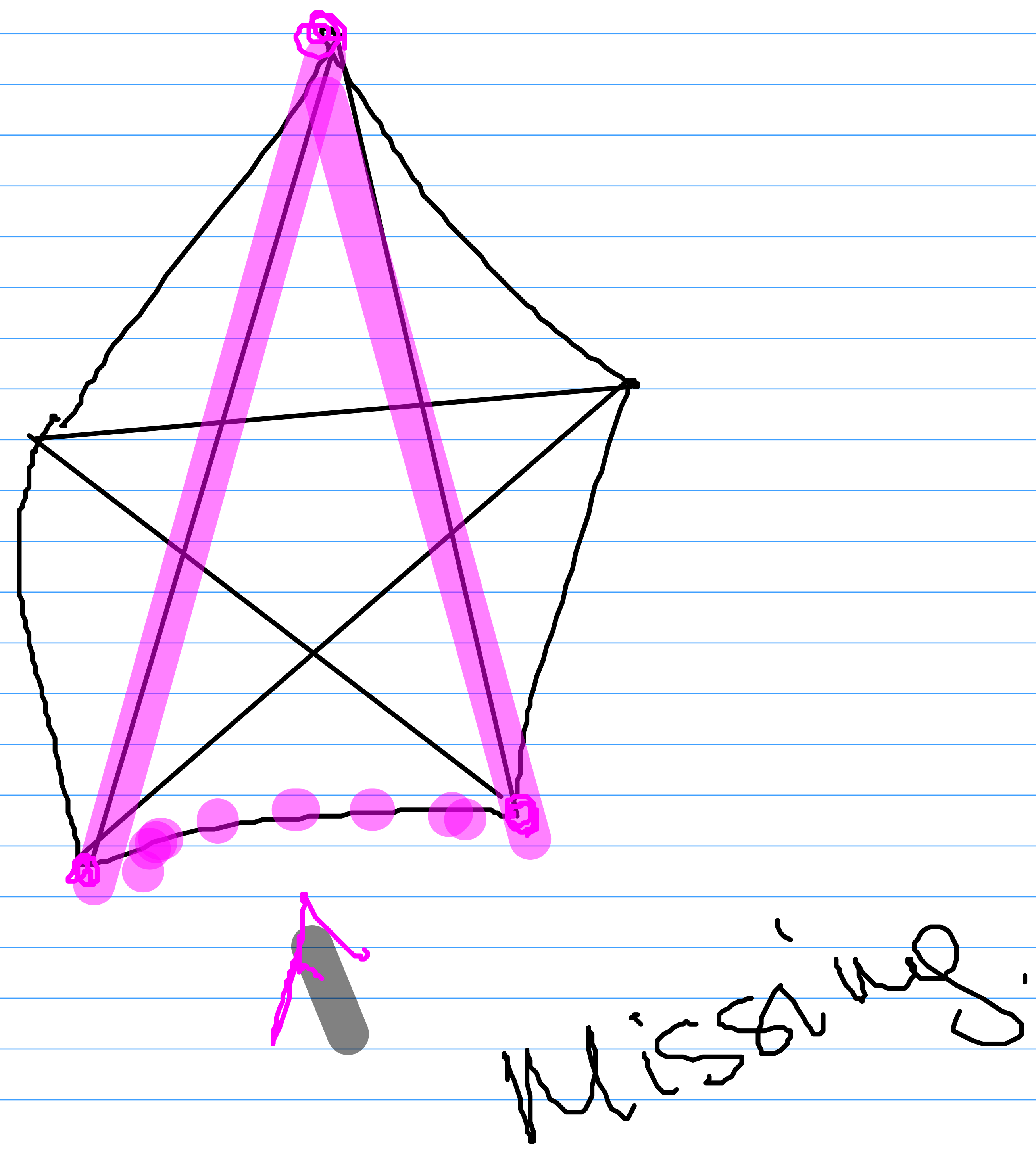
$\langle U \rangle_G$ is $(U, E \cap P(U))$

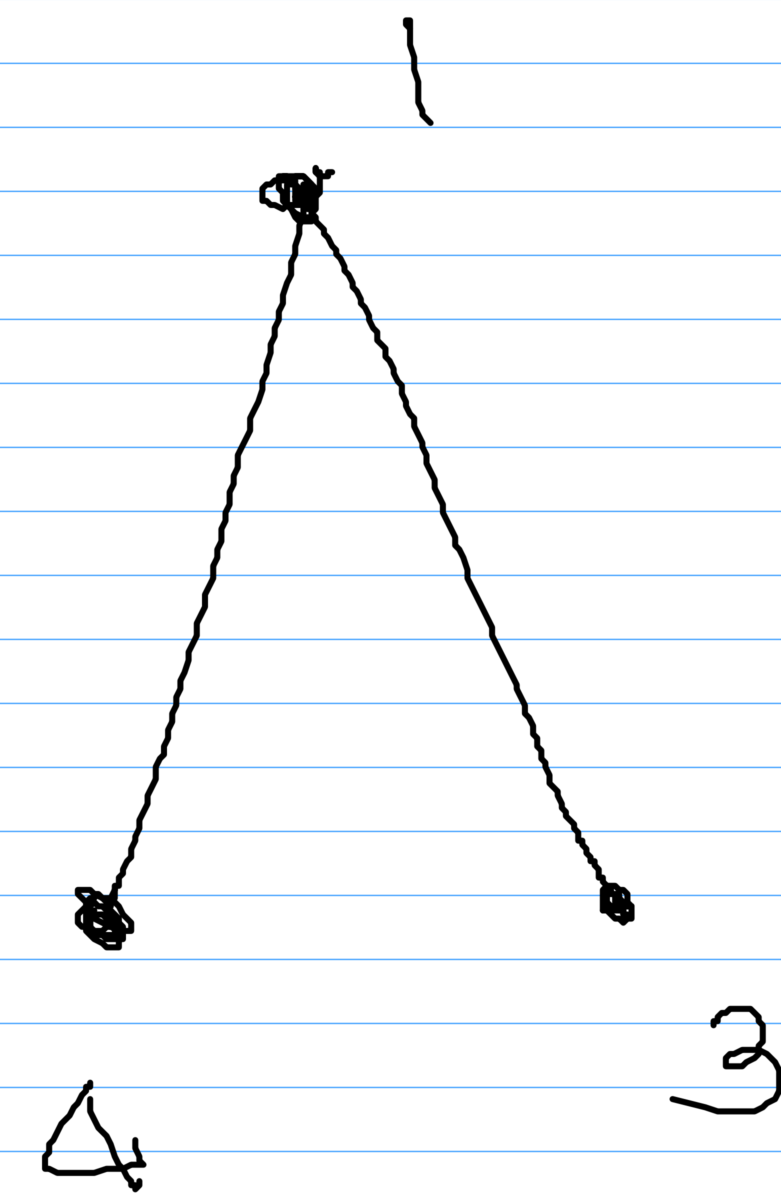


$U = \{1, 2, 3\}$

We say $G' \subset G$ is induced $\Longleftrightarrow$

$G' = \langle U \rangle_G$ for some $U \subseteq G$



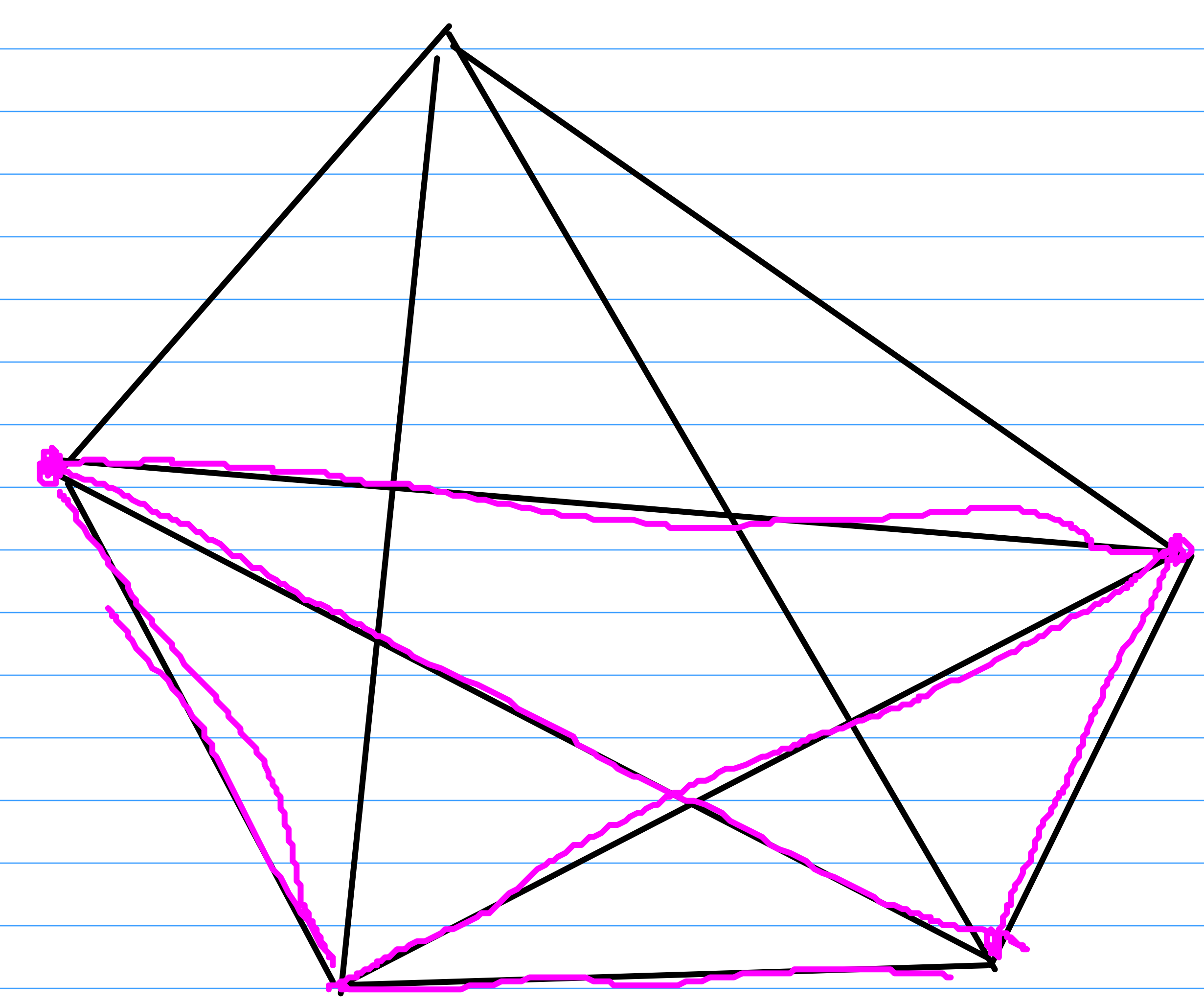


not induced

G graph (V, E) $\quad v \in V$

$$G - v := \langle V \setminus \{v\} \rangle_G$$

1

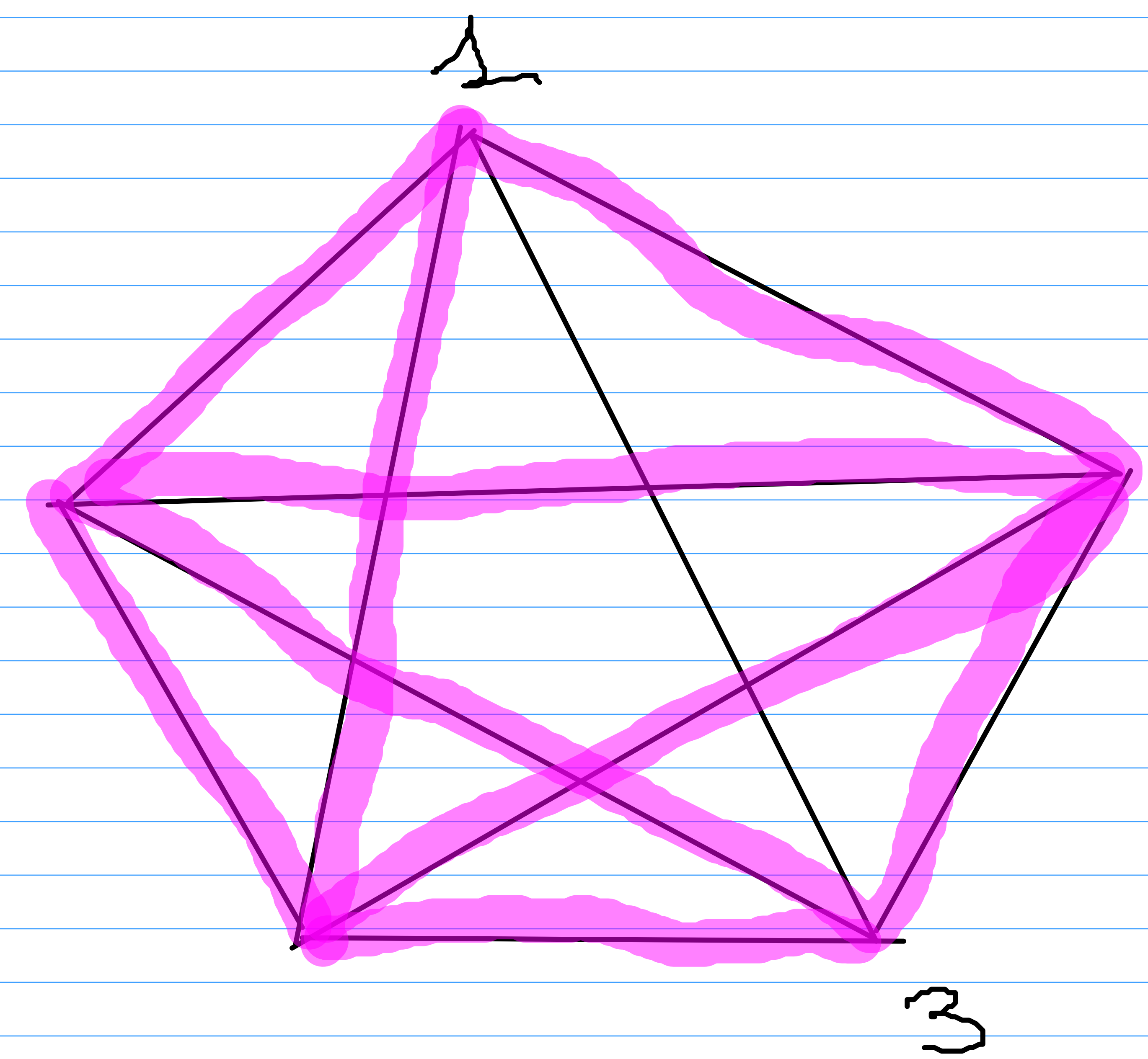


$G - 1$

K_4

$G = (V, E)$ graph $\quad e \in E$

$$G - e := (V, E \setminus \{e\})$$



$6 - (1,3)$

Important: many proofs induction on $|E|$, $|V|$, $|E+V|$

"Structural induction"

Def $G = (V, E)$ graph loop free $(G \subseteq K_n)$

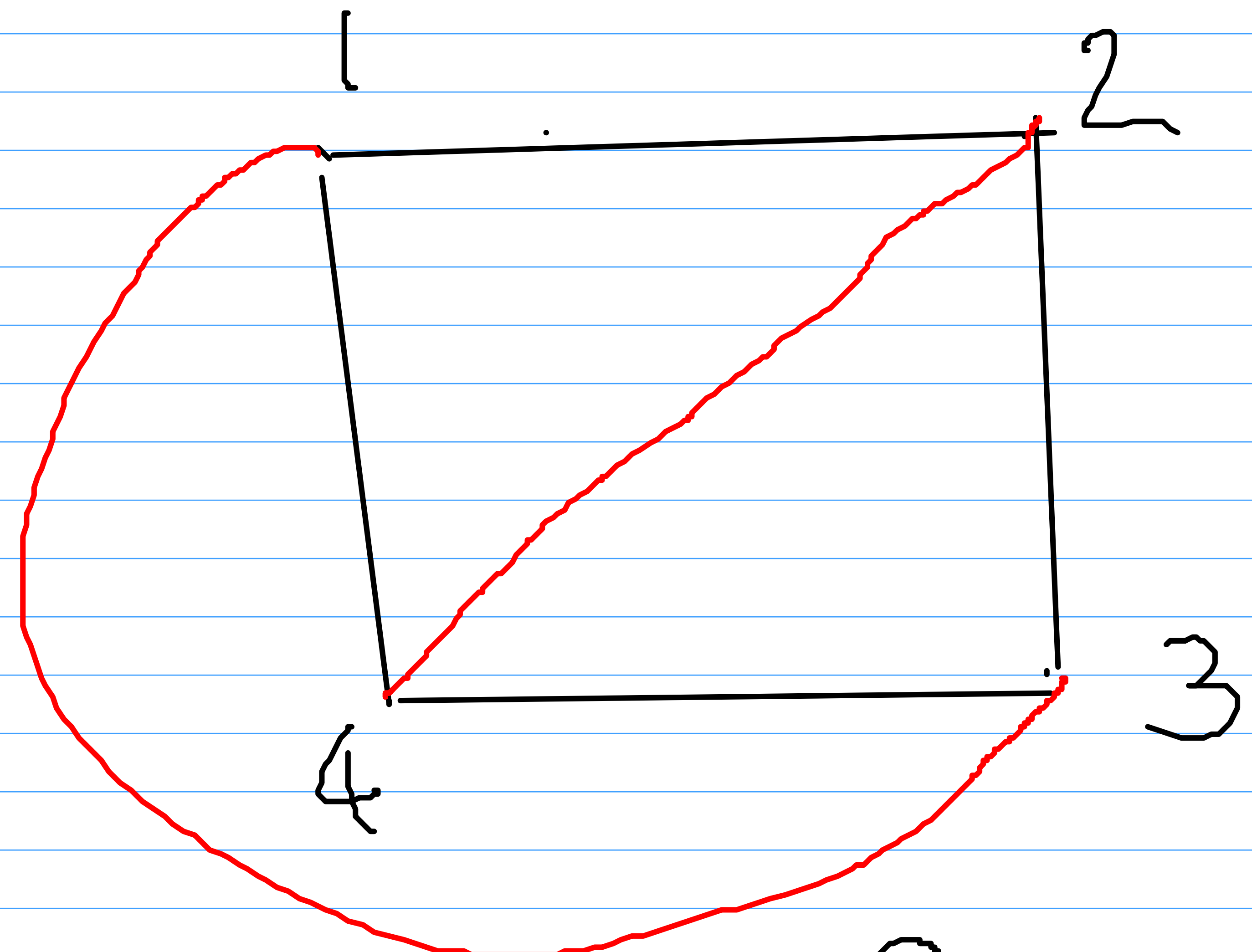
$n = |V|$

The complement of G is

$$\overline{G} := \left(V, \{A \in \mathcal{P}(V) \mid |A| = 2\} \setminus E\right)$$

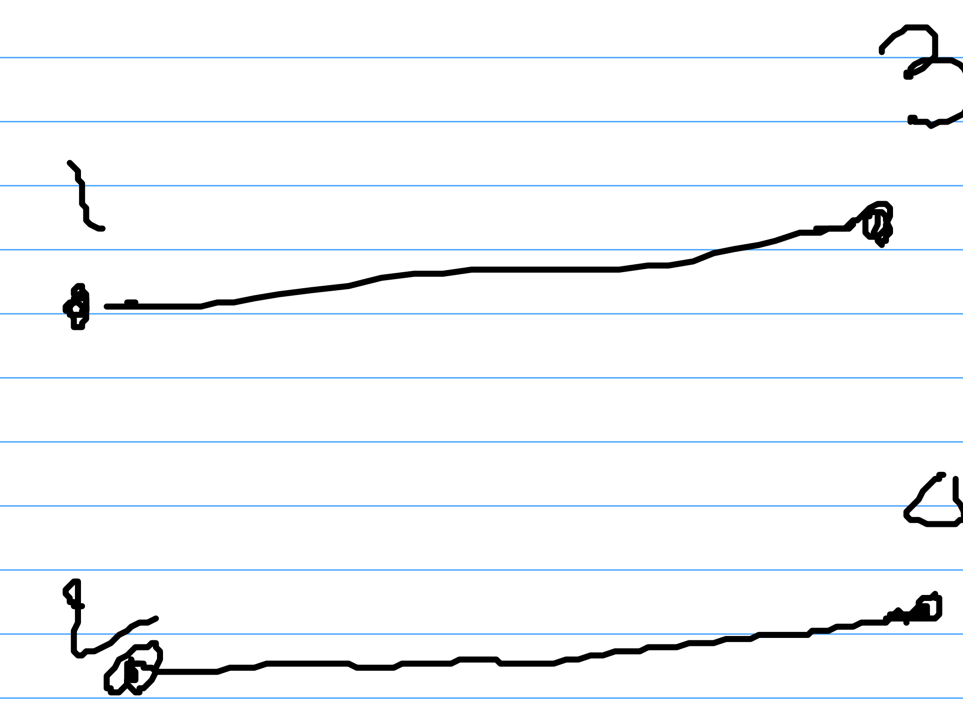
directed $\quad (V, V \times V \setminus E)$

Example



$\subseteq K_4$

$\overline{C_4} =$



Connectedness in not preserved by taking complements

Homomorphism and isomorphism

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

a graph homomorphism : $\varphi : G_1 \longrightarrow G_2$ is

given by $\varphi : V_1 \longrightarrow V_2$

such that $\underline{\varphi \times \varphi (E_1) \subseteq E_2}$ (directed)

$\underline{\varphi(\overset{\in V_1}{e}) \in E_2}$ for all $e \in E_1$

V_2 ?

$$\varphi : V_1 \longrightarrow V_2 \qquad \varphi \times \varphi : V_1 \times V_1 \longrightarrow V_2 \times V_2$$
$$(a, b) \longmapsto (\varphi(a), \varphi(b))$$

Roughly speaking

$\varphi : V_1 \longrightarrow V_2$ send vertices of G_1 to vertices of G_2

such that edges are preserved

Down to Earth

vertices are mapped to vertices

edges are mapped to edges between the corresponding vertices (maintaining direction)

An homomorphism φ is an isomorphism.
if φ is bijective and

- $\varphi \times \varphi|_{E_1}$ is bijective (directed)
- for every $e \in E_1$ there is a unique $e' \in E_2$ such that $\varphi(e) = e'$

EXERCISES 11.2

1. Let G be the undirected graph in Fig. 11.27(a).

   a) How many connected subgraphs of G have four vertices and include a cycle?

   b) Describe the subgraph G_1 (of G) in part (b) of the figure first, as an induced subgraph and second, in terms of deleting a vertex of G.

   c) Describe the subgraph G_2 (of G) in part (c) of the figure first, as an induced subgraph and second, in terms of the deletion of vertices of G.

   d) Draw the subgraph of G induced by the set of vertices $U = \{b, c, d, f, i, j\}$.

   e) For the graph G, let the edge $e = \{c, f\}$. Draw the subgraph $G - e$.

2. **a)** Let $G = (V, E)$ be an undirected graph, with $G_1 = (V_1, E_1)$ a subgraph of G. Under what condition(s) is G_1 *not* an induced subgraph of G?

   b) For the graph G in Fig. 11.27(a), find a subgraph that is not an induced subgraph.

3. **a)** How many spanning subgraphs are there for the graph G in Fig. 11.27(a)?

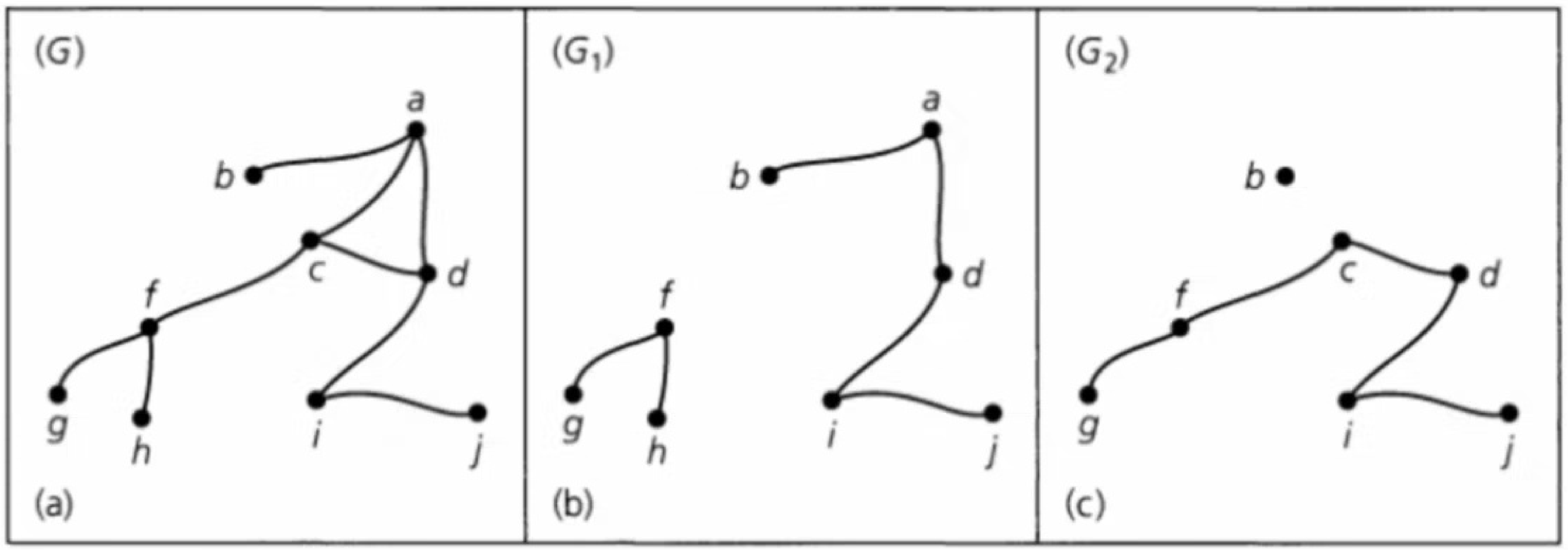


Figure 11.27

NOT ISOMORPHIC.

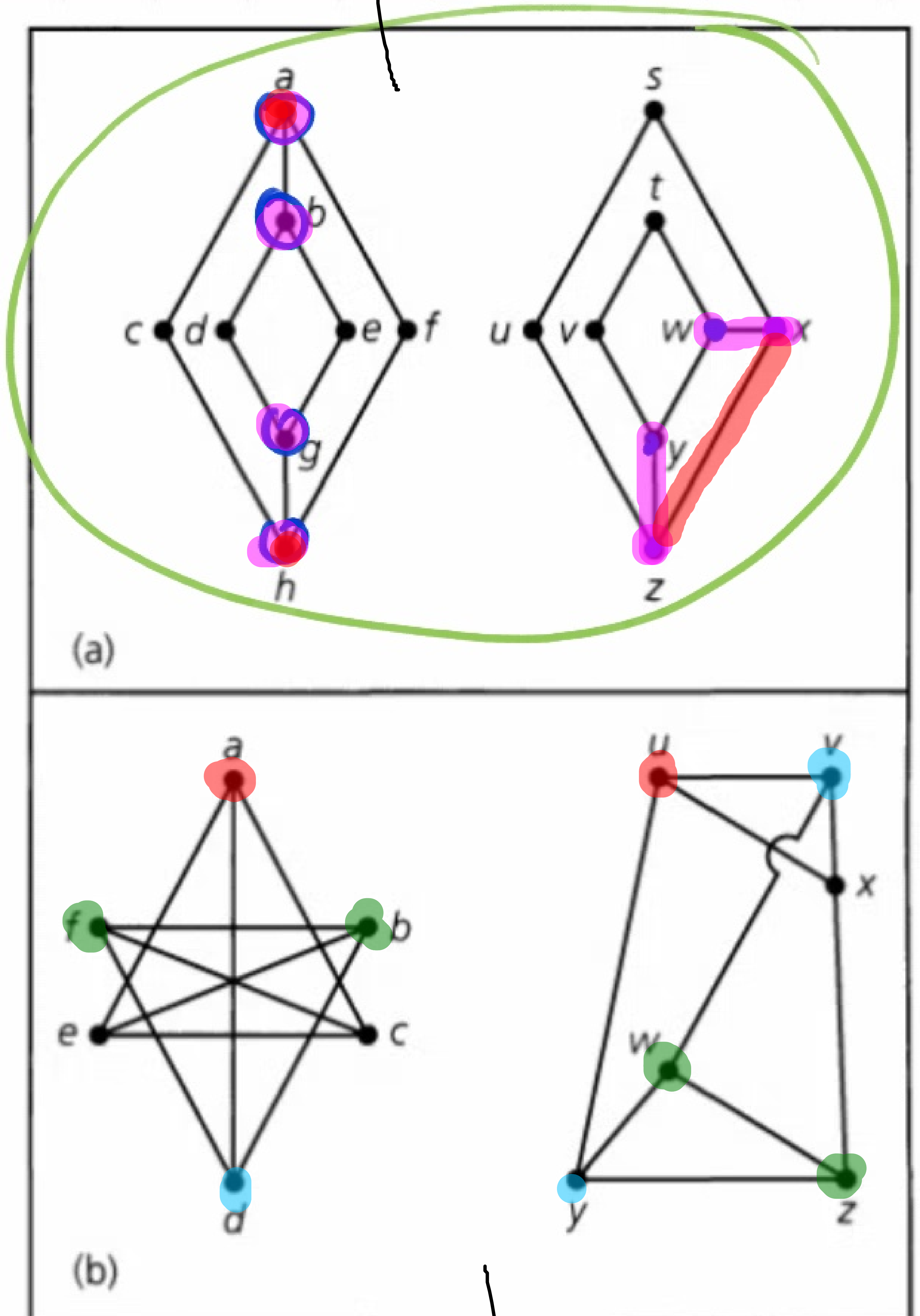


Figure 11.29

NOT ISOMORPHIC

Definition

$G = (V, E)$ graph or a multigraph

$v \in V$

$\deg v =$

$\parallel$

# of edges with v as one of the extremes (loop count 2).

Isom preserve the degree

$\deg v = \deg \varphi(v)$