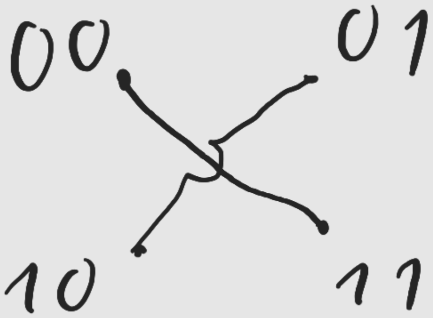
11.1 <u>Exercise 4</u>:

For $n \geq 2$, let $G = (V, E)$ be the loop-free undirected graph, where V is the set of binary n-tuples and $E = \{\{v, w\} \mid v, w \in V$ and v, w differ in (exactly) two positions$\}$. Find $\kappa(G)$.

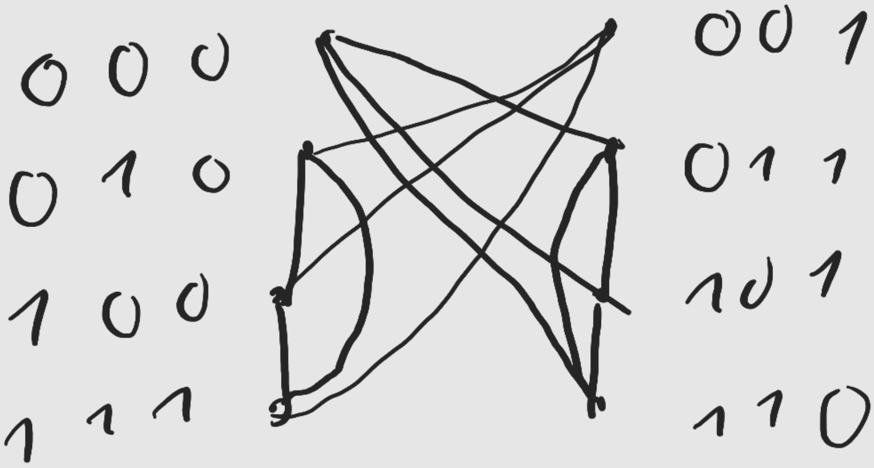
<u>Solution</u>:

$n = 2$:



$\kappa(G) = 2$

$n = 3$:



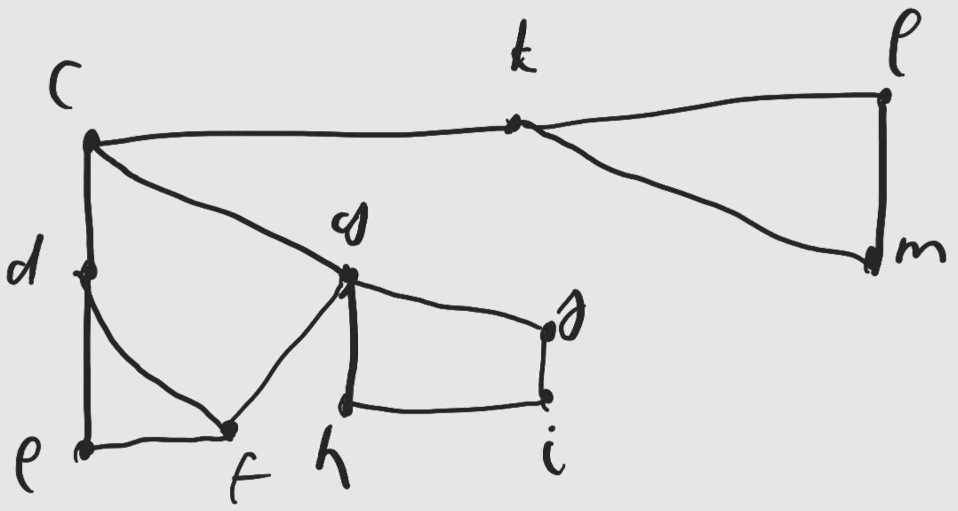
$\kappa(G) = 2$

For a general n:

We will have $\boxed{k(G) = 2}$: one connected component includes all vertices with an even number of digits 0, and the other component includes all vertices with an odd number of digits 0.

- First, it is easy to see that these two components are disconnected, since any edge flips two positions and therefore the parity remains the same. On the other hand, if you are in the first component you can flip two positions at a time until you reach the vertex $11\ldots1$, so all the vertices in this component are connected, and similarly on the second component with $011\ldots1$.

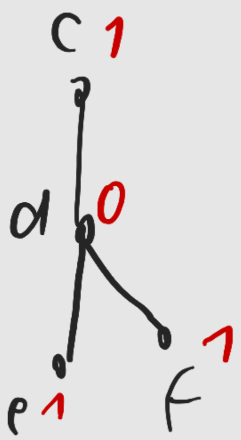
Exercise 6:



Find the distances from d to the other vertices in G.

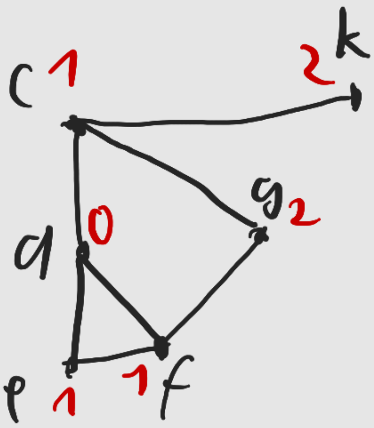
Solution:

We start with the vertex d. Its neighbors are {c, e, f}, so each of them have distance 1.



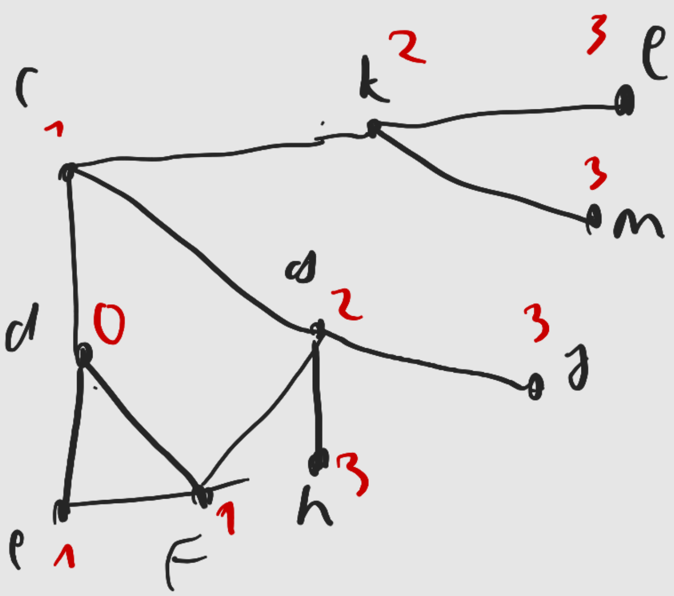
Then make the list of all neighbors of these four vertices: $\{c, d, e, f, k, g\}$

k, g were not listed previously, so they have distance 2:

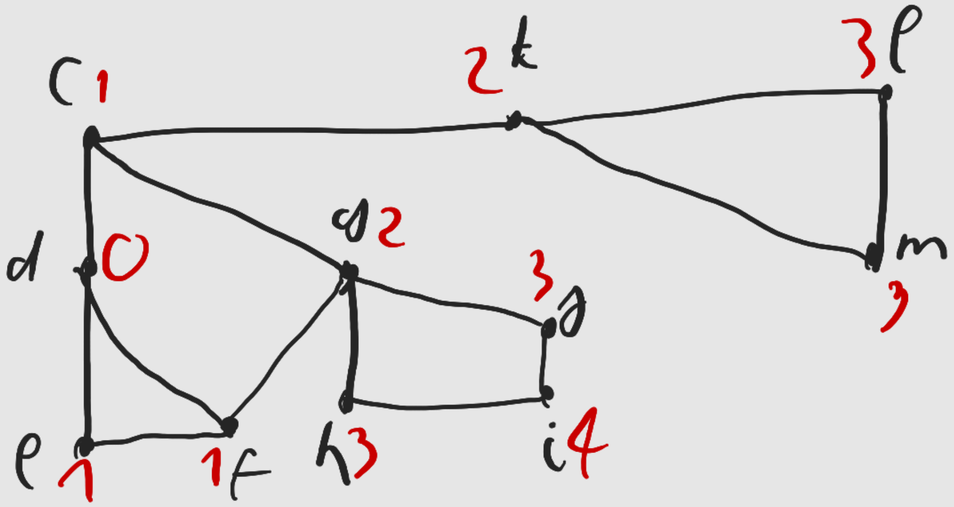


We continue and obtain now

$\{c, d, e, f, g, h, j, k, l, m\}$; the new vertices have distance 3.



Finally, the next iteration give us the remaining vertex i, so we have



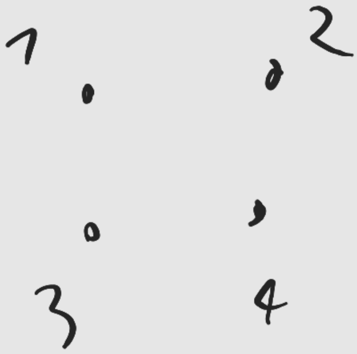
11.2 Exercise 6:

Find all (loop-free) non isomorphic undirected graphs with four vertices.
How many of these graphs are connected?

Solution:

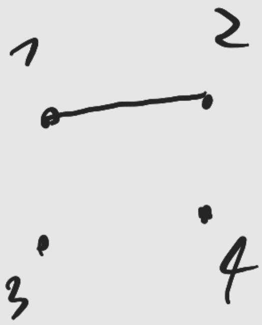
We sort these graphs by number of edges:

0 edges:

1) 

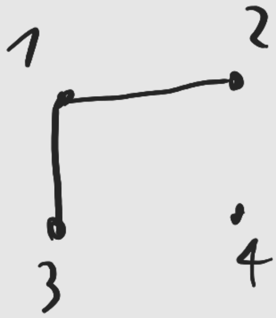
not connected

1 edge:

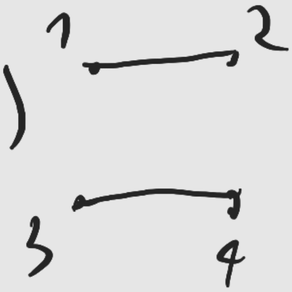
2) 

not connected.

2 edges:

3) 

not connected

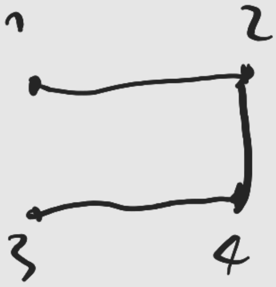
4) 

not connected

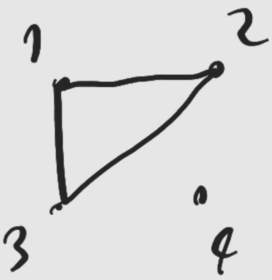
3 edges:

5) 

connected

6) 

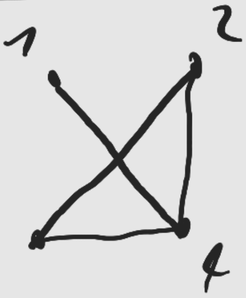
connected ↻ self-complement

7) 

not connected

complement graphs

4 edges:

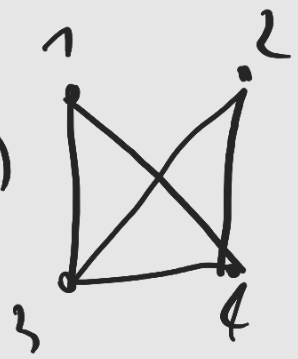
8) 

connected ← complement to 3)

9) 

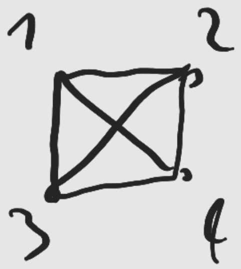
connected ← complement to 4)

5 edges:

10) 

connected

6 edges:

11) 

connected

There are 11 non isomorphic graphs,

6 of them are connected.

Exercise 8:

a) How many paths of length 4 are there in the complete graph K_7?

b) How many paths of length m are there in K_n?

Solution:

a) A path is described by its sequence of vertices, and since the graph is complete every sequence defines a path. So we must count the number of sequences of 5 different vertices in K_7.

There are exactly $\dfrac{7!}{(7-5)!}$ such sequences.

But every path gets counted twice, so there are $\boxed{\dfrac{7!}{2!\cdot 2}} = 1260$ paths

b) By the same argument as in the previous case, the number of paths is

$$\boxed{\frac{n!}{(n-(m+1))!\cdot 2}}$$

Exercise 10:

Let G be an undirected graph with v vertices and e edges. How many edges are there in $\overline{G}$?

Solution:

By definition, $G \cup \overline{G} = K_v$, which has $\binom{v}{2}$ edges, since the edges of G and $\overline{G}$ are disjoint, $\overline{G}$ must

$$\binom{v}{2} = \frac{v(v-1)}{2}$$

$\binom{v}{2} - e$ edges.

Exercise 12:

a) Let G be an undirected graph with n vertices. If G is isomorphic to its own complement $\overline{G}$, how many edges must G have?

b) Find an example of a self-complimentary graph on four and one on five vertices

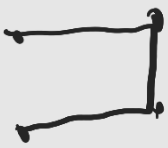
c) If G is a self-complimentary graph on n vertices, where $n>1$, prove that $n = 4k$ or $4k+1$.

Solution:

a) Since G is isomorphic to $\bar{G}$, we have $|E(G)| = |E(\bar{G})|$, meaning that

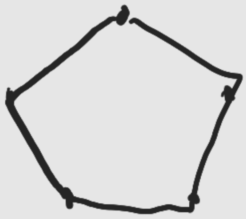
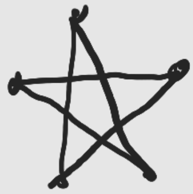
$$e = |E(G)| = \binom{n}{2} - e \rightsquigarrow$$

$$e = \binom{n}{2}/2$$

b) On 4 vertices: $G =$  , with dual

$\bar{G} =$ 

On 5 vertices: we must $|E(G)| = \binom{5}{2}/2 = 5$, we can pick

$G =$  , $\bar{G} =$ 

c) If G is self-complementary, we computed that we must have $|E(G)| = \binom{n}{2}/2$. However, $|E(G)|$ is an integer, so $\binom{n}{2}$ must be even.

If $n = 4k+2$ or $n = 4k+3$, then we get in the first case

$$\binom{n}{2} = \frac{(4k+2)(4k+1)}{2} = (2k+1)(4k+1)$$

is a product of two odd numbers, so it is odd.

In the second case

$$\binom{n}{2} = \frac{(4k+3)(4k+2)}{2} = (4k+3)(2k+1)$$

is again a product of two odd numbers.