Attention Starting 1 pm the students of the class I thought in the period AB will be taking the re-exam. So:

1) I will have my phone on with audio during the lecture

2) If they call me I need to answer.

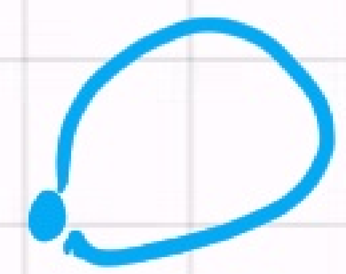
3) I might be late in publicing the notes & videos.

Lectures 9/10 - Graphs 2 & 3

- Euler circuit (A ok)
- Planar Graphs
- Hamilton path & cycles
- Coloring.

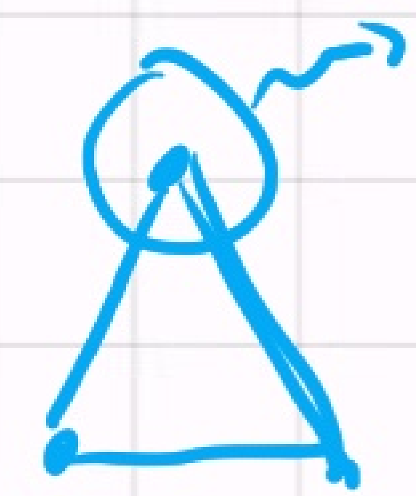
Euler circuit

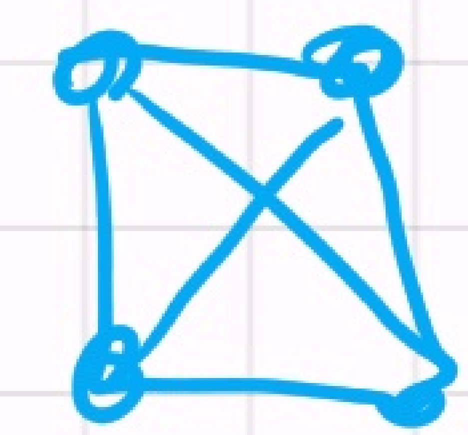
<u>Def</u> G a (multi)graph $\deg(v) = \#$ of vertices adjacent to v counted with multiplicity (loops count twice)



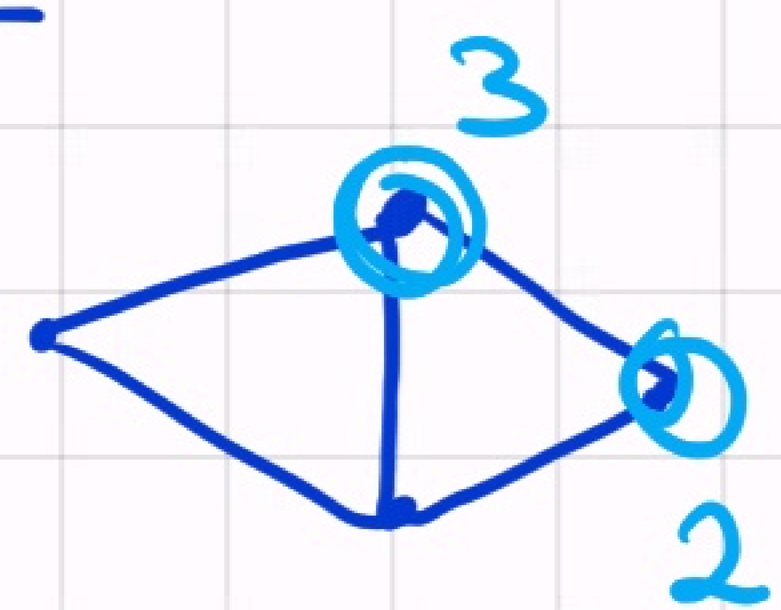
$\deg(v)=2$

A graph is n-regular if all the vertices have degree n





<u>Example</u> K_n is $(n-1)$-regular



is not n-regular

Proposition $\sum \deg(v) = 2|E|$ Proof: AoK

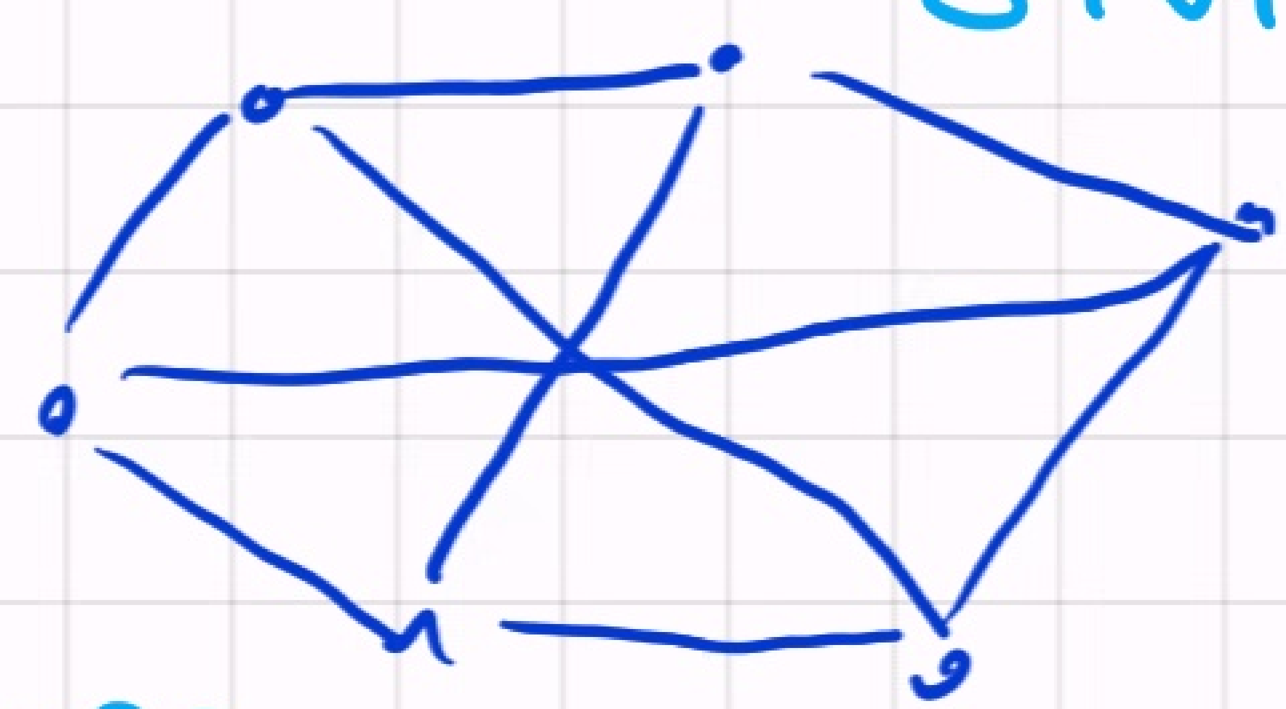
↳ even

Corollary there is an even number of vertices of odd degree.

$$\sum \deg(v) = |V| n = 2|E|$$

Example G is n-regular

- No 3-regular with 10 edges $\quad 3|V| = 20$
- $|E| = 9$, $|V| = 6$ ⇝ 3 reg graph with 9 edges

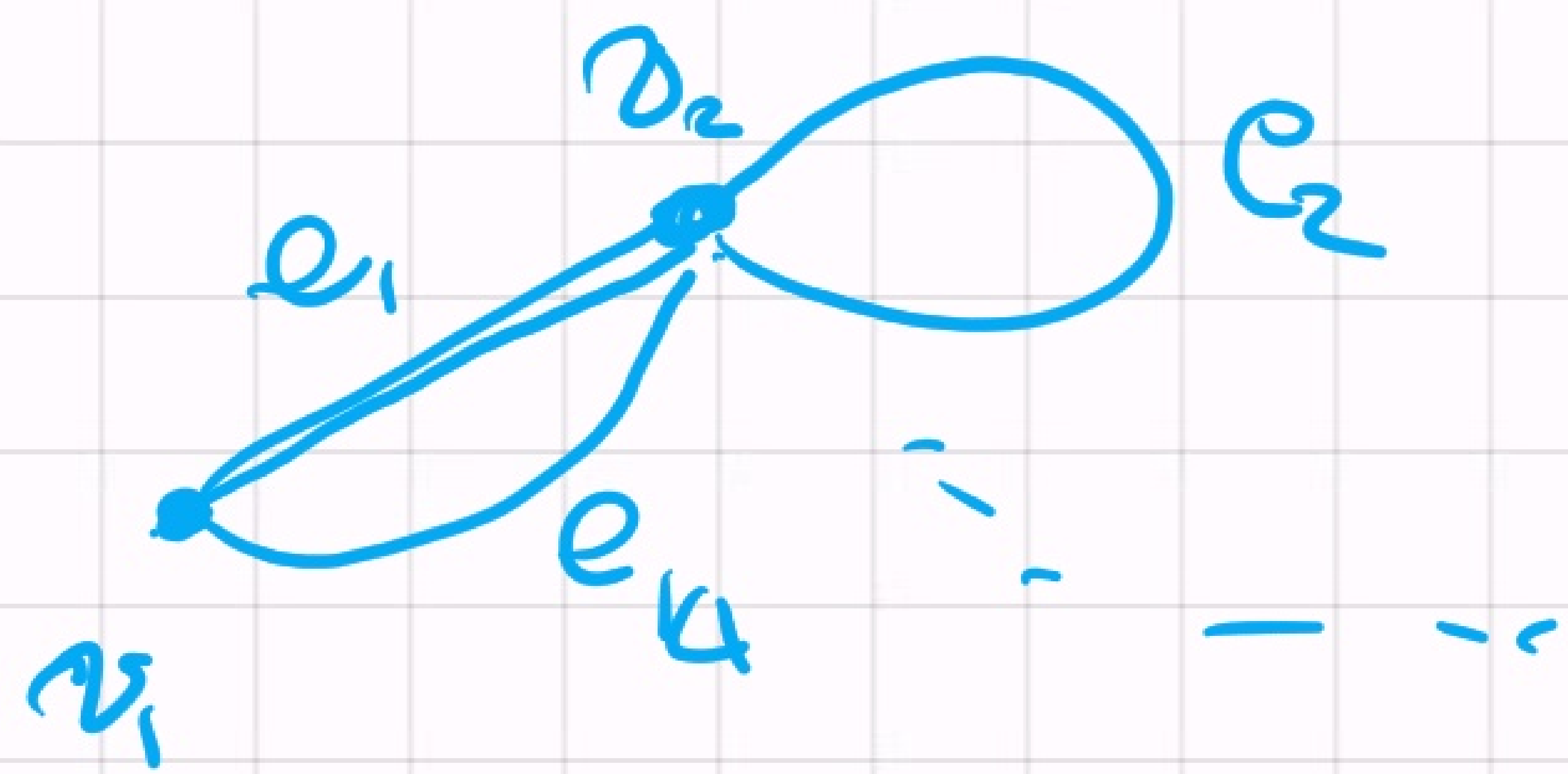


Def An Euler circuit (trail) on a Graph G is a circuit (trail) which passes all the vertices

Rmk walk on a multigraph $\rightsquigarrow$ keep track of edges

$$(v_1 \; e_1 \; v_2 \; e_2 \; v_2 \; \ldots \; v_{n-1} \; e_{n-1} \; v_n)$$

$$f(e_i) = \{v_i \; v_{i+1}\}$$



Theorem There is an Euler circuit on G (finite) $\Longleftrightarrow$ G is connected & all the vertices have even degree.

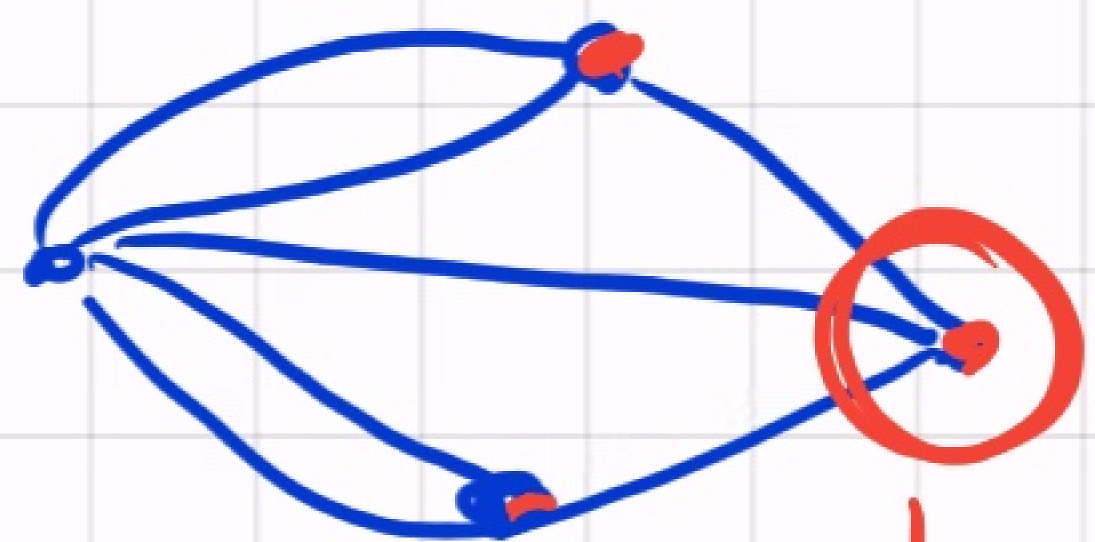
Proof AoK

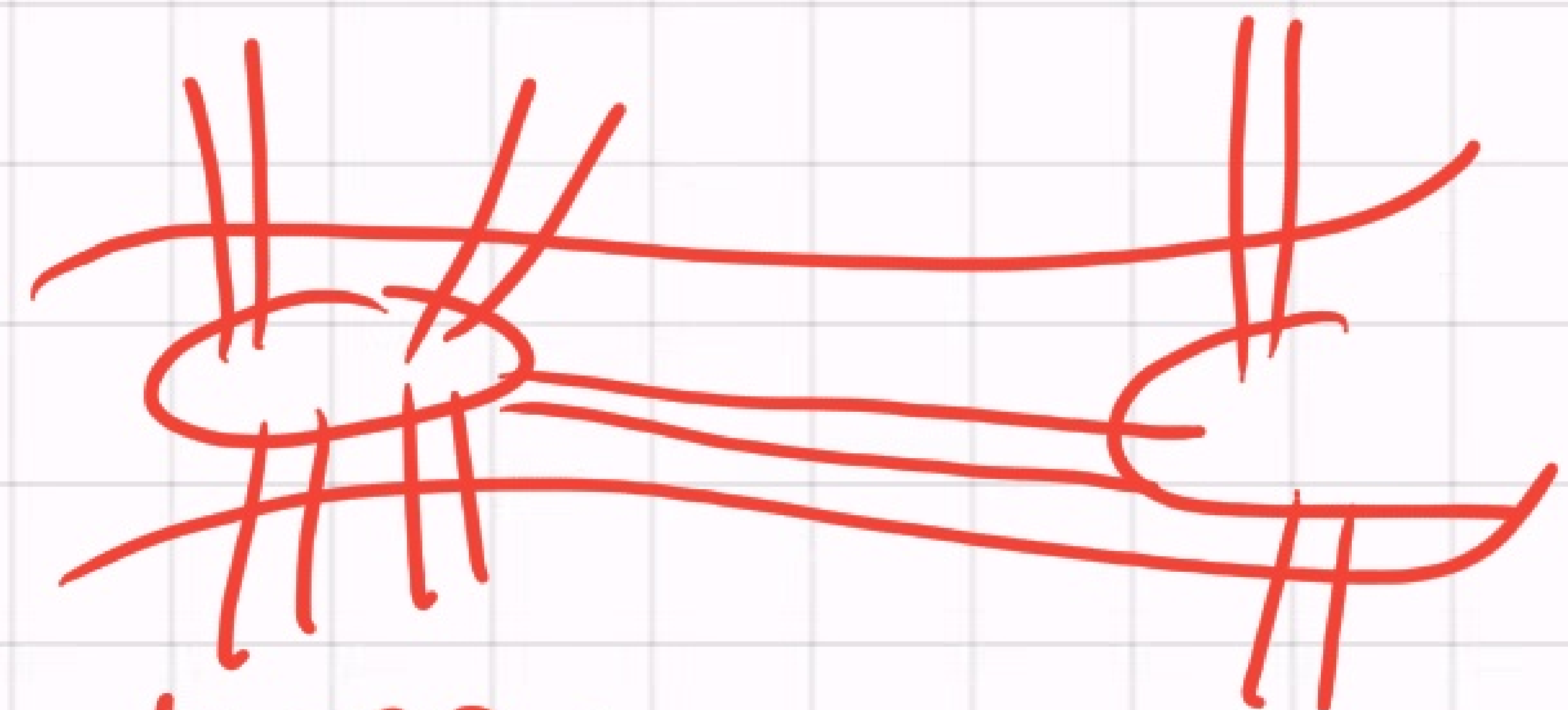
Example $V = \mathbb{Z}$ $\quad E = \{\{n, n+1\} \mid n \in \mathbb{Z}\}$

Connected, even degree but no trail.

Example

The graph has no Euler circuit.





Planar Graphs

Def $G=(V,E)$ Graph. Its geometric representation is the metric space $|G|$

- Set $V \amalg (\vec{E} \times [0,1]) \Big/ \begin{matrix}(e,0)\sim s(e)\\(e,1)\sim t(e)\end{matrix}$

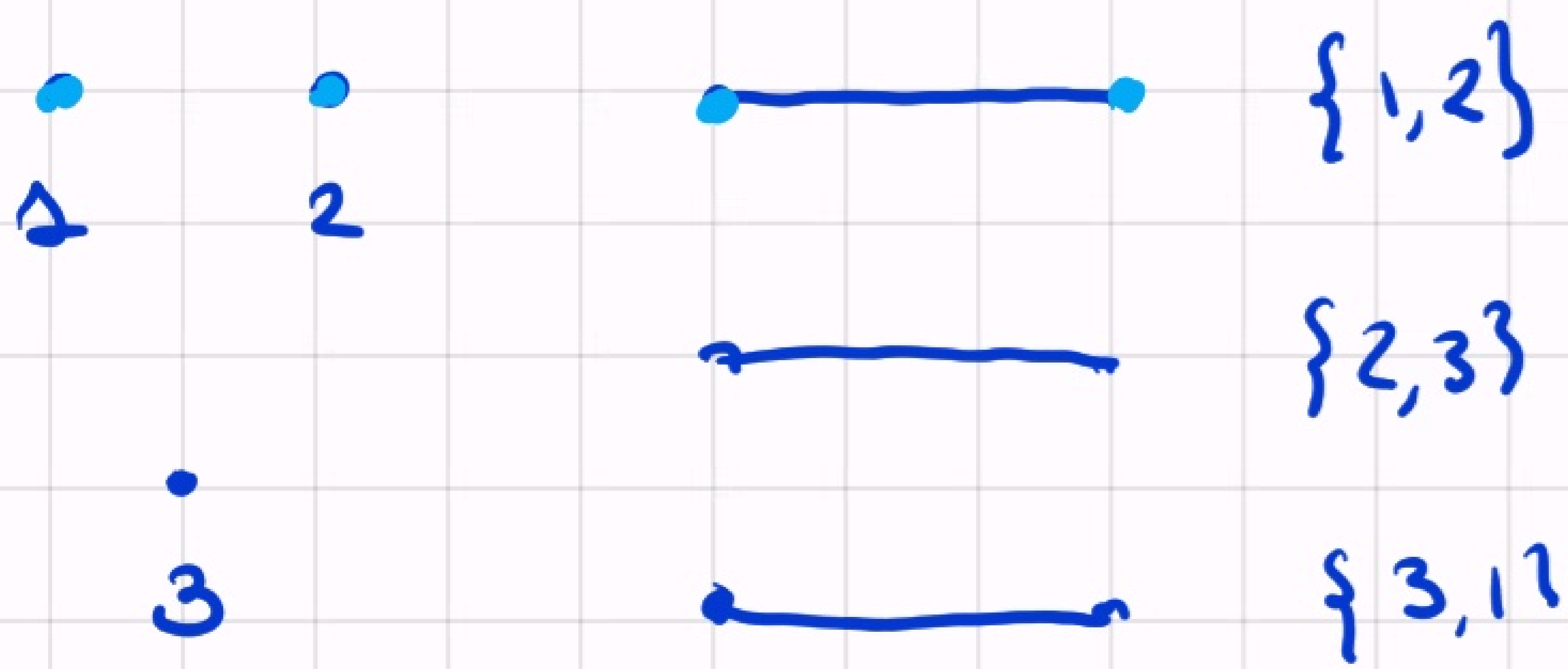
  where $(V\vec{E})$ is an orientation of G

- metric induced by the interval

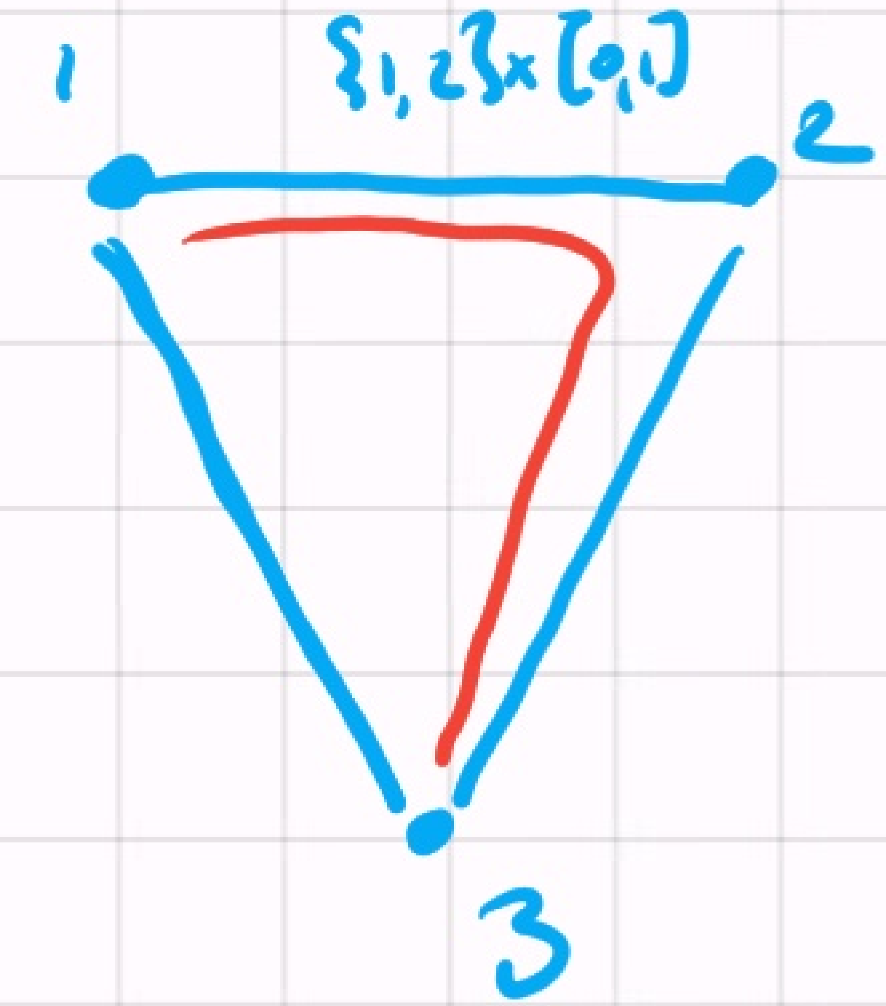
Example

$G = (\{1,2,3\}, \{\{1,2\}\ \{2,3\}\ \{3,1\}\})$

$V \sqcup E \times [0,1]$



$E \times [0,1]$



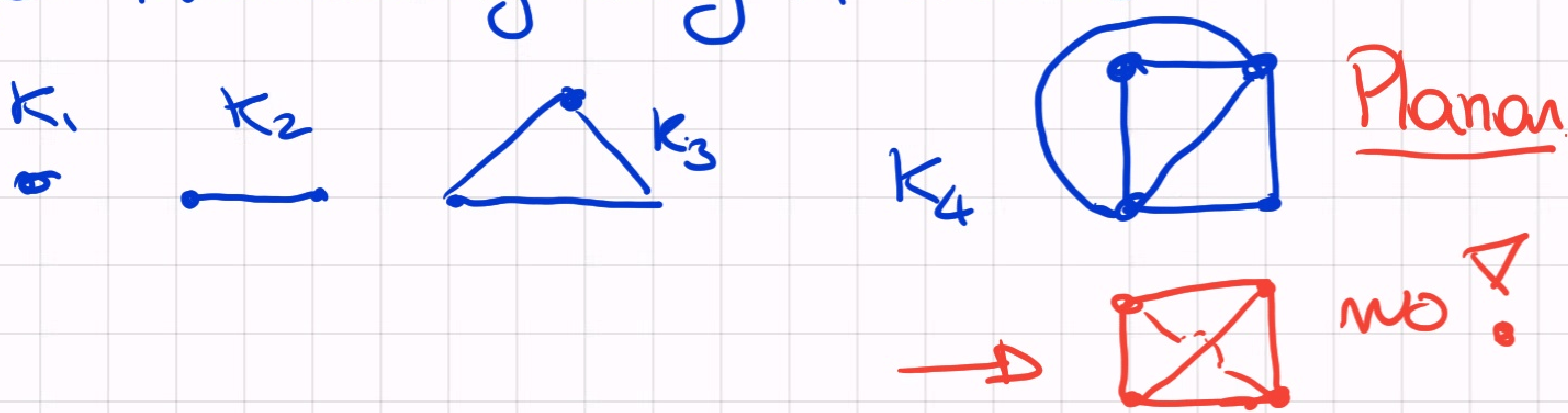
$V \sqcup E \times [0,1]$

A Graph is called planar if there is an injective (continuous) map

$$|G| \longrightarrow \mathbb{R}^2$$

In fewer words we can draw it on a plane with edges intersecting only in vertices

Example :



We are going to see that

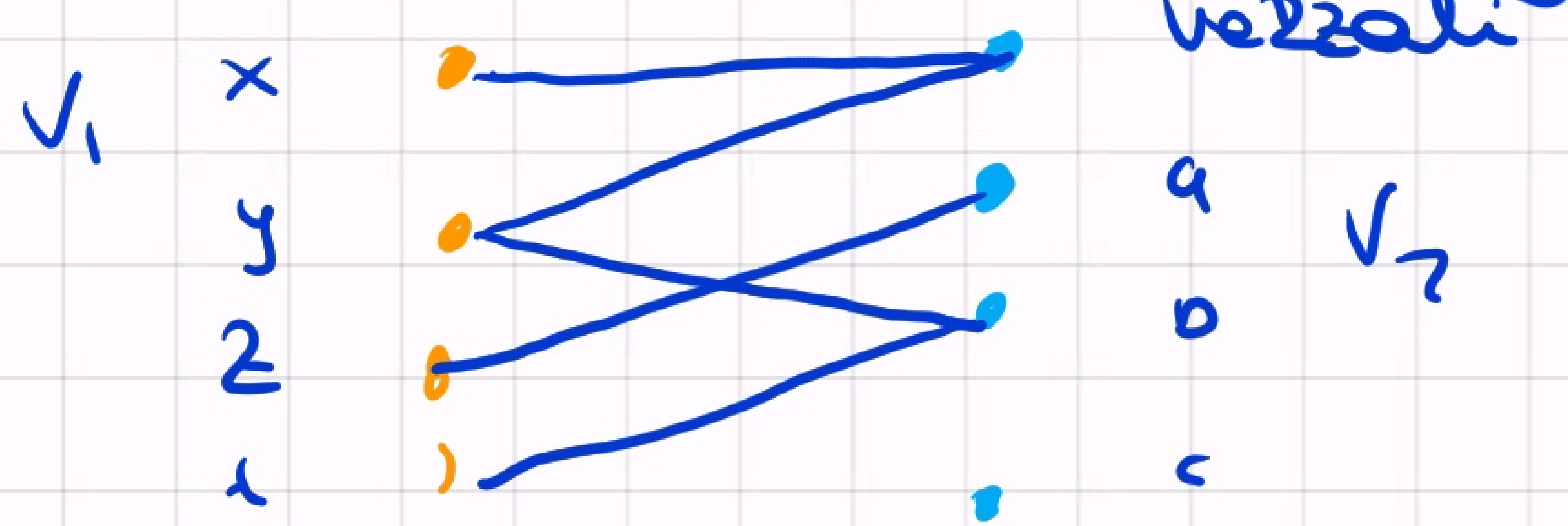
K_5 is not planar (We are going to see a proof of this)

Aim: To characterize planar graph intrinsically

• **Def** $G=(V,E)$ is bipartite if we can write $V=V_1\cup V_2$ with $V_1\cap V_2=\emptyset$ and every edge is of the form $\{a\ b\}$ with $a\in V_1$ & $b\in V_2$

Example Team fencing $V=\{\text{participants}\}$

$\{v_1, v_2\}\in E \iff v_1$ went against v_2

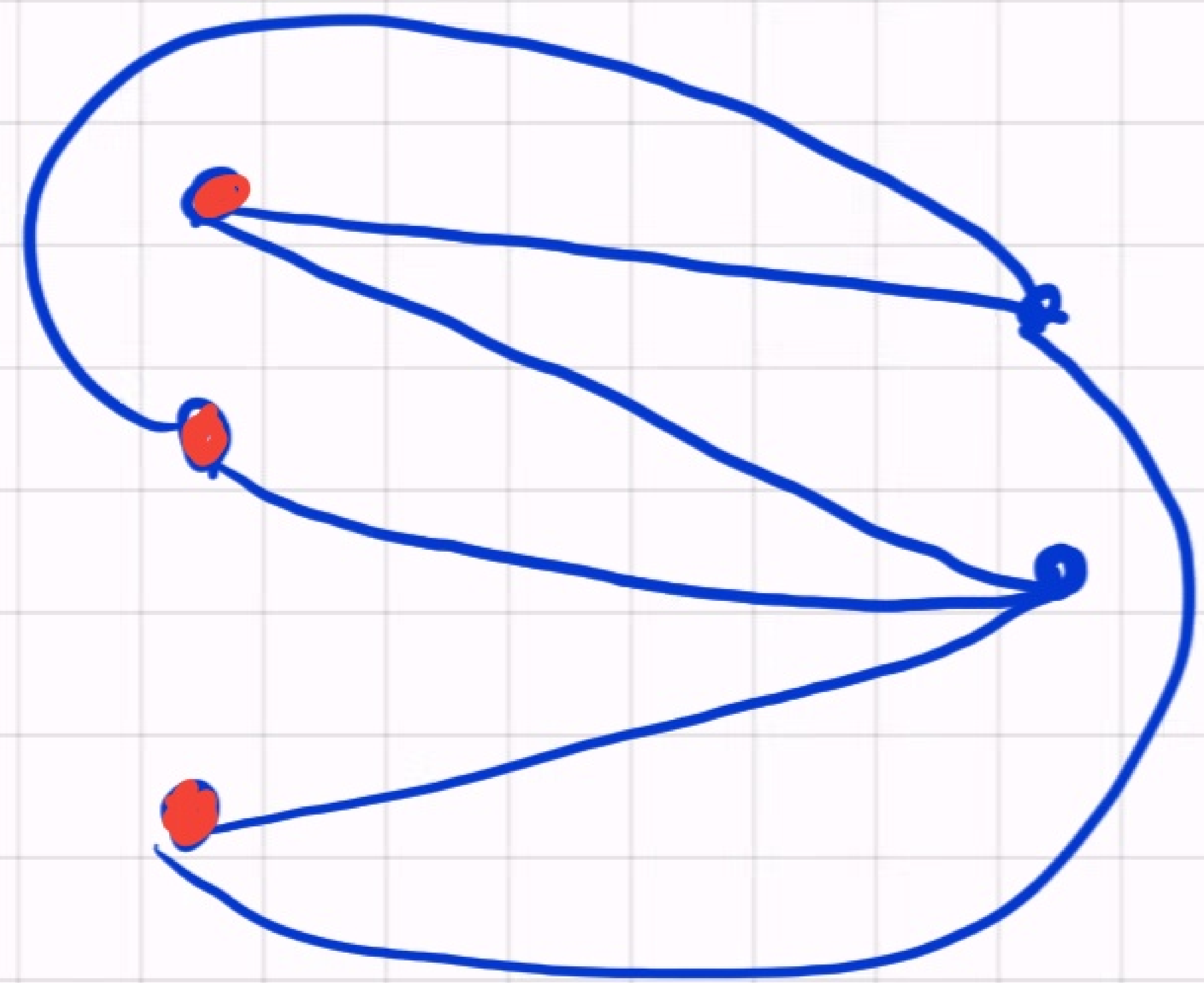


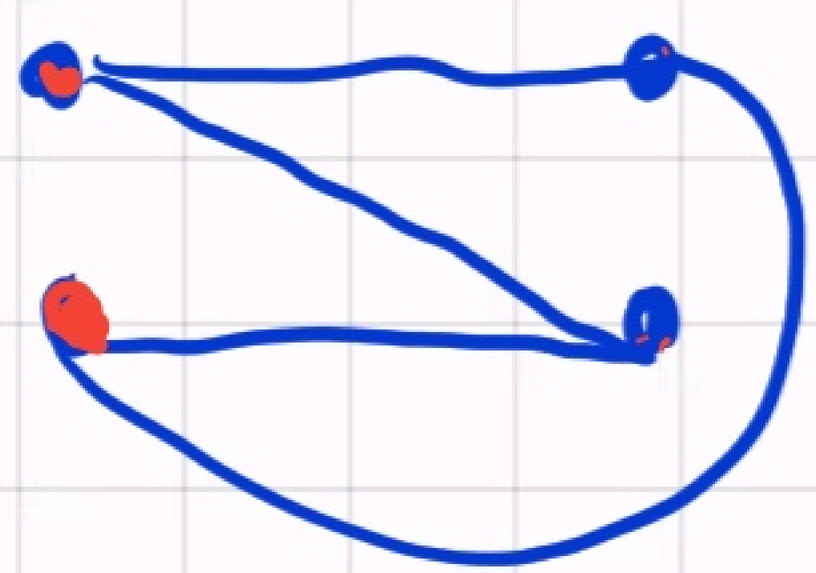
Complete bipartite graph $K_{m,n}$

$|V_1| = m$ $|V_2| = m$ all possible edges

$K_{1,1}$ 

$K_{2,1}$ 

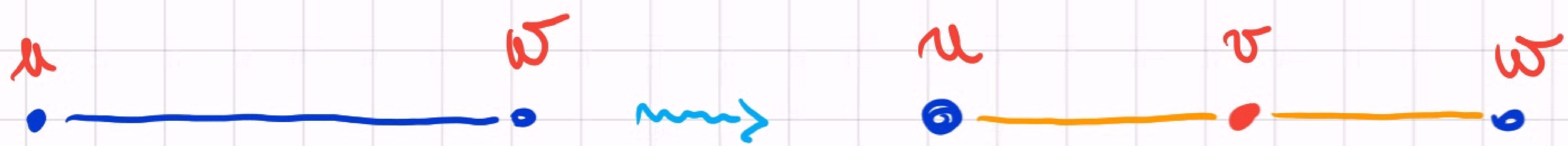
$K_{3,2}$ 

$K_{2,2}$ 

$K_{3,3}$: not planar. (To be proved)

• <u>Def</u> two graphs are <u>homeomorphic</u> if the geometric realizations are ($\exists$ $f : |G| \longrightarrow |G|'$ bijective continuous with inverse continuous)

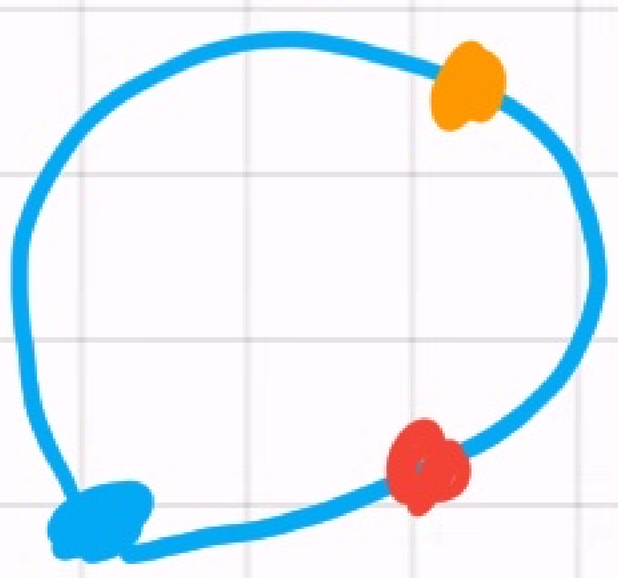
• <u>Def</u>: An elementary subdivision of a graf G is the graph G' where an edge $e = \{u, w\}$ is replaced by $\{u\ v\}$ $\{v, w\}$ where $v \notin V$



Prop

Two graphs are homeomorphic iff they can be obtained from the same graph with a sequence of elementary subdivisions

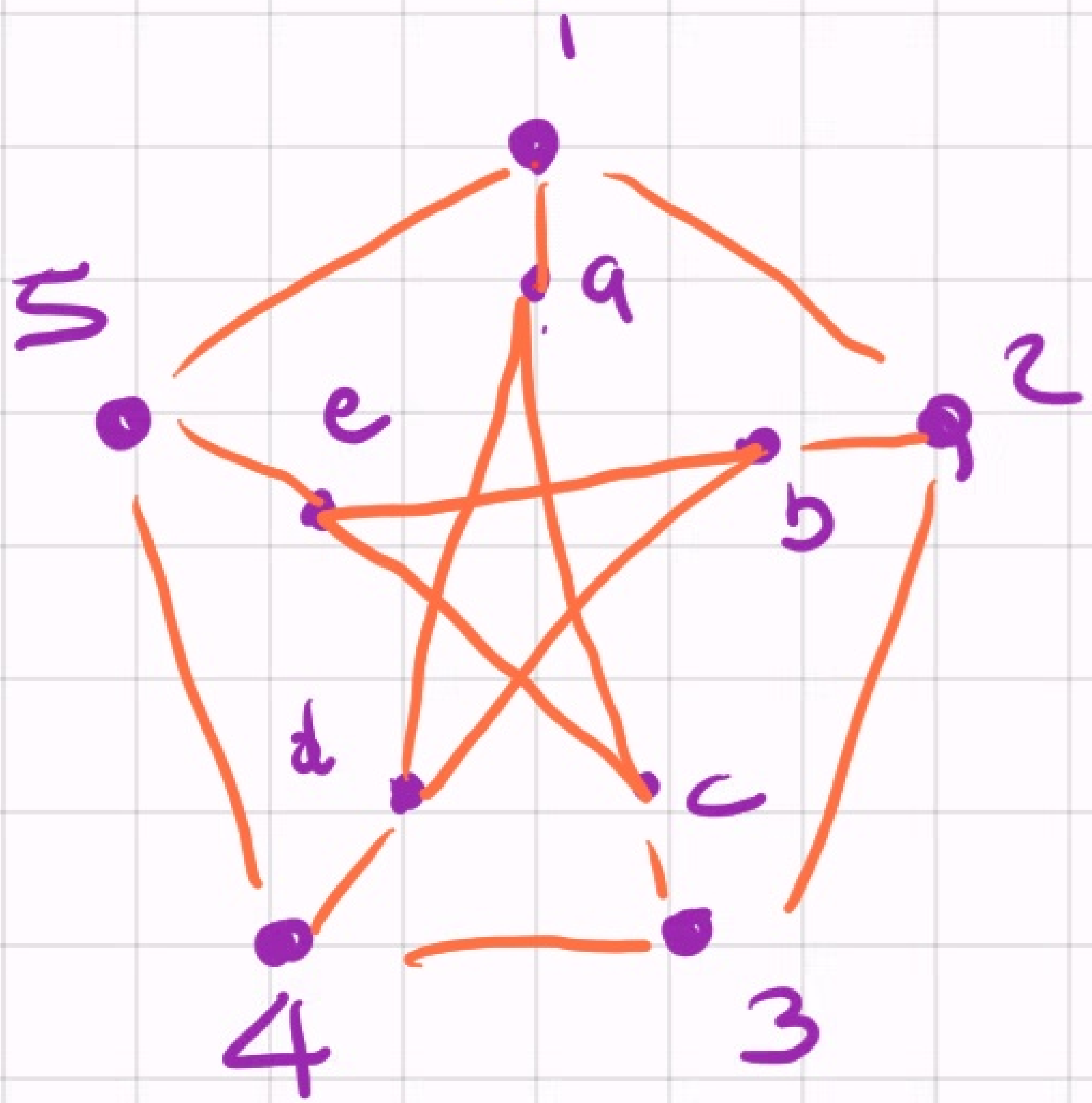
Example Any graph is homeomorphic to a loop free graph.



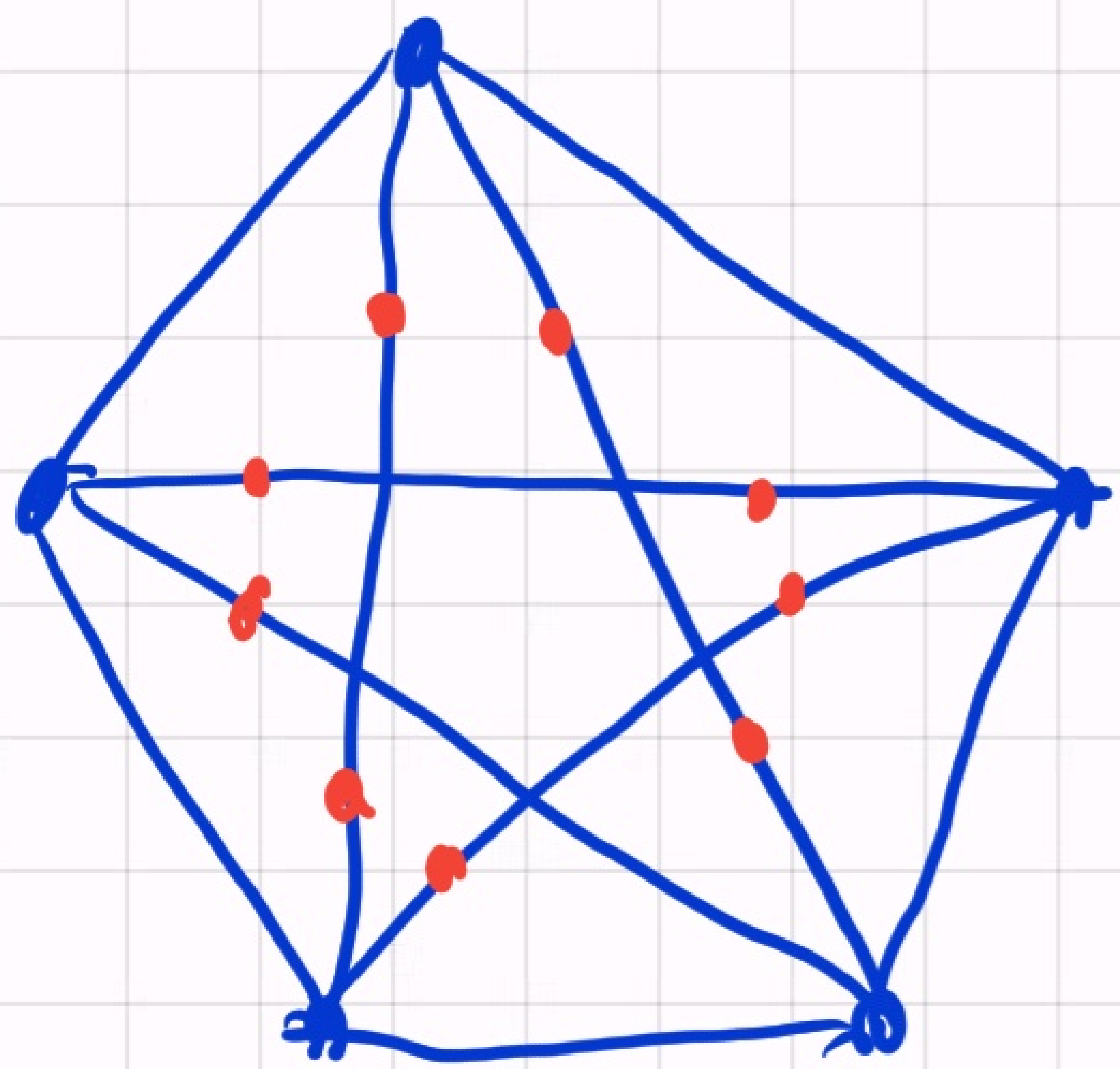
$\rightsquigarrow K_3$

Theorem (Kuratowski)

A Graph is not planar $\Longleftrightarrow$ it contains a subgraph homeomorphic to K_5 or $K_{3,3}$



is not planar



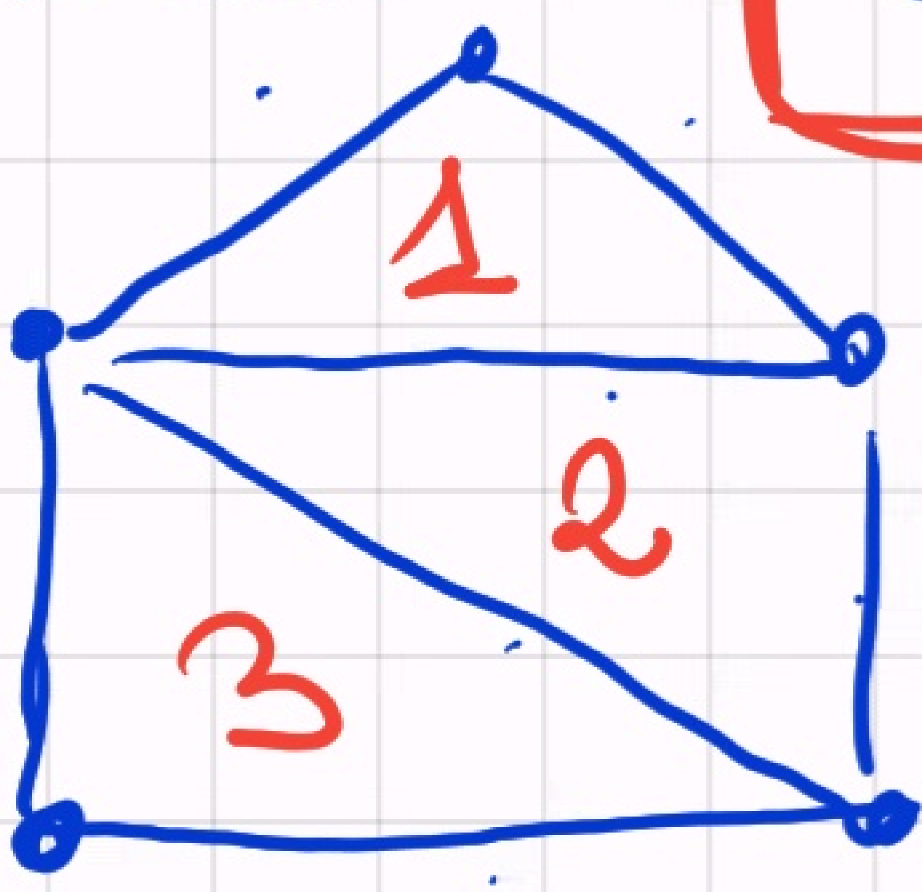
Theorem G planar & connected with v vertices and e edges. Let $\iota : |G| \longrightarrow \mathbb{R}^2$ embedding & r the number of connected cpp of $\mathbb{R}^2 - |G|$ (r is the number of closed areas $+1$)

then $\boxed{v - e + r = 2}$ $\left(= \chi(S^2)\right)$ if you know topology

Graph ~ CW complex

χ (Graph)

4

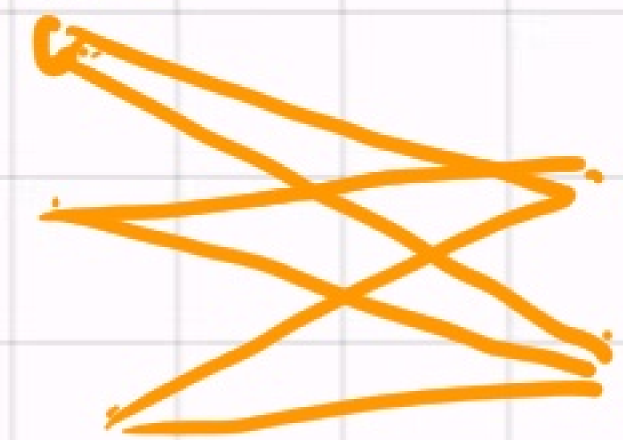


$5 - 7 + 4 = 2$

Corollary G loop free connected planar graph

$\Rightarrow$ $e \leq 3v - 6$ & $3r \leq 2e$

G is bipartite $4r \leq 2e$



Example

K_5 $v = 5$

$e = \frac{4(5)}{2} = 10$

$3v - 6 = 15 - 6 = 9 < 10$

$\rightsquigarrow$ NOT PLANAR.

$K_{3,2}$ $v = 5$

$e = 2^3 = 8$

$r = 2 - v + e = 5$

$20 \leq 16$?

Proof Thm: In the book by induction IE,

Proof cor: The boundary of any enclosed area is made up by at least 3 edges (4 in the bipartite case)

(loop free & simple)

$$3r \leq 2e \qquad \left(4r \leq 2e \iff 2r < e \right)$$

$$6 = 3 \cdot 2 = 3(v - e + r) = 3v - 3e + 3r$$

$$\leq 3v - 3e + 2e$$

$$= 3v - e$$

$$\boxed{3v - 6 \geq e}$$

$$4r \leq 2r$$

$$4 = 2(v - e + r) = 2v - 2e + r \leq 2v - 2e + e$$

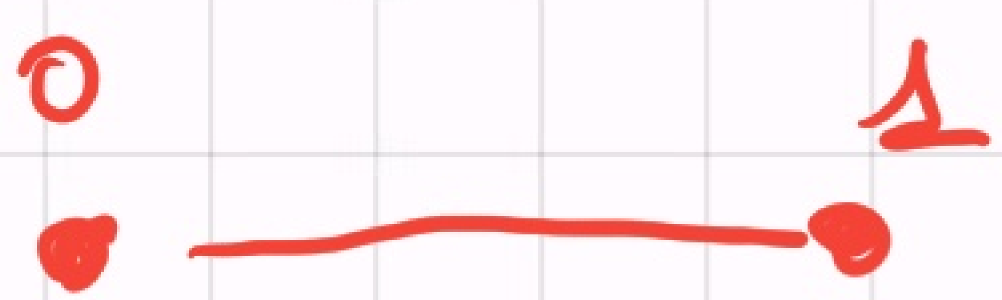
$$\boxed{2v - 4 \geq e}$$

Hamilton Cycles

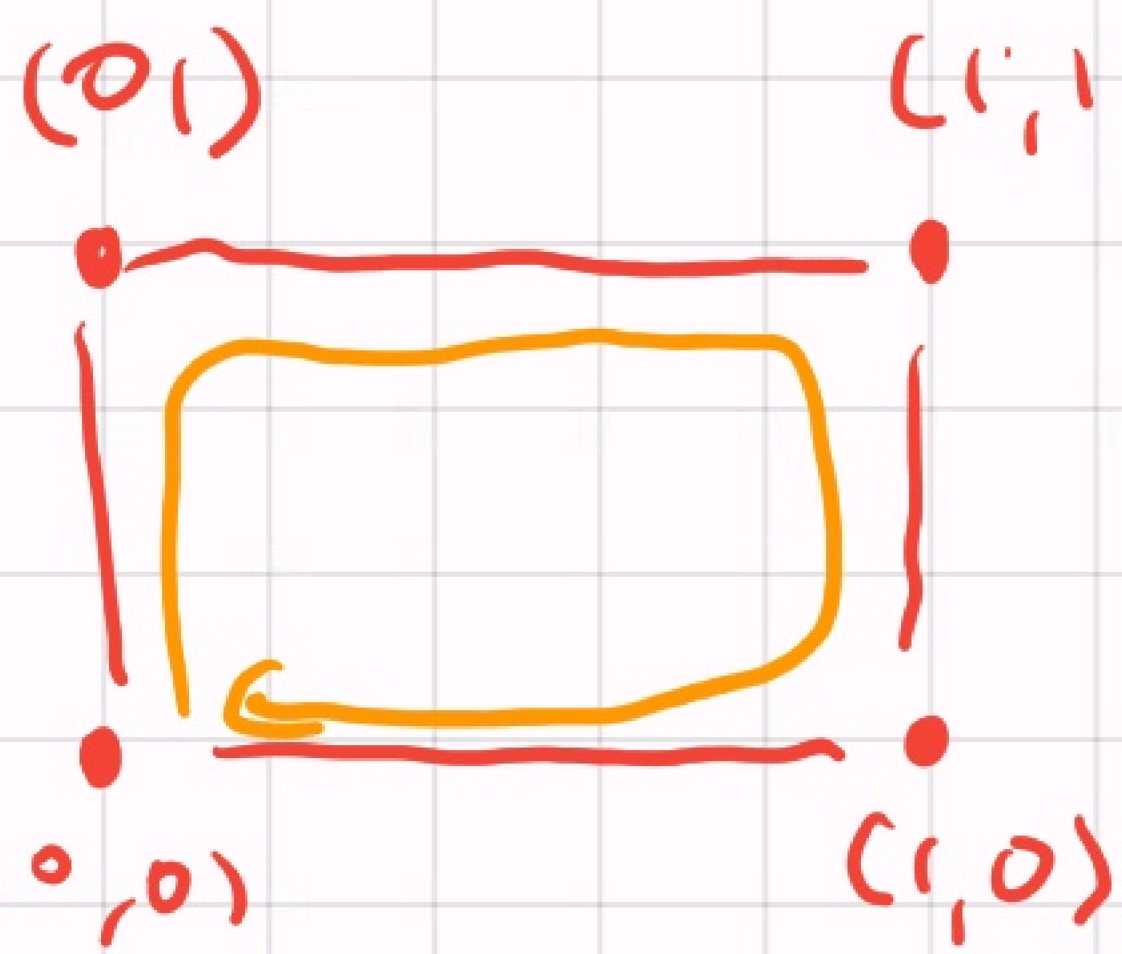
A cycle or path in a (multi) graph is **Hamilton** if it visit every vertex

Example $Q_n = \left(\{0,1\}^n \,/\, \{ \{v_1, v_2\} \,/\, v_1 \,\&\, v_2 \text{ differs in 1 coord} \} \right)$

$n = 1$



$n = 2$



There is one for every n!

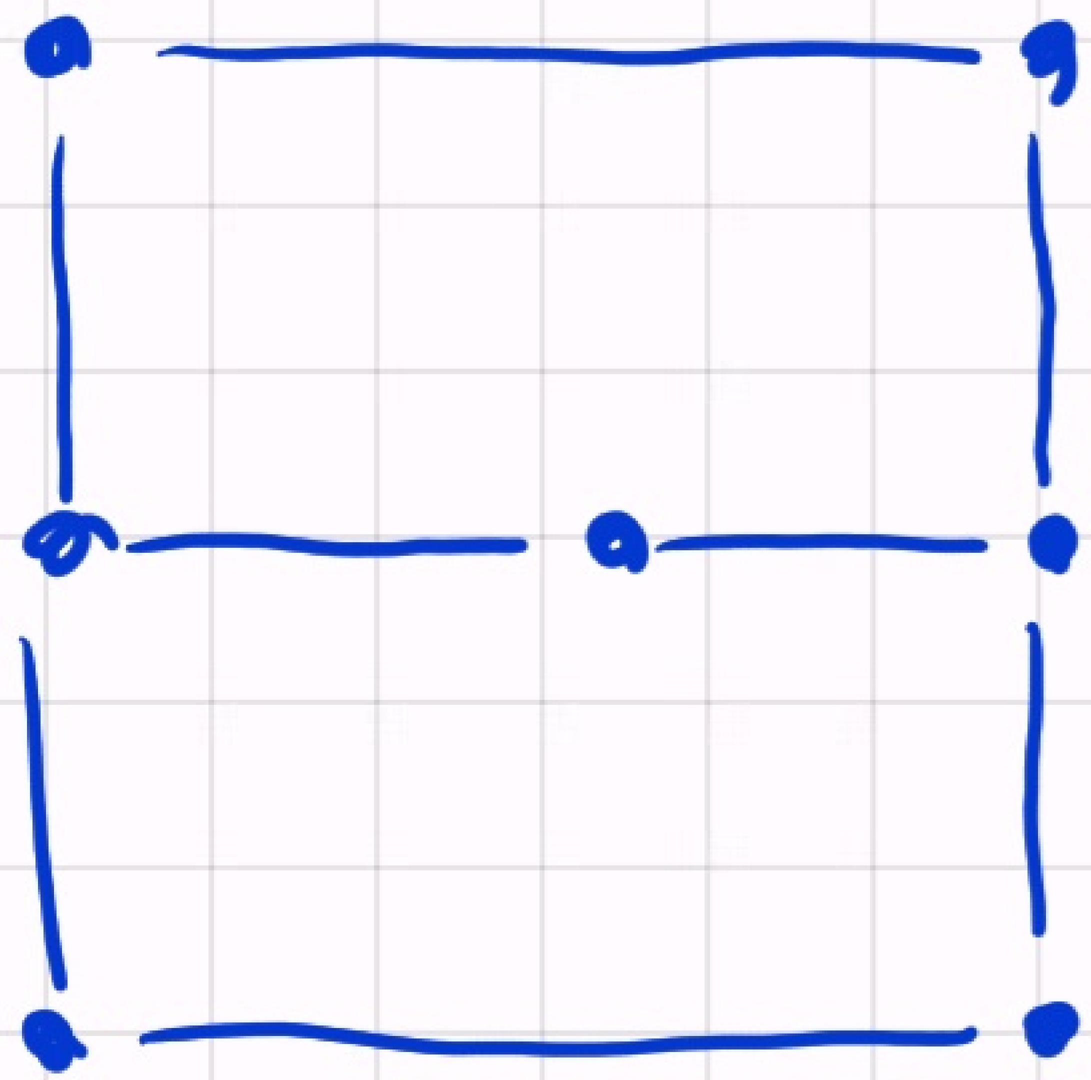
If $(v_1 \dots v_n, v_1)$ is an Hamilton cycle in Q_n

$$((0v_1) \dots (0, v_n), (1, v_n)\ (1, v_{n-1}) \dots (1 v_1), 0v_1)$$

is an Hamilton cycle in Q_{n+1}

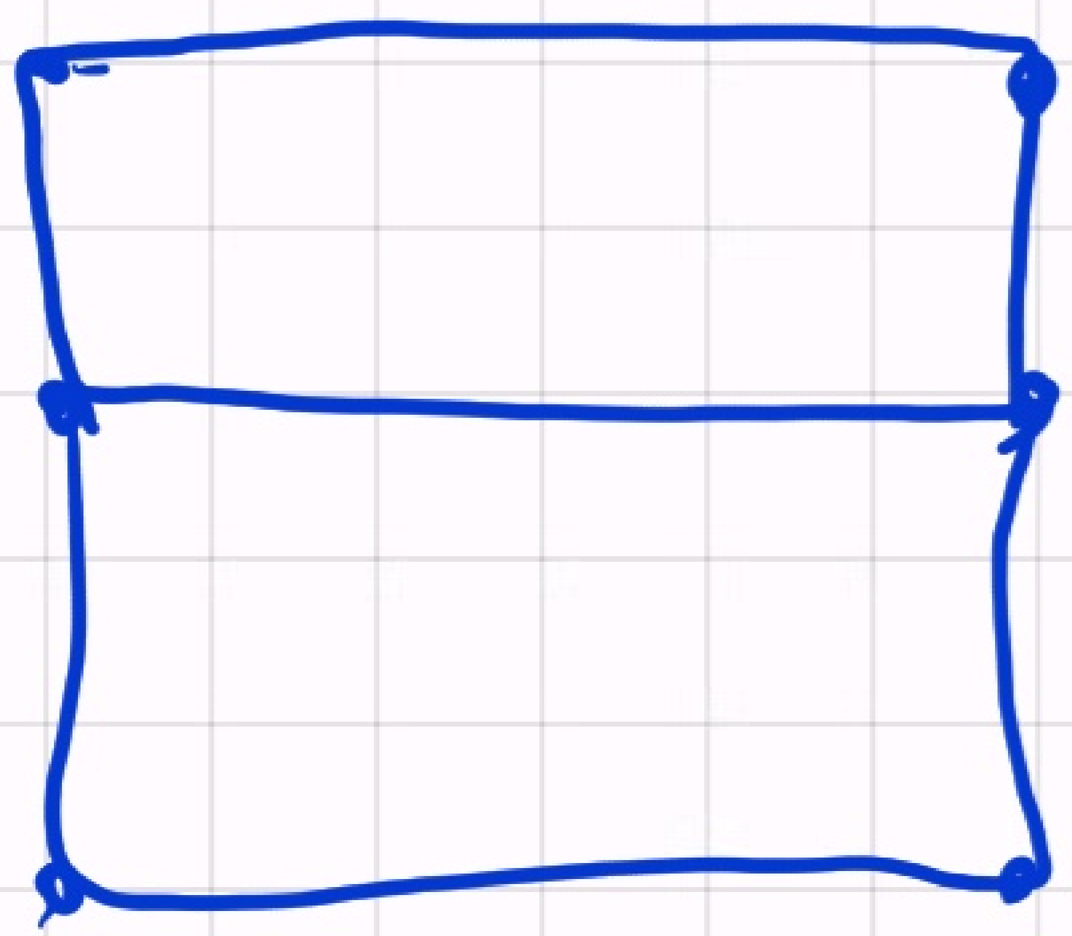
Q_n has an Hamilton cycle.

Example



does not have an HC.

Having an HC is not constant in the homeo class



but this does!

Theorem $G=(V,E)$ with

- $|V| = n \geq 3$
- $\forall v, w \in V \; v \neq w$ not adjacent

$$\deg(v) + \deg(w) \geq n$$

$$\Downarrow$$

G has an Hamilton cycle.

Proof : We prove the contrapositive :
We assume there is no HC and deduce
that $\exists\, v, w$ not adjacent such
that $\deg(v) + \deg(w) < n$

We will use : K_n has an HC.

$G \longleftrightarrow K_n$ is a subgraph.
We start adding edges to G and we will get
H a subgrapo of K_n without an HC
but such that $H+e$ will have

an HC. If $\exists\ v, w$

$$\deg_H(v) + \deg_H(w) < n$$

$$\begin{array}{cc} \leq & \leq \\ \deg_G(v) & \deg(w) \end{array}$$

Let $e = \{v, w\}$ an edge in K_n and not in H

$H + e$ has an HC

$$\begin{tikzcd} \overset{v_1}{v} \arrow[r, "e"] & \overset{v_2}{w} \arrow[r] & v_3 \arrow[r] & v_4 \cdots\cdots & v_n \arrow[llll, bend left] \end{tikzcd}$$

$3 \leq i \leq n$ $\{w, v_i\}$ $\{v, v_{i-1}\}$

cover all the edges from v to w. For every i at most 1 of these is in $E(H)$ or you can construct an HC in H

$$\deg_H(v) + \deg_H(w) < n$$

QED.

$\deg v + \deg w \geq \frac{n}{2} + \frac{n}{2} = n \quad \forall v, w$

Corollary

- $\deg(v) \geq \frac{n}{2} \quad \forall v$ then there is an Hamilton cycle.
- if $|E| \geq \binom{n-1}{2} + 2$ then it contain an Hamilton cycle

Proof: a, b not adjacent in G

$G' = G \setminus \{a, b\}$

$|V(G')| = |V(G)| - 2$

$|E(G')| = |E(G)| - \deg(a) - \deg(b)$

$G' \subseteq K_{n-2}$ $\qquad |E(G')| \leq |E(K_{n-2})|$

$= \binom{n-2}{2}$

$$\binom{n-1}{2} + 2 \leq E \leq \binom{n-2}{2} + \deg(a) + \deg(b)$$

$$\deg(a) + \deg(b) \geq \binom{n-1}{2} - \binom{n-2}{2} + 2$$

computation

$$= n$$

QED

Coloring An n-coloring of $G = (V, E)$ is $\rho : V(G) \longrightarrow \{1 \ldots n\}$ such that $\rho(v) \neq \rho(w)$ if v and w are adjiacent

Prop $\exists$ an n-coloring $\Longleftrightarrow$ $\exists$ graph homomorphism $G \longrightarrow K_n$

$\rho : V(G) \longrightarrow \{1 \ldots n\}$ morphism $\rho : G \to K_n$

$e \in E(G)$ $e = \{a, b\}$ $\{\rho(a), \rho(b)\} \in K_n$

$\Longrightarrow \rho(a) \neq \rho(b)$

$\rho : V(G) \longrightarrow \{1 \dots n\}$ coloring

$e = \{a, b\} \in E(G)$ the $\rho(a) \neq \rho(b)$

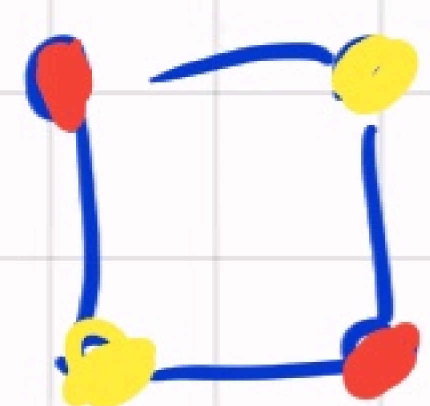
which means $\{\rho(a), \rho(b)\} \in E(K_n)$

One can extend ρ to a graph morphism

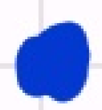
$\min \{ n \in \mathbb{N} \mid G$ has an n-coloring $\}$

is the chromatic number of G

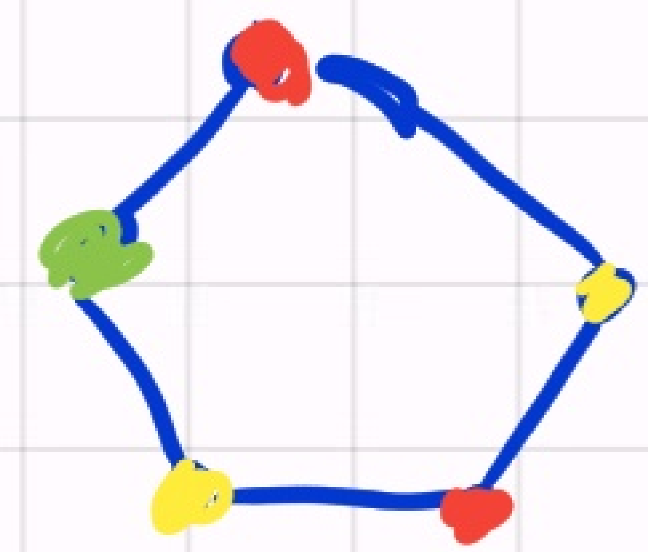
Example

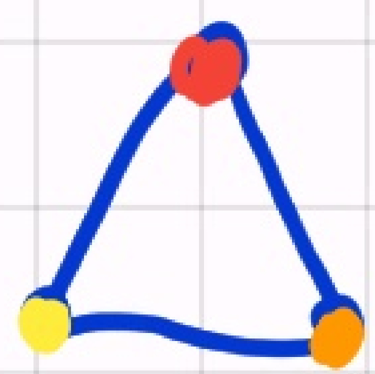
- K_m has cromatic number m and has $\frac{m!}{m!}$ m-coloring
- The chromatic number of G is $\leq |V|$
- The Chromatic # of  is 2

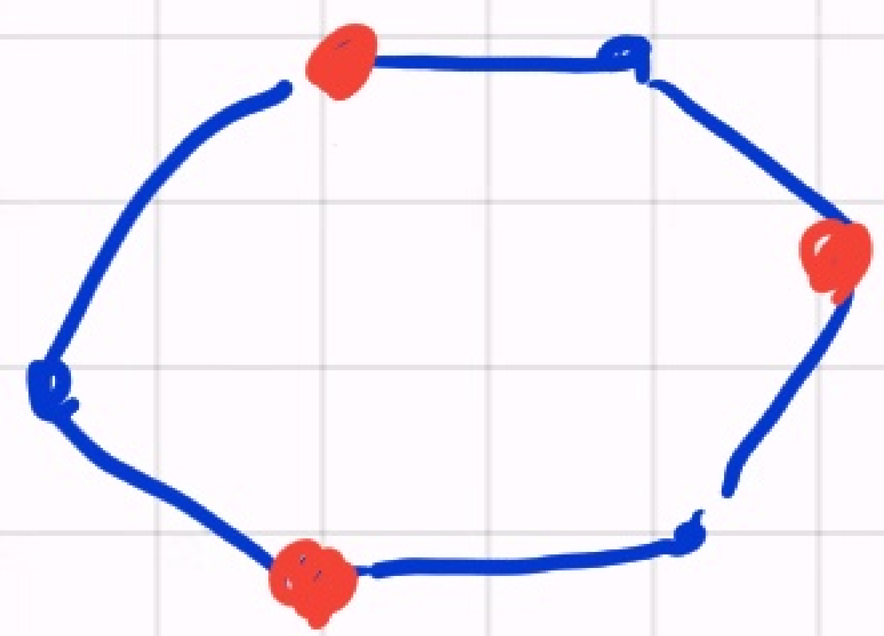
The cyclic graph on n vertices has chromatic # $=$

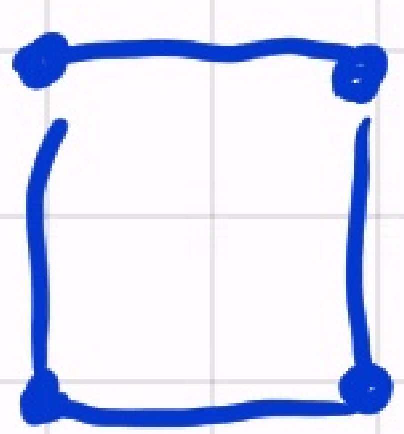
 1

 2

 $= 3$

 $=3$

 $=2$

 $=2$

2 if n is even
3 if $n>1$ is odd
1 if $n=1$

Prop G has cn $\leq 2 \iff$ is bipartite

Proof If G is bipartite then we can find a 2 coloring

$V = V_1 \cup V_2$ & $V_1 \cap V_2 = \emptyset$ & no edges in V_i

$$\rho : V \longrightarrow \{1,2\}$$
$$v \longmapsto \begin{cases} 1 & \text{if } v \in V_1 \\ 2 & \text{if } v \in V_2 \end{cases}$$

Conversely, if $cn \leq 2$ then $\exists$ a 2-coloring

$$\rho : V \longrightarrow \{1, 2\}$$

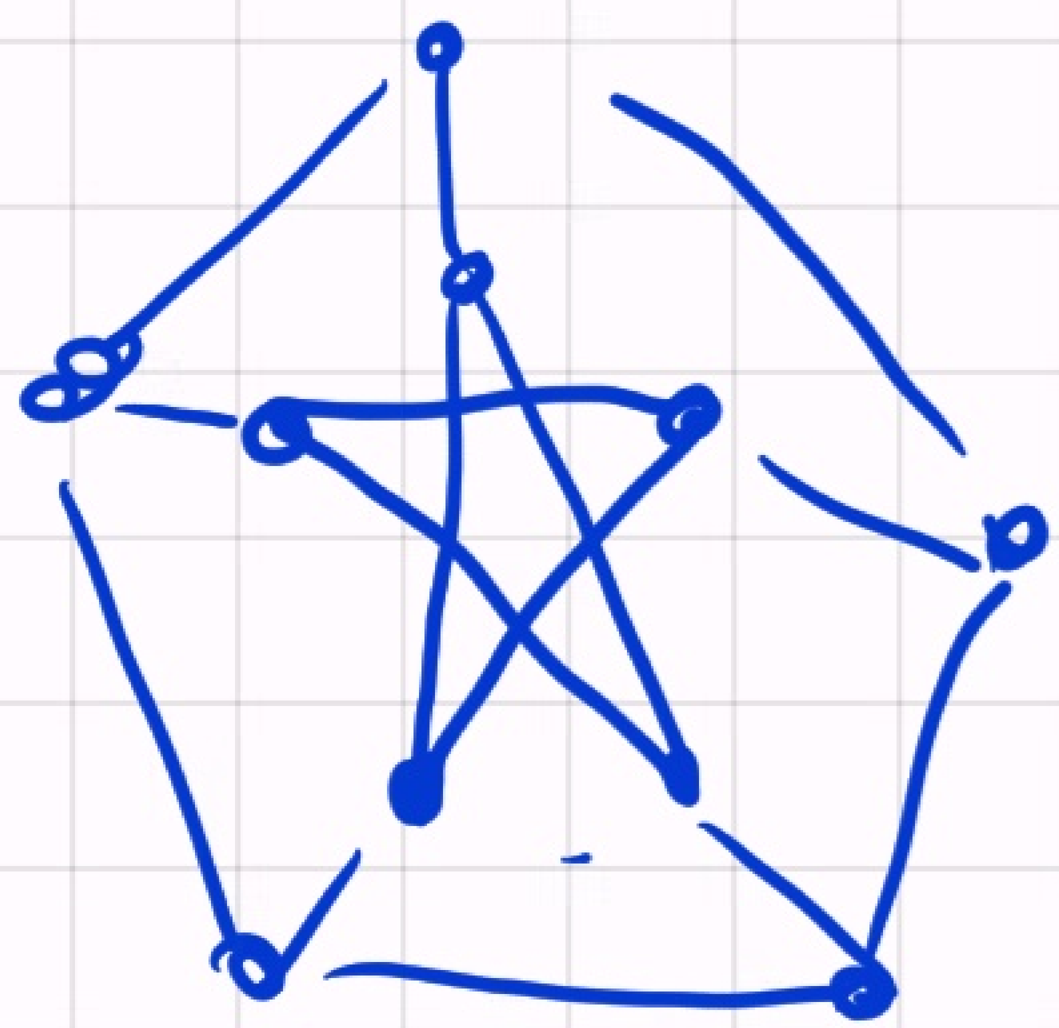
$$V_i := \rho^{-1}(i) \qquad i = 1, 2$$

$$V = V_1 \cup V_2 \qquad V_1 \cap V_2$$

if $a, b \in V_i$ they are colored with the same color $\Rightarrow$ they are not adjacent

$\Rightarrow$ there is no edge connecting them.

QED



has an 3

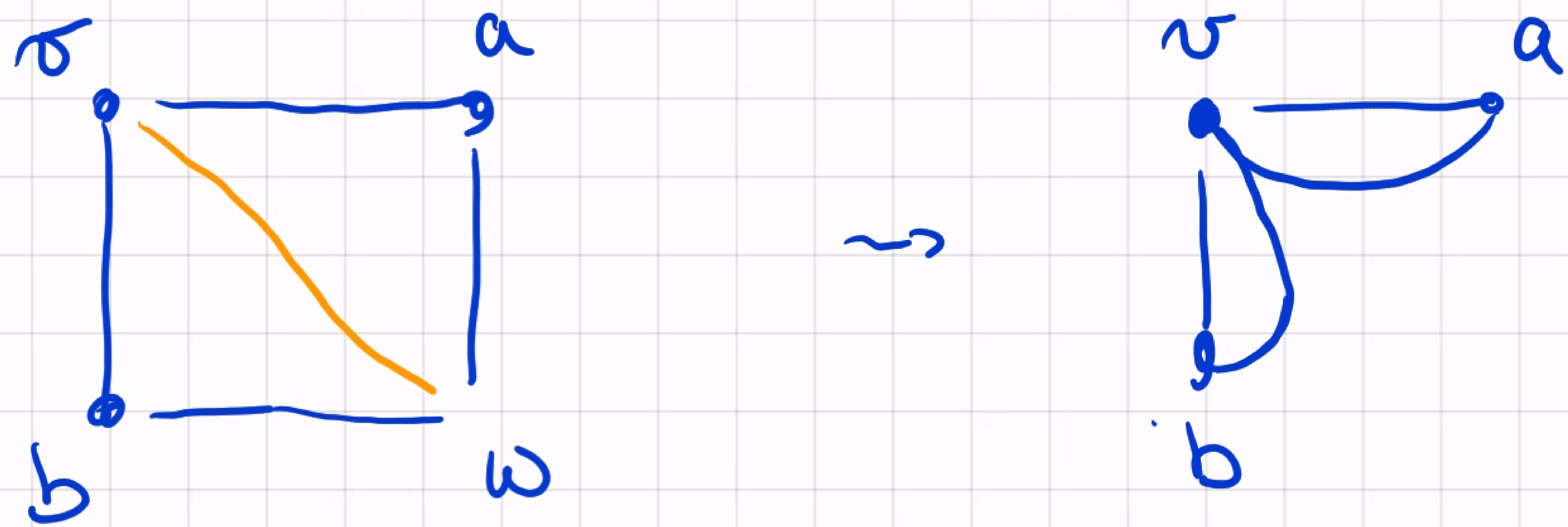
Example in the book.

The n-th cromatic # of a graph G is

$$\chi_G(n) := \left|\{ f : G \to K_n \text{ graph homo} \}\right|$$
$$= \left|\{ n\text{-coloring of } G \}\right|$$

G a graph $e = \{v, w\}$ an edge the graph obtained by collapsing e is

$$\left(V/_{v \sim w} \quad E(G) \setminus e /_{v = w} \right)$$



$\boxed{\text{Theorem}}$ $\exists$ a unique polynomial $P(6, x)$ such that $\forall\, n$ $P(6, n) = \chi_6(n)$ for every $n \in \mathbb{N}$

Proof

$\nabla!$ Suppose that $p(x)$ and $q(x)$ are such that $p(n) = \chi_6(n) = q(n)$

$$\deg\big(p(x) - q(x)\big) \leq \max\{\deg(p), \deg(q)\}$$

but $\forall\, n \in \mathbb{N}$ $[p-q](n) = 0$

$p-q$ has ∞-~~ma~~ many roots

$\Downarrow$

$p-q=0 \implies p=q$

3) is G has a loop $\Rightarrow P(G,x)\equiv 0$

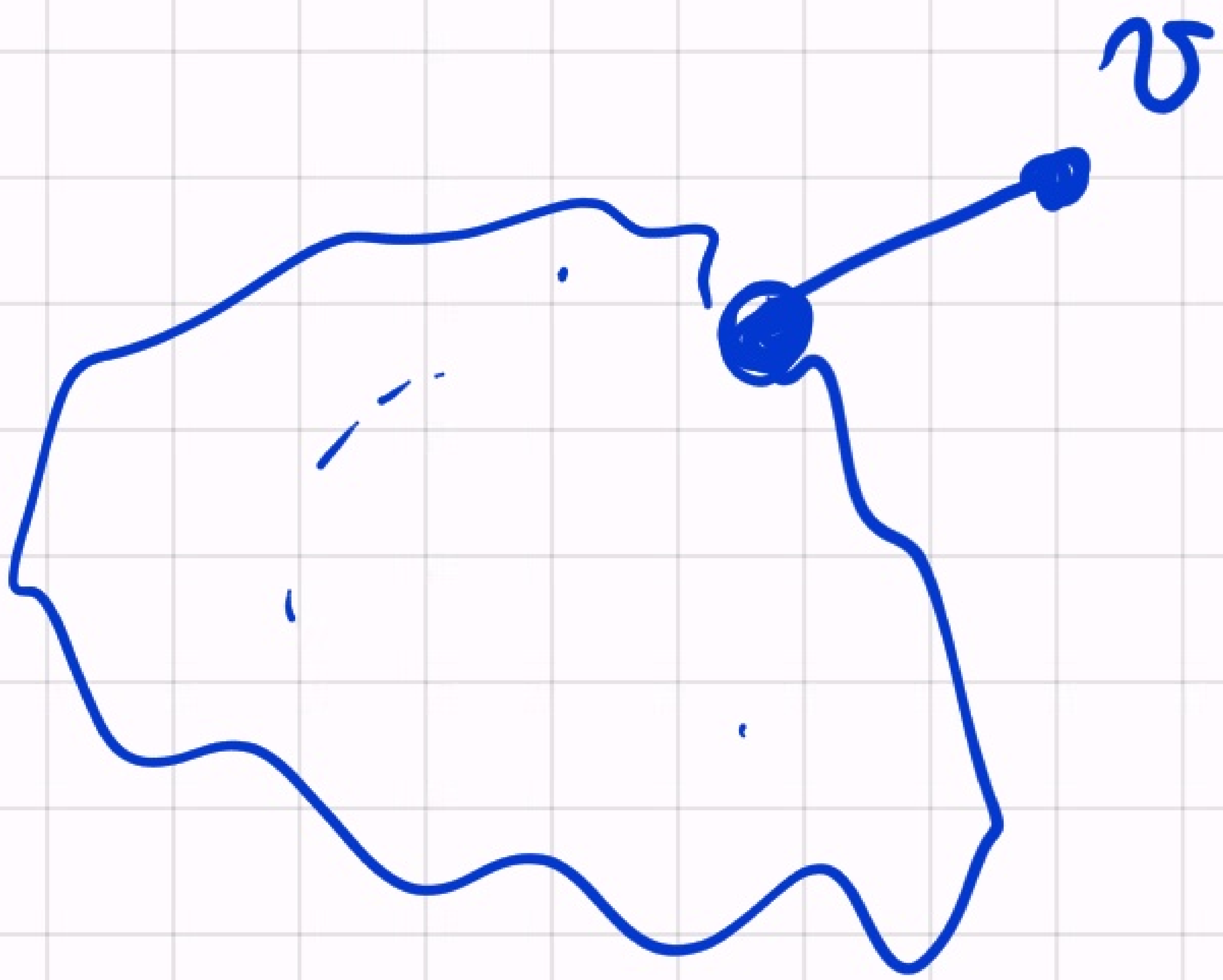
if G is loop free with just one vertex

$G = \bullet$ You can take $\underline{\underline{P(G,x)=x}}$

$P(G,n) = n = \chi_G(n)$

We proceed by structural induction.

- if G has a terminal vertex



$$\chi_G(n) = \chi_{G \setminus v}(n)\,(n-1)$$

$$P(G, x) = P(G \setminus v, x) \cdot (x-1)$$

IH this is a poly.

- $e = \{a, b\}$ is a non terminal edg.

$$\chi_{G-e}(n) = \chi_G(n) + \chi_H(n)$$

↳ you collapsed e.

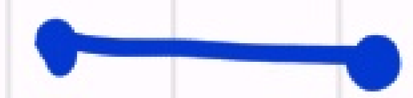
$$\chi_G = \chi_H - \chi_{G-e}$$

$$P(G,x) = P(H,x) - P(G-e,x)$$

polynomial.

Example

x



$x(x-1)$

$x(x-1)(x-1)$

$x(x-1)^{n-2} \cdot (x-1)$

$x(x-1)^{n}$

$P(K_n, x) =$

Theorem Let $G = G_1 \cup G_2$ & $G_1 \cap G_2 \cong K_m$

then

$$P(G, x) = \frac{P(G_1, x) \cdot P(G_2, x)}{P(K_m, x)}$$