Mm5023 lecture 9 –

Graphs II

Plan

- Euler circuits (Necessary & sufficient conditions for $\exists$)
- Planar graphs

Degree

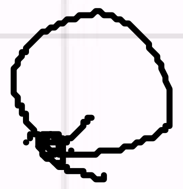
Given a (directed or undirected) graph $G = (V E)$ and $v \in V$ we define the degree of v to be

$$\deg v = |\{ w \in V / w \neq v \ \& \ w \sim v \}| + 2 |\{\text{loops at } v\}|$$

If G is directed we can also define the incoming & outgoing degrees (Important for lecture 13)

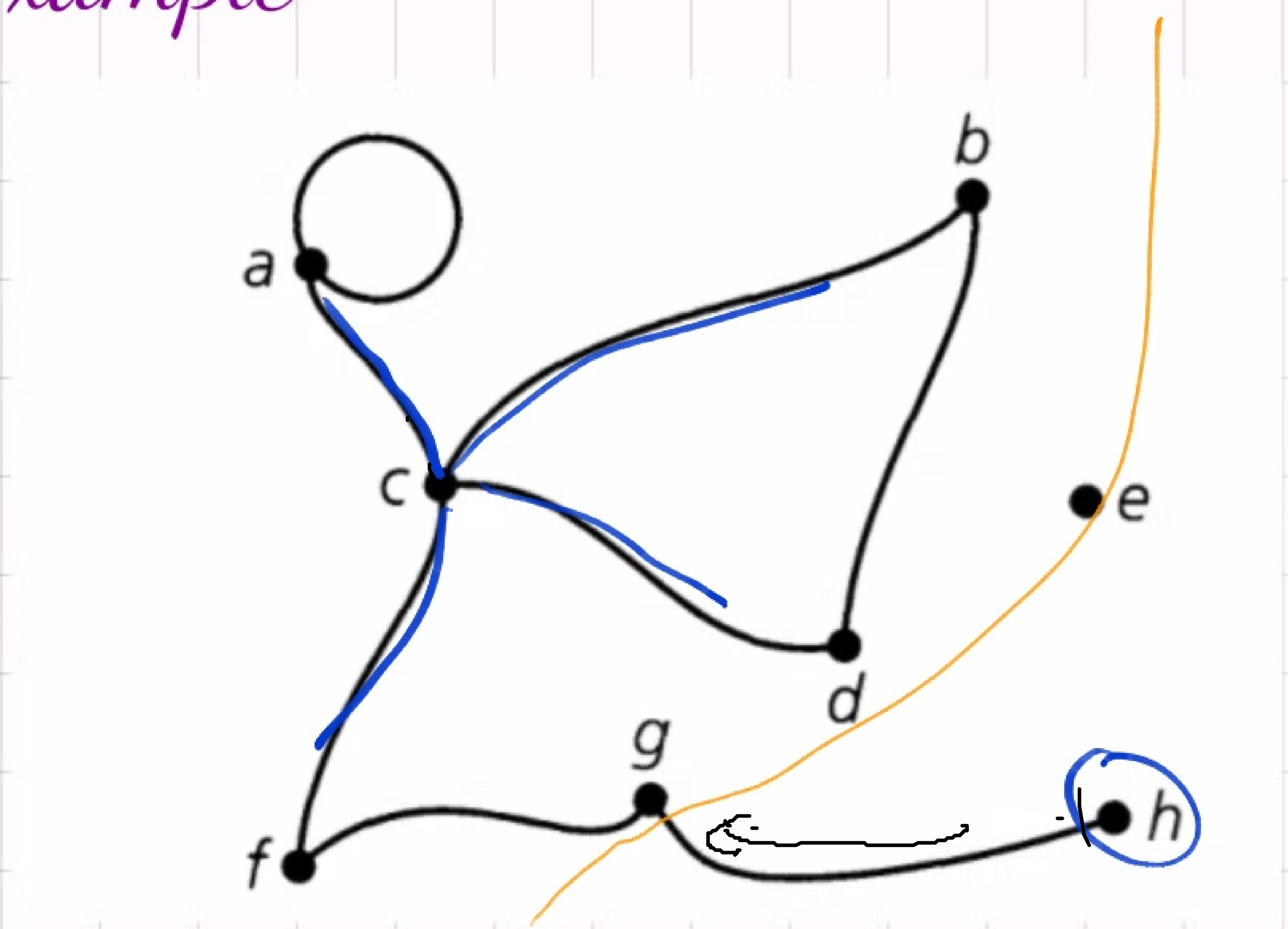
$$\deg_{IN} v = |\{ w \in V / (w, v) \in E \}|$$ — # of edges with arrow pointed at v

$$\deg_{OUT} v = |\{ w \in V / (v, w) \in E \}|$$ # of edges with tail out v



$\deg f = 2$

Example



$\deg(a) = 3$

$\deg(b) = 2$

$\deg(c) = 4$ $\deg(h) = 1$

$\deg(d) = 2$

$\deg(e) = 0$

$\deg(g) = 2$

A vertex with $\deg(v) = 1$ is said to be extremal/pendant (important in proofs)

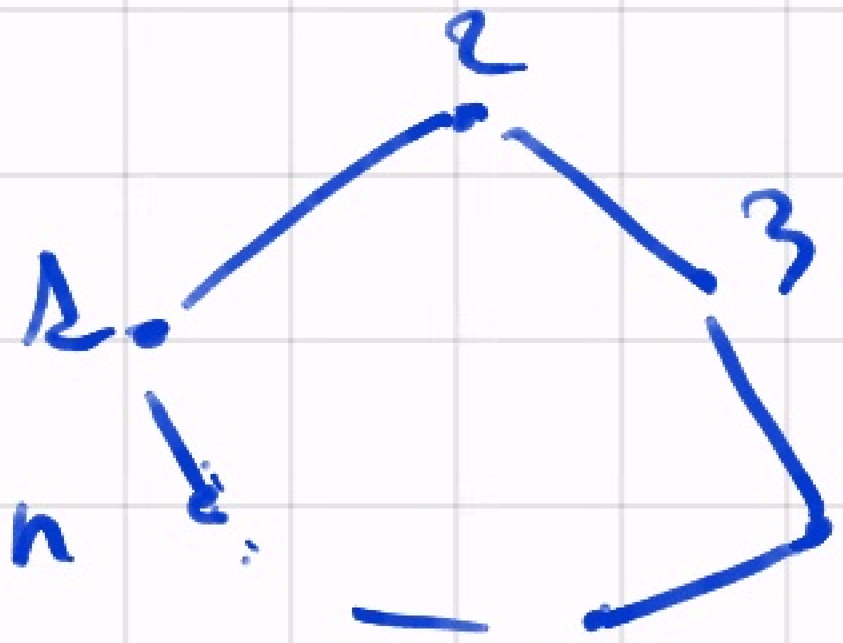
N- regular graphs

Let $n \in \mathbb{N}$, we say that a graph is n-regular if $\deg(v) = m$ for all $v \in V$

Examples

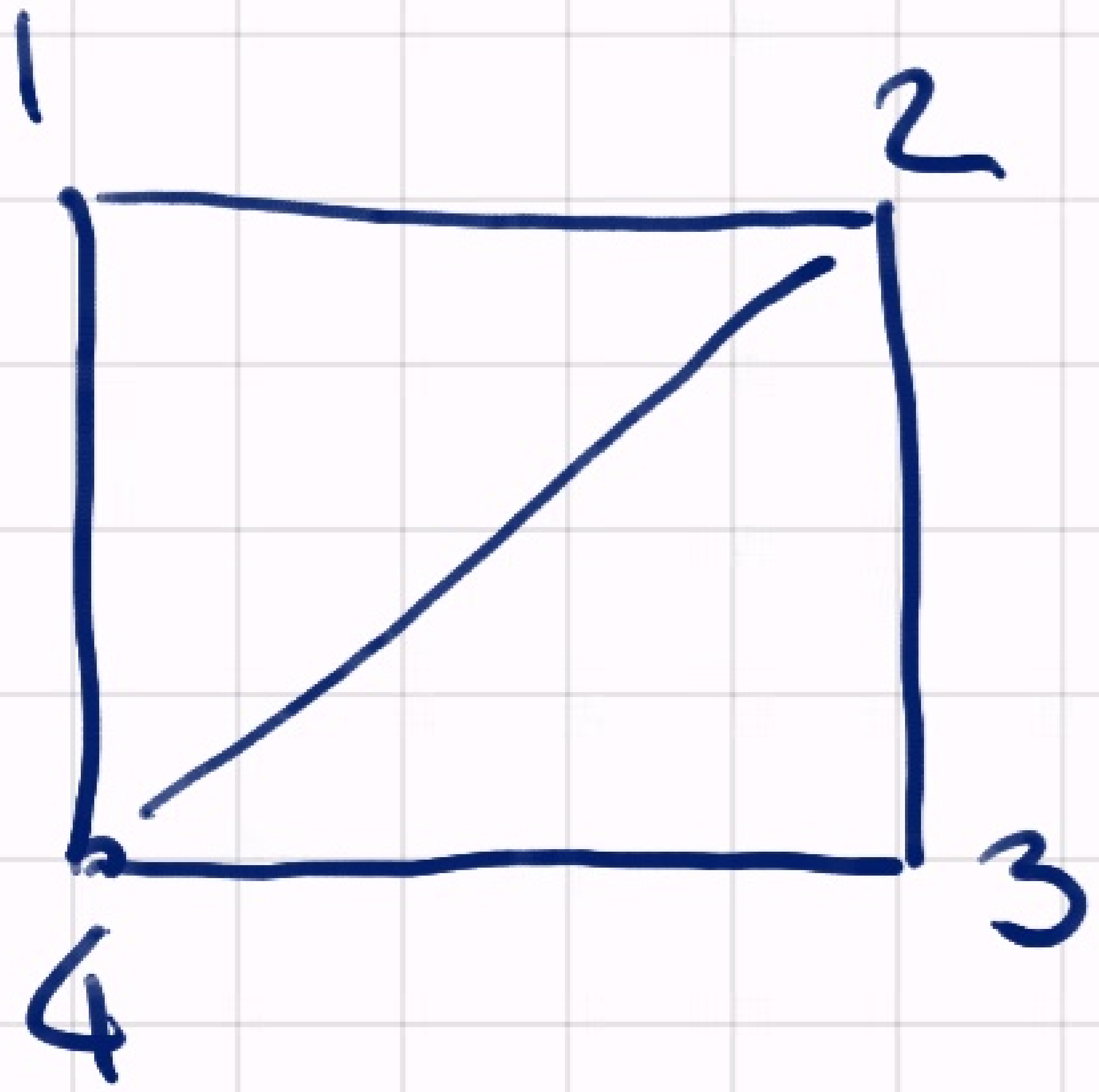
- K_n $m-1$ regular

- C_n



2 - regular.

Example



deg (1) = 2 deg (3) = 2

deg (2) = deg (4) = 3

This is not regular for any n

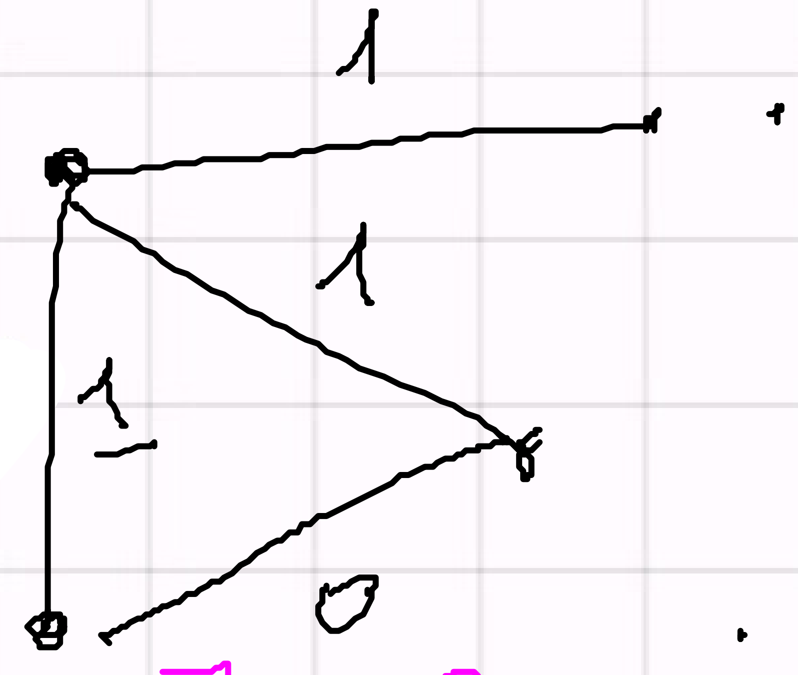
Edges and vertices

Prop Given a (multi) graph $G=(V,E,p)$ we have that

$$2|E| = \sum_{v\in V} \deg v$$

~~If G loop free.~~

$$e \sim v \iff v \in p(e) \subseteq \mathcal{P}(V)$$



Proof:

$$\sum_{v\in V(G)} \deg v = \sum_{v\in V(G)} \sum_{\substack{e\in E(G)\setminus\text{loop}\\ e\sim v}} 1 + \sum_{\text{loop at } v} 2$$

$$= \sum_{\substack{e\in E(G)\\ \text{no loop}}} \underbrace{\left(\sum_{\substack{v\in V(G)\\ e\sim v}} 1\right)}_{=2} + \sum_{\text{loop at } v} 2$$

$$= \sum_{e\in E(G)} 2 = \boxed{2|E|}$$

$\#$

Corollary: In a graph or multigraph

$$\sum_{v \in V} \deg(v) \text{ is EVEN}$$

$\Rightarrow$ there is an even number of vertices with odd degree

How many edges has K_n ?

We already counted this

$$\frac{1}{2} n(n-1)$$

Let us count it again using the formula

K_n is $(n-1)$ regular

$\deg(v) = n-1$ for every $v \in V(K_n)$

$$2|E(K_n)| = \sum_{v\in V} \deg v = \sum_{v \in V} (n-1) = n(n-1)$$

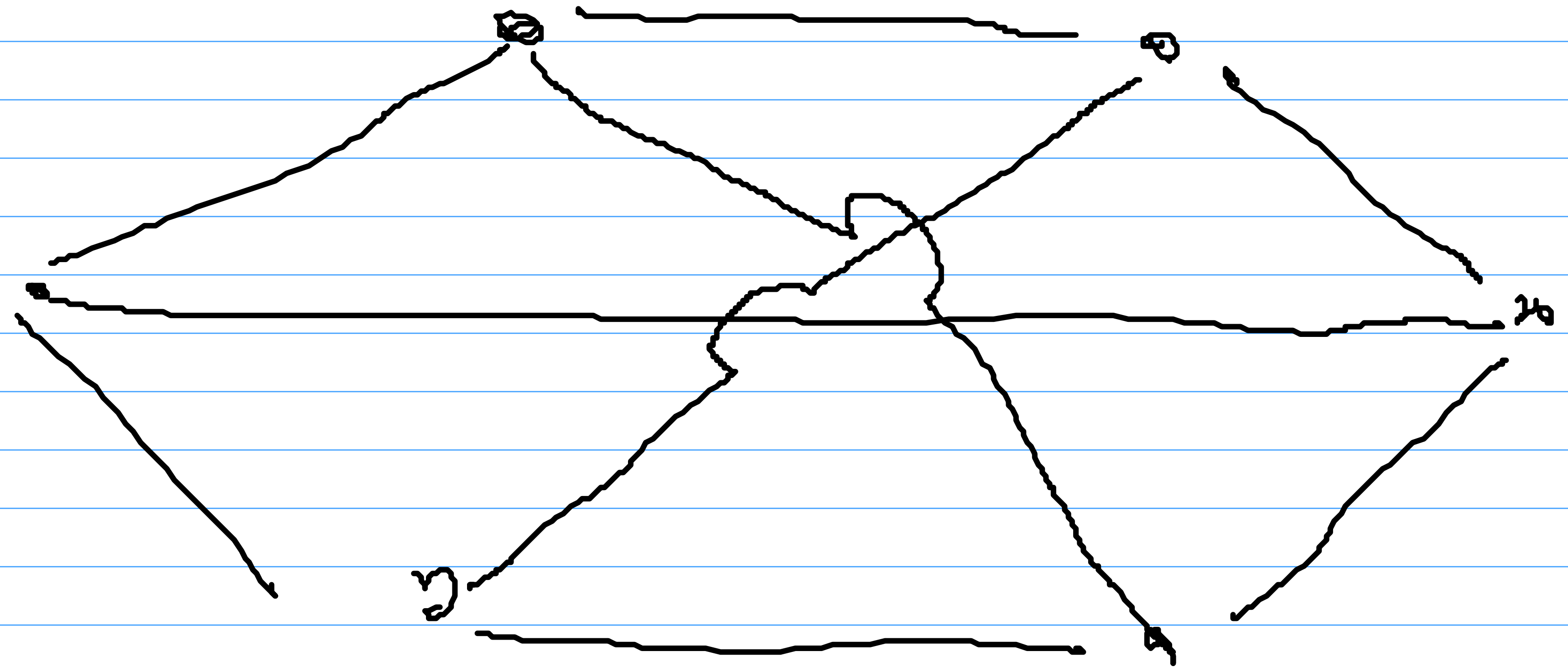
$$E(K_n) = \frac{1}{2} n(n-1)$$

Is there a 3-regular graph with 9 or 10 edges?

$|E| = 9$

$$2|E| = 18 = \sum_{v \in V} \deg v \overset{3\text{-reg}}{=} \sum_{v \in V} 3 = |V| \cdot 3$$

$|V| = 6$



$|E| = 9$ YES

$|E| = 10$

$$20 = \sum_{v \in V} \cdot 3 = |V| \cdot 3$$

but $3 \nmid 20$

$\Rightarrow$ Impossible.

Euler circuits and trails

Recall: A trail is a walk with no repeated edges

A circuit is closed trail

Def A trail or circuit is called Euler if it passes through all the edges

The main result

Given a finite (multi) graph $G=(V,E)$, without isolated vertices, then it has an Euler circuit (trail) $\iff$

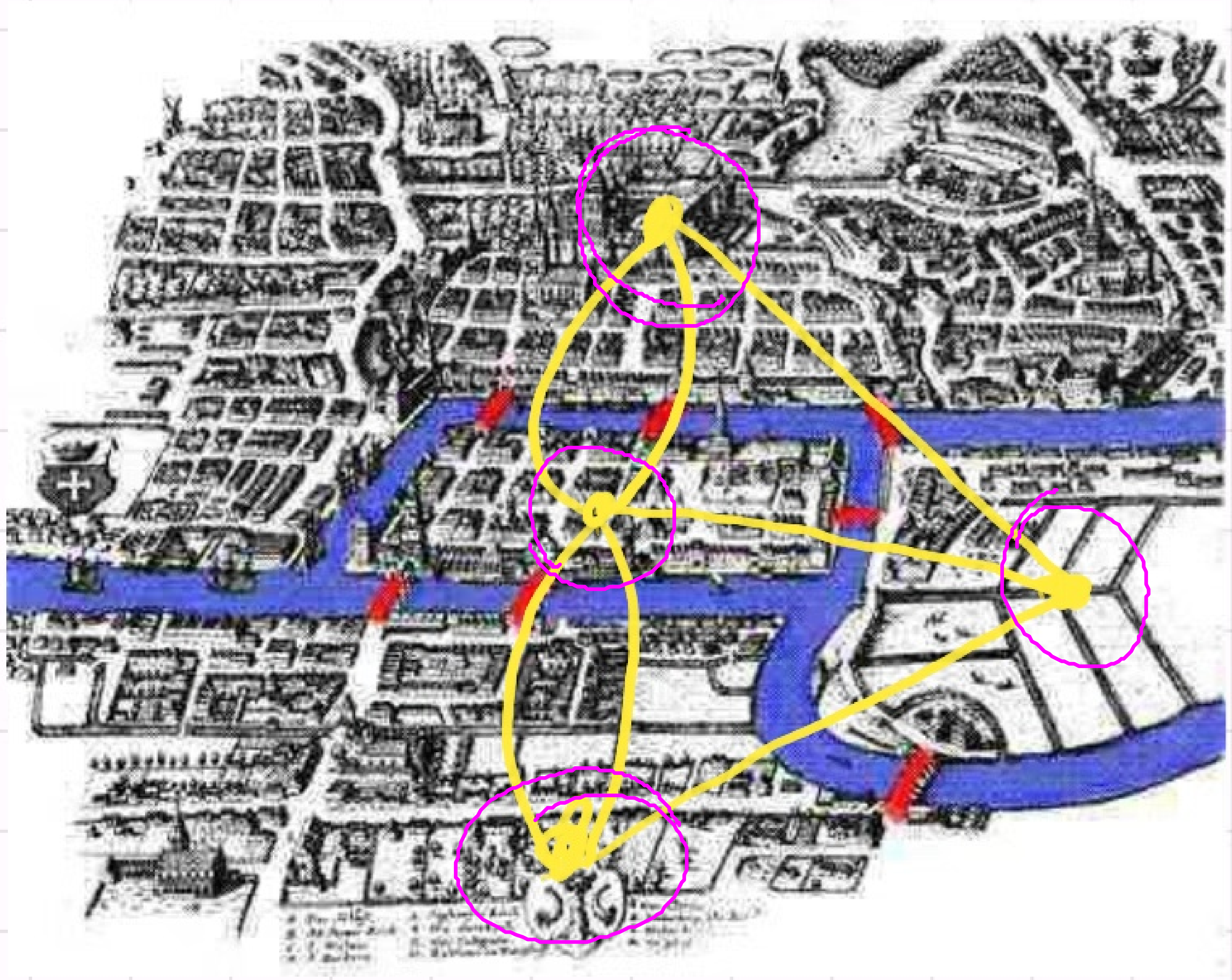
① G is connected

② All vertices of V have even degree (circuit)

(There are at most 2 vertices of odd degree)

trail

The original problem



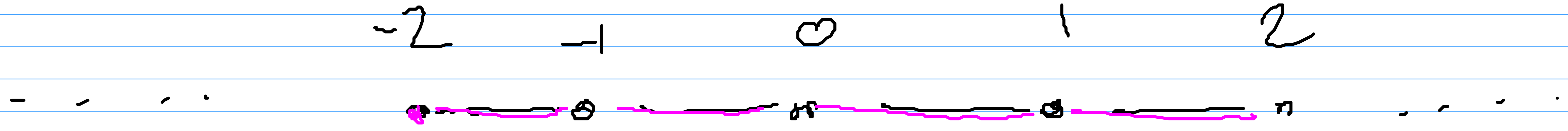
There are more than two vertices with odd degree : neither Euler trail nor circuit exist

So you cannot walk and visit all the bridges just once

Example: the finiteness condition is important

$V = \mathbb{Z}$

$E = \{ \{i, i+1\} \mid i \in \mathbb{Z} \}$



$\deg v = 2$ for all $v \in V$

but there is no Euler circuit.

can never go back.

Lem : $G = (V, E, p)$ finite connected non-trivial with vertices of even degrees
$\Rightarrow$ then there is a non trivial circuit

Proof : We are going to show that a trail of maximal length is a circuit.

Note that a trail of maximal length always exist (by the well ordering)

The length of any trail in G is $\leq |E|$

$l : T = \{ \text{trails in } G \} \longrightarrow \mathbb{N}$

if $T \neq \emptyset$ then $l(T)$ bounded above
$\Rightarrow$ it has max

$\exists\ t_0 \in T$ with $l(t_0)$ maximal.

Why T is not empty?

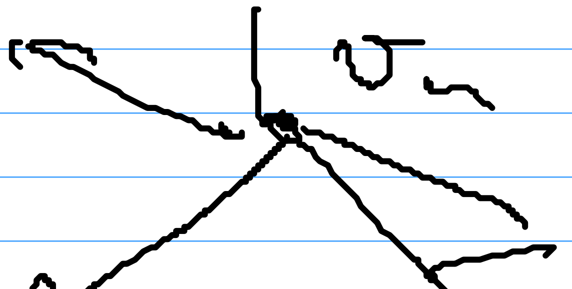
if there are two or more vertices then G has an edge by connectedness.

e_1 $\qquad$ $p(e_1) = \{v, w\}$

$(v\ e_1\ w)$ is trail (does not repeat edges).

There is a trail of maximal length if there are circuit $\Rightarrow$ there is a circuit

let $(v_1\ e_1\ v_2\ \dots\ e_{n-1}\ v_n)$ a trail of maximal length. If $v_1 \neq v_n$ I walked just an odd numb of edges out of v_1 or v_n



there is an unused edge out of v_n then I can extend the trail $\Rightarrow$ contradict maximal length. #

Proof of the theorem. $\Rightarrow$))

we just consider circuit

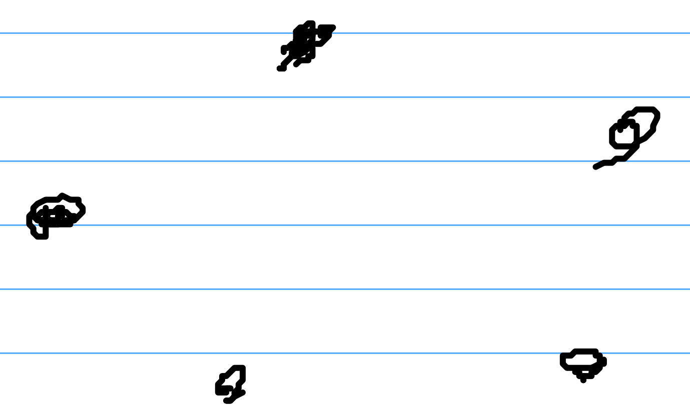
If a graph has 2 vertices with odd deg you consider the graph with same vertices and edges and an extra edge connecting the odd vertices

this has an Euler circuit. (if you prove the thm in this case). If you remove the added edge from the graph $\Rightarrow$ you get an Euler trail connecting the two odd vertices

We need to prove the $\exists$ of EC in the all even case

by induction on $|E|$

if $|E| = 0$



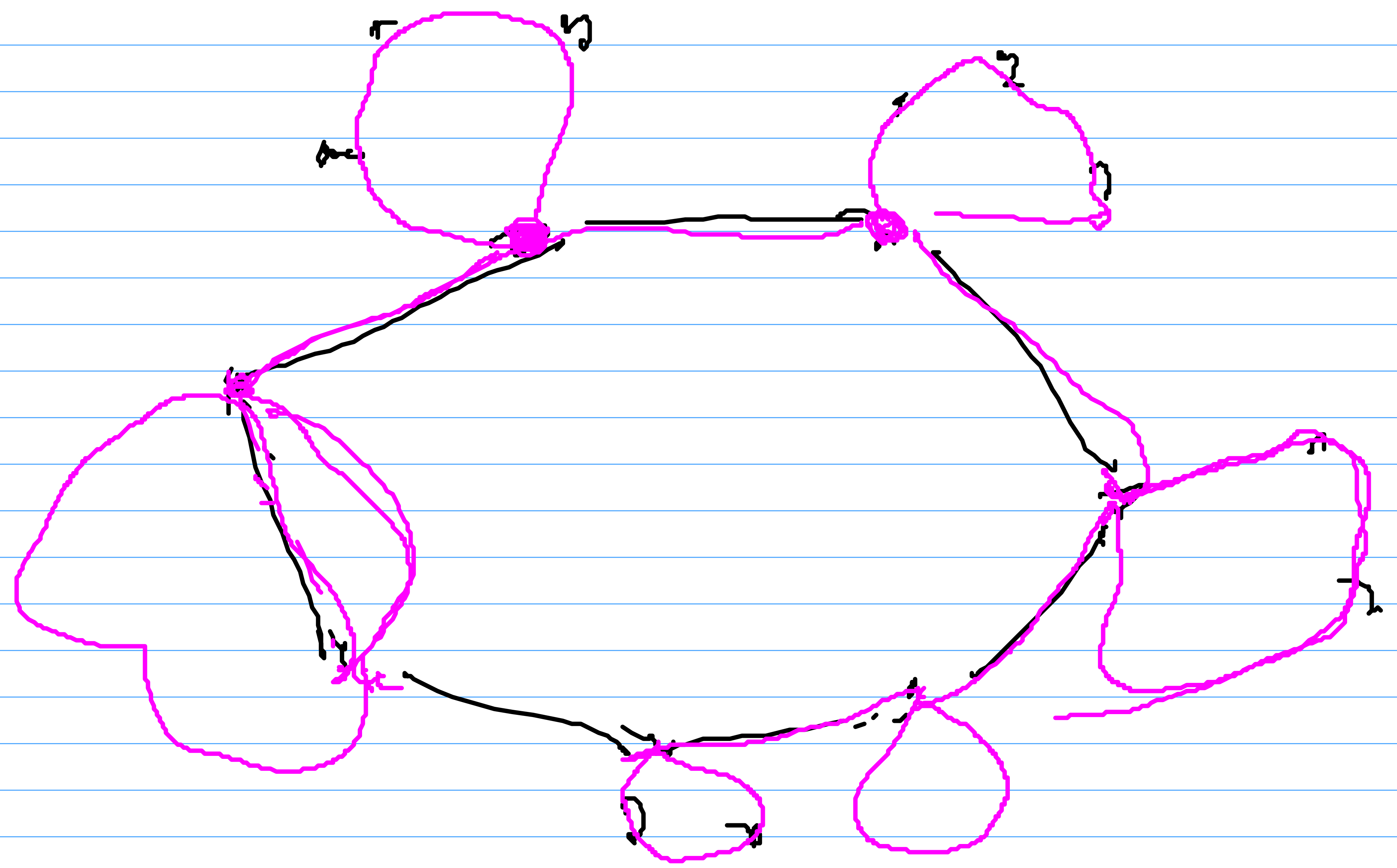
(v)

$v \in V$ is an EC.

Suppose the result true for $|E| \leq k$ we prove it for $|E| = k+1$

Lemma $\Longrightarrow$ $\exists$ non trivial circuit

$(v_0\, e_1\, v_1 \ldots e_n\, v_n)$



$G' = G \setminus \text{circuit} = (V, E \setminus \{\text{used edges}\})$

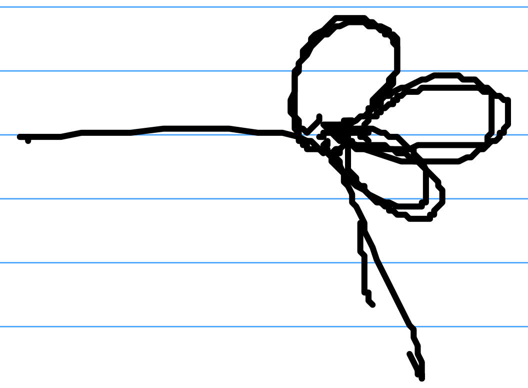
$\| \quad \bigcup_{i=1}^{m} G_i$ $\qquad G_i$ conn compon the G_i's are connected

$|G_i| \leq k$

has an Euler circuit.

You use the initial circuit to creat an euler circuit

$\Leftarrow$) ~~G to~~ Can assume G loop free no isolate vertices $(v_0 e_1 \dots e_n v_0)$



$\Rightarrow$ G is connecte

An euler circuit visits every edge but then it visit every vertices you can join any two vertices with a path

use this to conclude

if you go in you go out

$$\deg(v) = \left|\{ i \in \{1 \dots n\} \mid v \sim e_i \}\right| = \underline{2} \{ i \in \{1 \dots n\} \mid v = v_i \}$$

Works also for loops

$\deg(v)$ is even.

initial & last one: can have odd deg.

Planar Graphs

$G=(V,E)$ graph. Its geometric realization is a metric space $|G|$

underlying set is

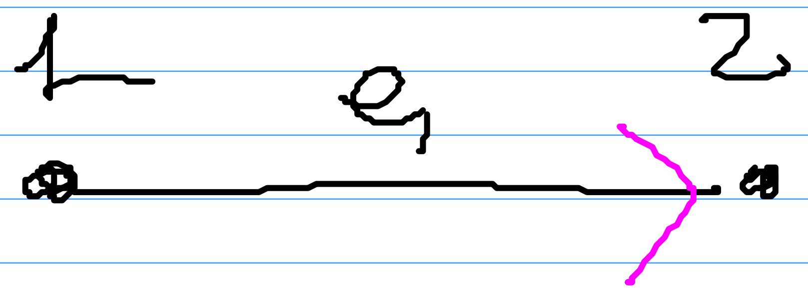
$$V \sqcup (E\times[0,1]) \Big/ \begin{array}{l}(e,0)\sim s(e)\\ (e,1)\sim r(e)\end{array}$$

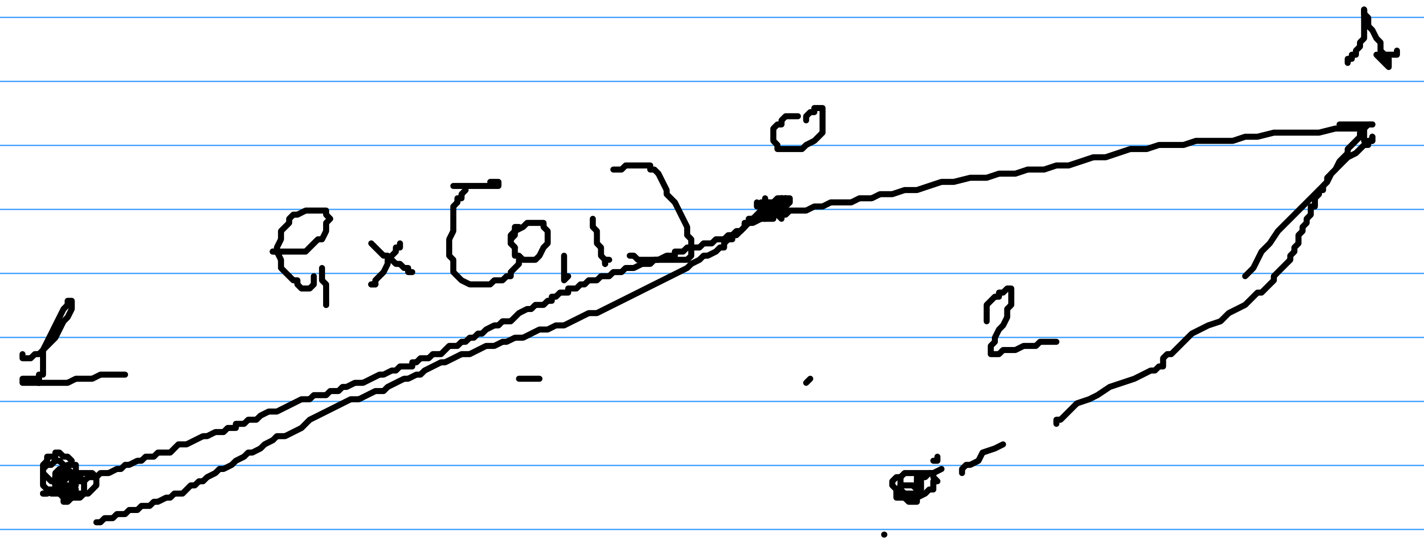
$(V,\widetilde{E})$ oriented graph that is G if we forget the orientation

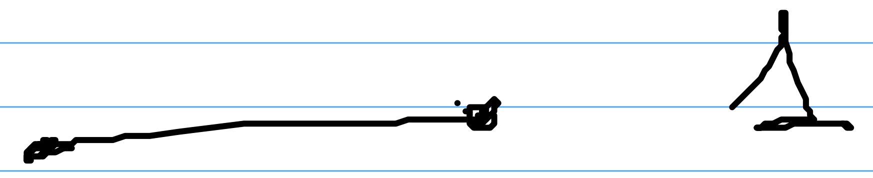
the distance d is the path distance.

Example

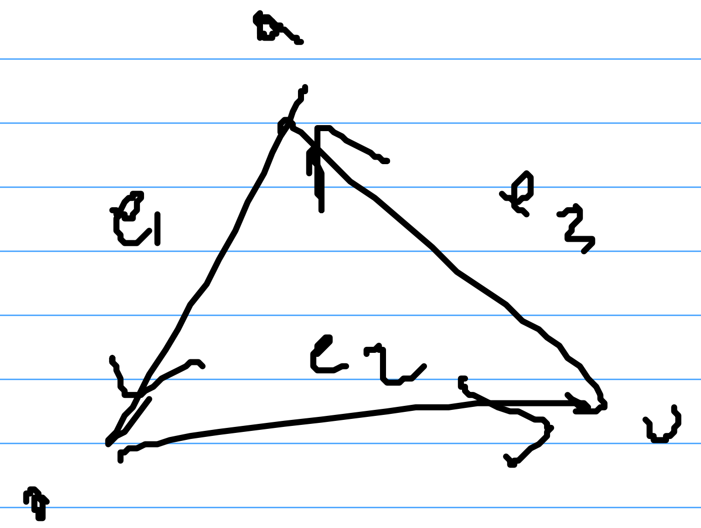
K_2

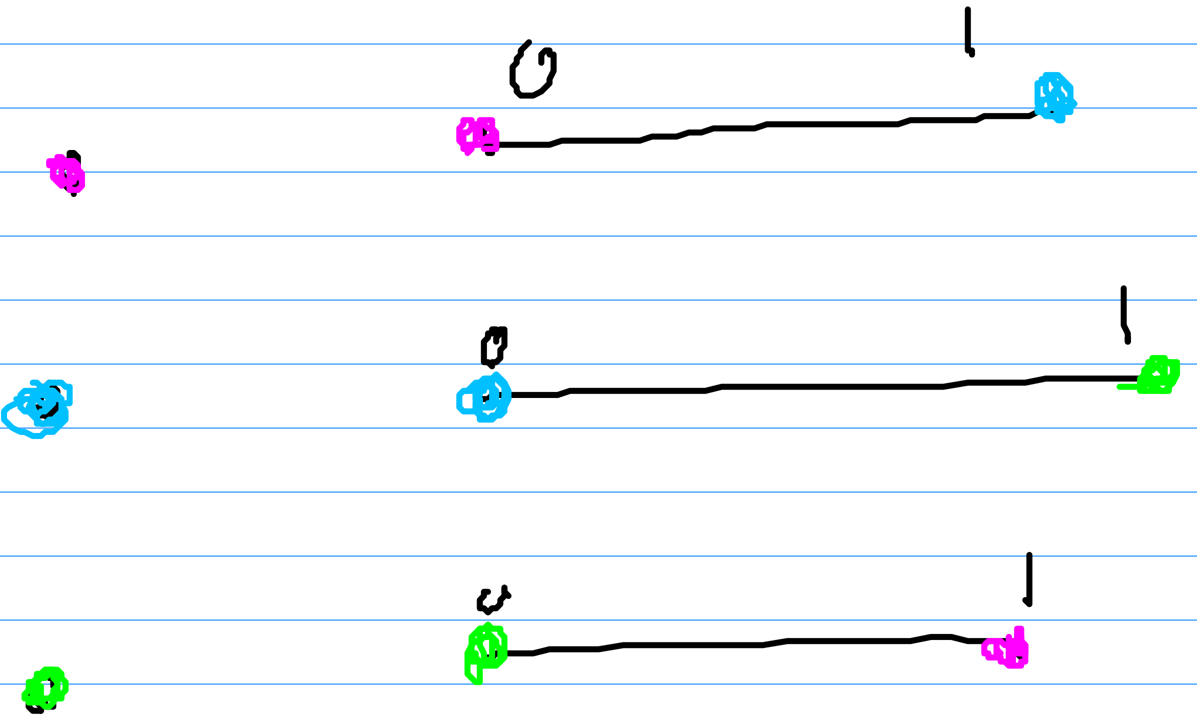


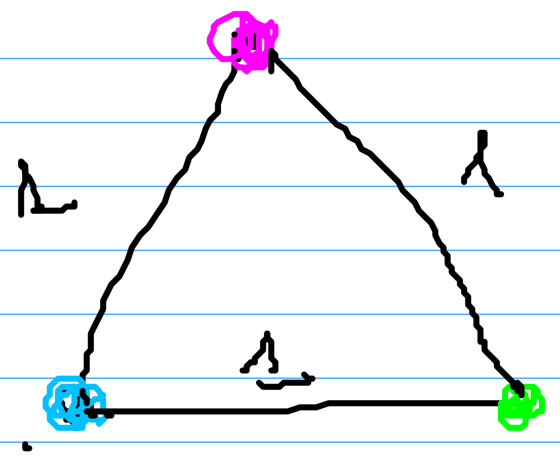




C_3







Geometric realization is the ~~one~~ graph that you draw.

- **Def** A graph $G=(V,E)$ is said to be planar if there is an injective & continuous map

$$i : |G| \hookrightarrow \mathbb{R}^2$$

metric space

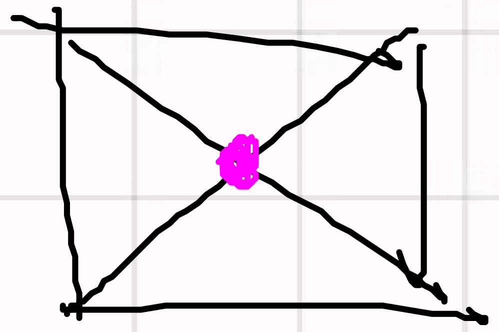
metric space

$$d(x,y) = \sqrt{x^2+y^2}$$

I can draw G in such a way that the only intersection of edges are on vertices

Q: Characterise planar graphs?

K_4



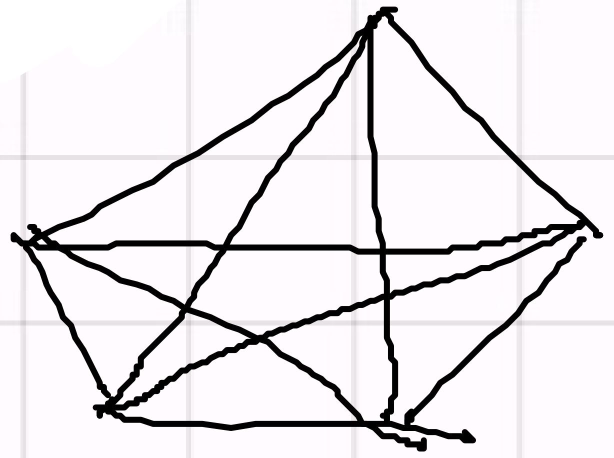
PLANAR

Bipartite graphs

A graph $G=(V E)$ is bipartite if $V=V_1 \sqcup V_2$ such that $v \sim w \iff v$ and w belongs to different sets of the partition

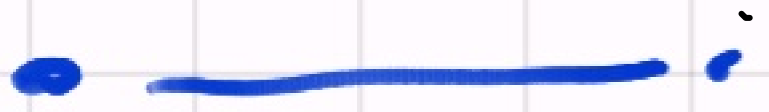
$K_{n,m}$ complete bipartite graph

K_5



not planar.

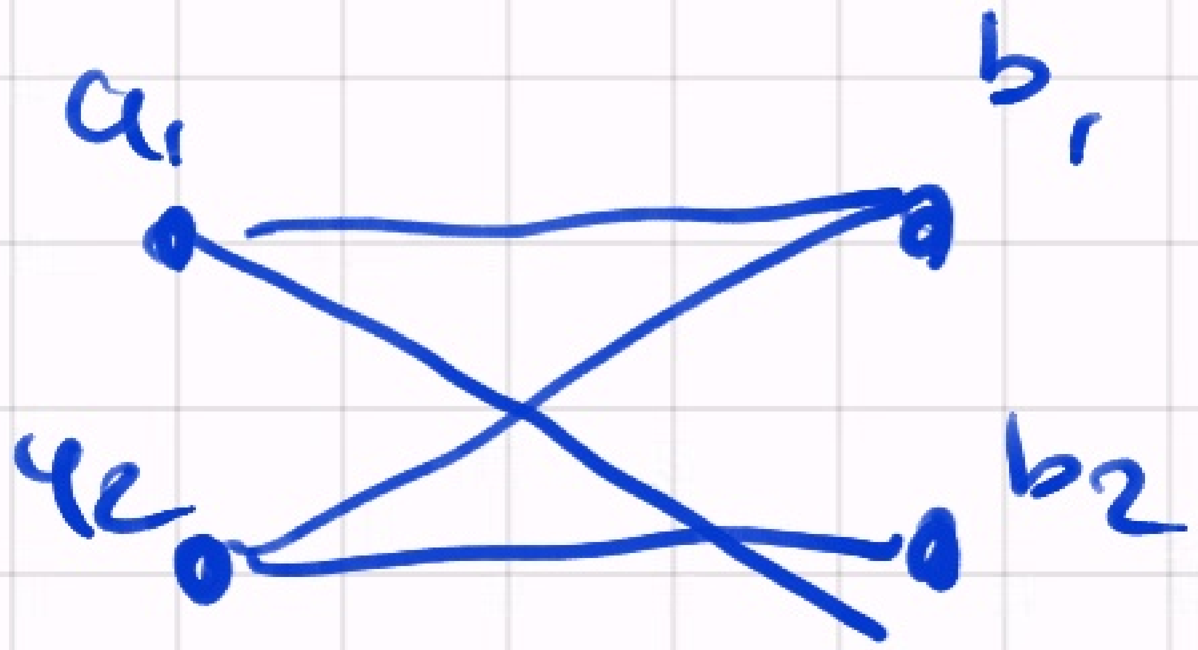
K_{11}

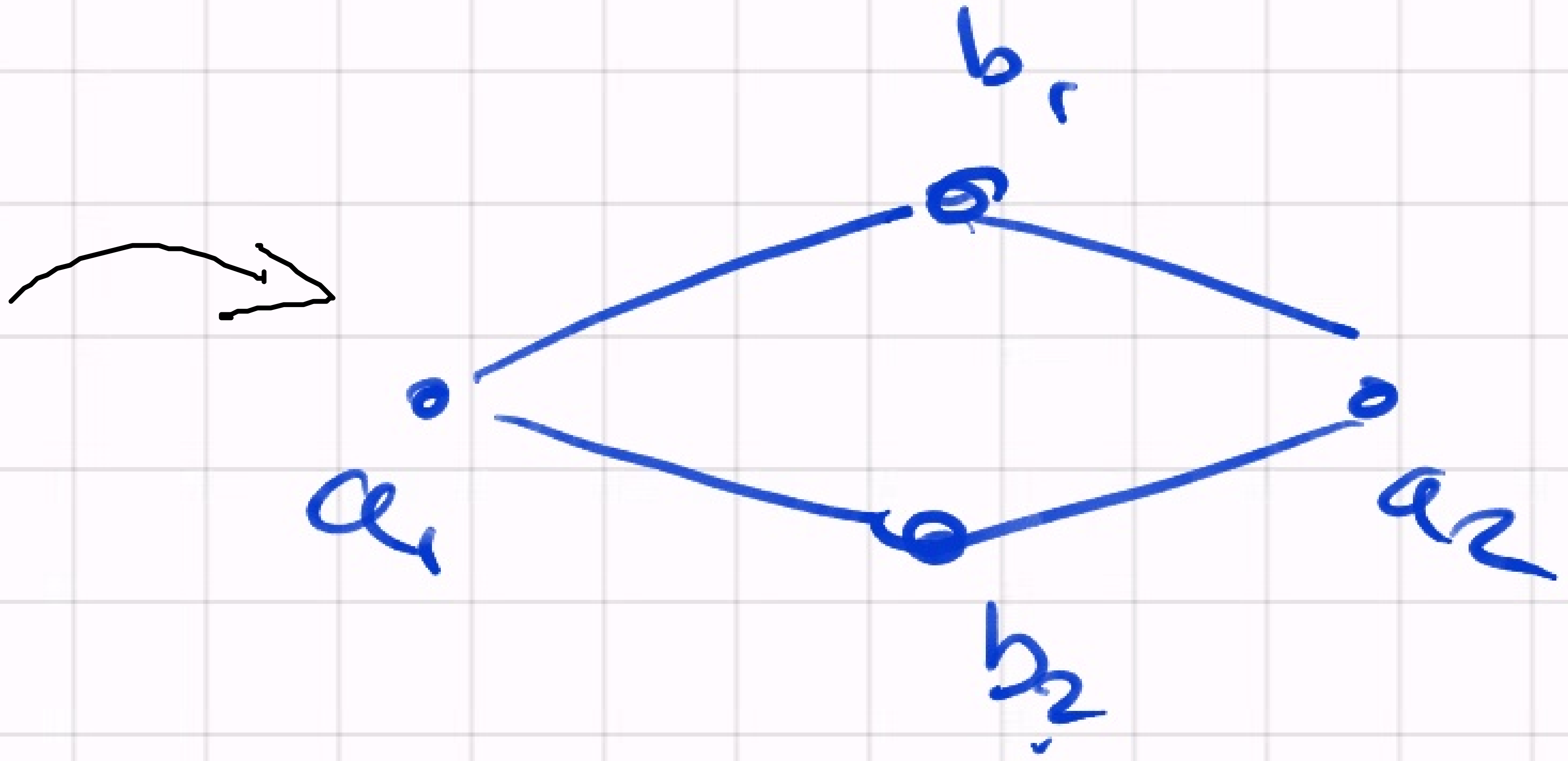


K_{12}

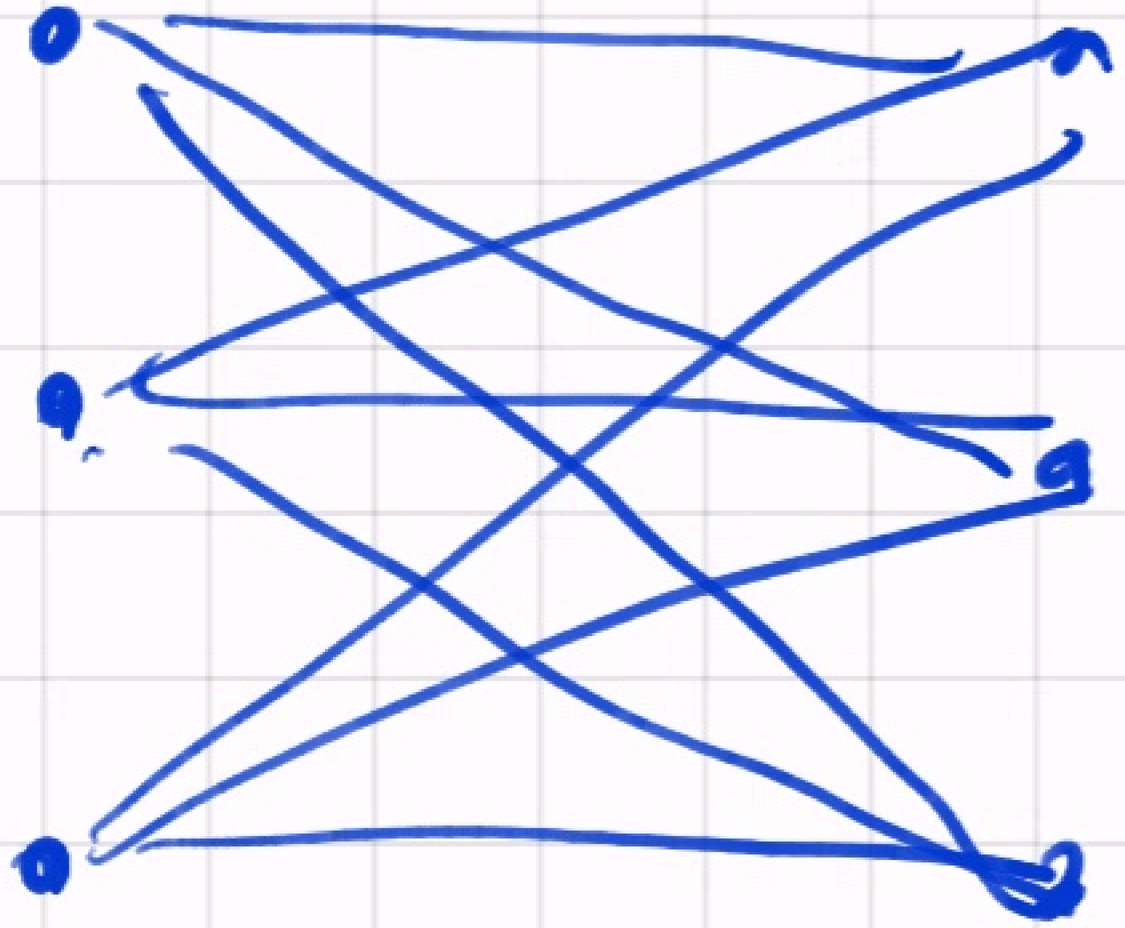
K_{13}

K_{22}





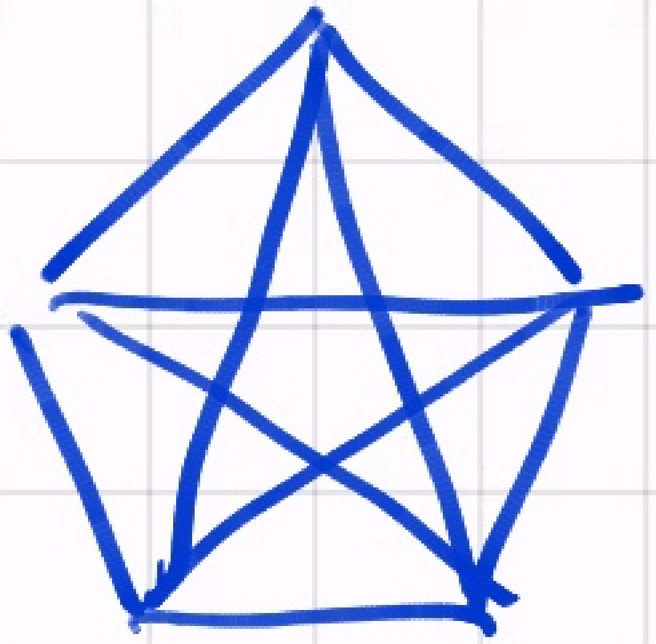
K_{33}



not plan

Step 1

We want to show that K_5 and $K_{3,3}$ are not planar.



Kuratowski Theorem.

We say that two graphs G_1 & G_2 are homeomorphic

$\Longleftrightarrow \quad \exists \ \varphi : |G_1| \longrightarrow |G_2|$ continuous bijective & φ^{-1} continuous

Combinatorial way to see it

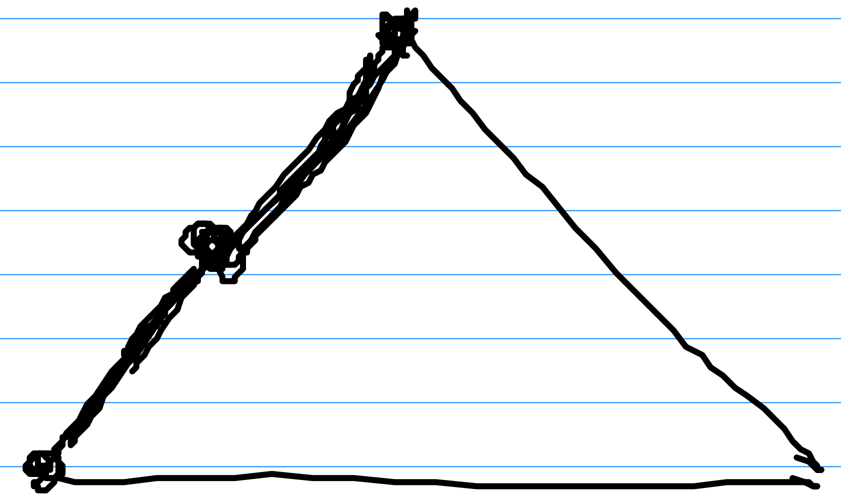
Def $G=(V,E)$ graph $E \neq \emptyset$ an elementary subdivision is an operation as follow

① remove $\{v,w\} \in E$

② add $u \in V$

③ add $\{v,u\}$ and $\{u,w\}$ to E

Example



split $\{v,w\}$ in two

Thm Two graphs are homeomorphic $\iff$ they are isomorphic after elementary subdivision

Proof is rooted in topology

Graphs are 1- simplexes

Geometric realization.

Thm A Graph is not planar iff it contains a subgraph homeomorphic to either K_5 or $K_{3,3}$.

$(\Leftarrow)$

We are going to show that K_5 & $K_{3,3}$ are not planar.

Theorem

Let $G=(V,E)$ a $\boxed{\text{connected}}$ planar graph with $|E|=e$; $|V|=v$ and denote by

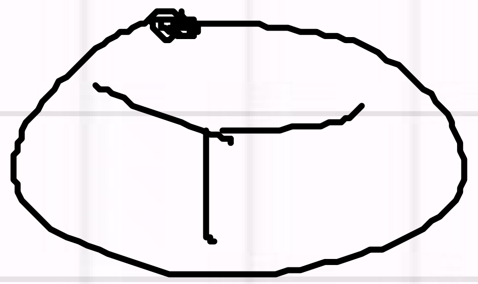
$$i : |G| \hookrightarrow \mathbb{R}^2 \quad \text{an immersion}$$

giving the planarity. Let r be the # of regions in $\mathbb{R}^2$ delimited by the edges of G. That is $r = \#$ connected cp of $\mathbb{R}^2 \setminus |G|$. Then

$$\boxed{v - e + r = 2}$$

↳ formula for Euler χ

Graph ⟺ triangulization of a sphere
of the spher



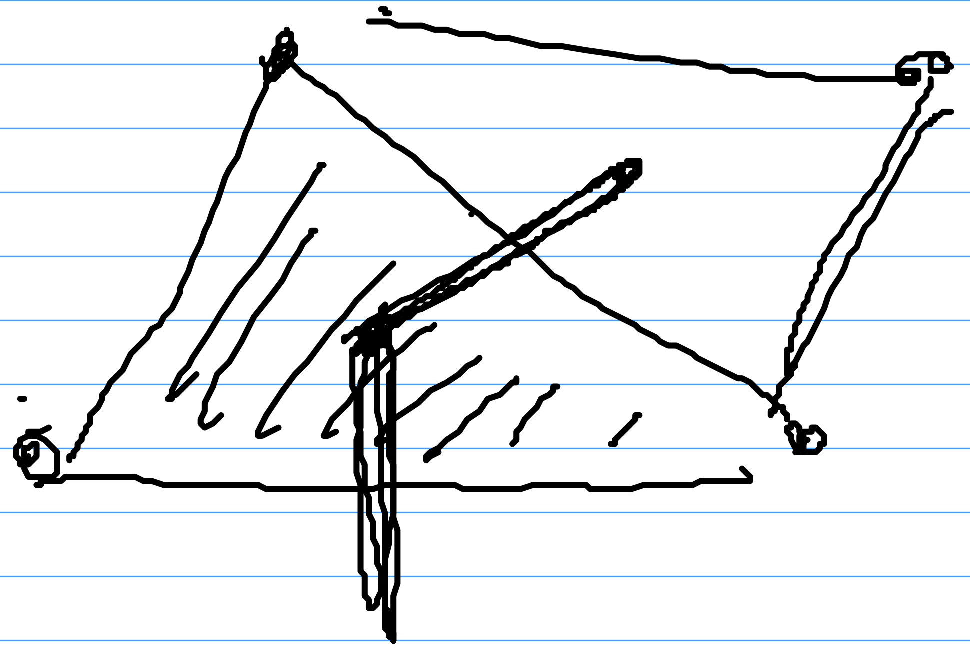
Cor 4 $G=(V,E)$ loop free simple planar

$3r \le 2e$ $e \le 3v-6$

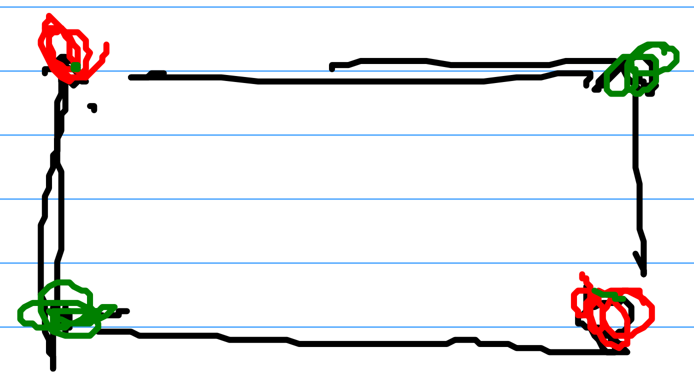
furthermore if G is bipartite

$e \le 2v-4$

Proof: Loop free + simple you need at least 3 edges to bound every region.



(for bipartite you need 4



)

$2e$ = sum of the degree of every regio

$\geq 3r$

$2 = v - e + r \leq v - \frac{1}{3}e$

$6 \leq 3v - e$

$4r \geq 2e$

$\Downarrow$

$e \leq 2v - 4$

Example

K_5 is not planar.

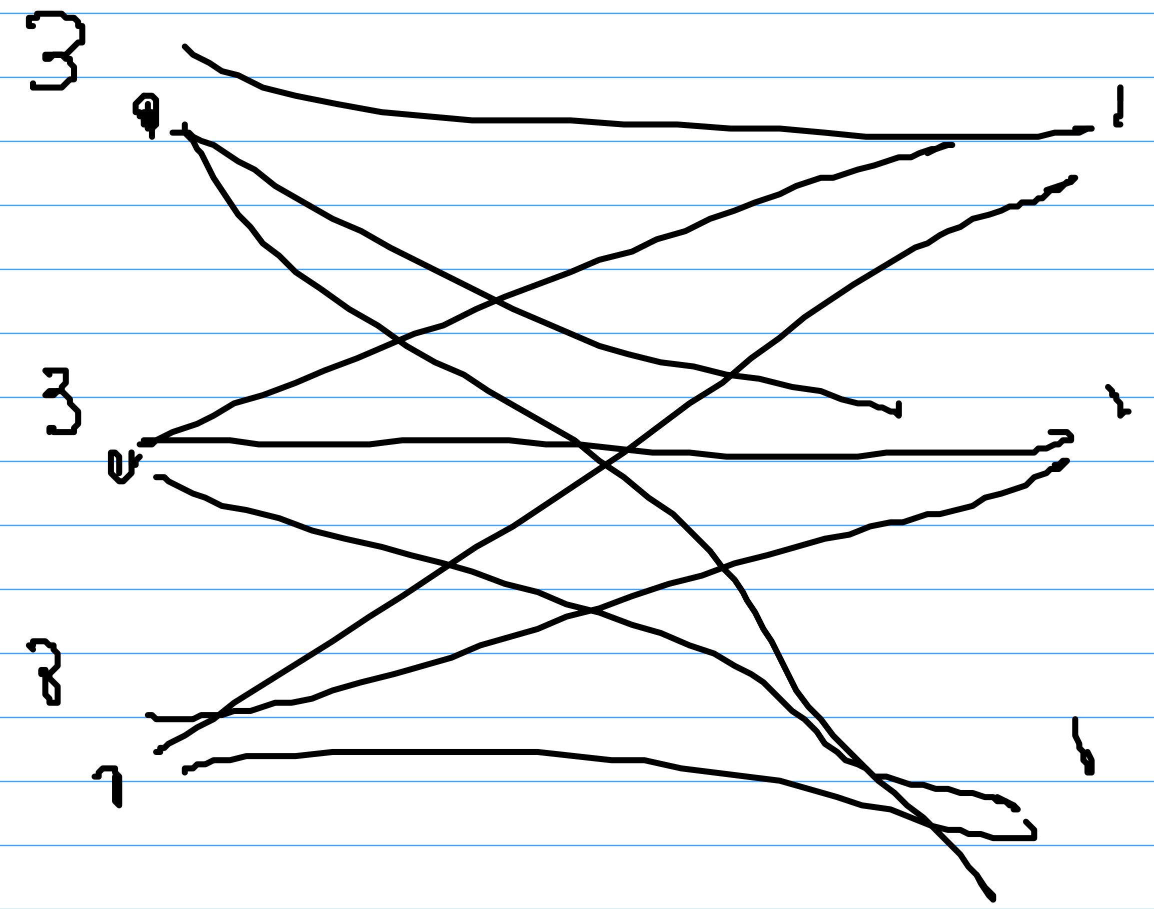
$|V| = 5$

$|E| = \frac{1}{2} 5(4) = 10$

$3 \cdot 5 - 6 = 9 < 10$ Not planar

Examples :

$K_{3,3}$ not planar.



$|V| = 6$

$|E| = 9$

$2 \cdot 6 - 4 = 8 < 9$

Observe : this "numerical" considerations go just one way.

If the numeric fit that of a plane you do not know that the graph is indeed planar.

Corollary

If $G=(VE)$ is loopfree, planar, simple then $3r \leq 2e$ & $e \leq 3v-6$. If furthermore G is bipartite then $e \leq 2v-4$

Examples

Graph homeomorphism

Two graphs G_1 and G_2 are homeomorphic if there is

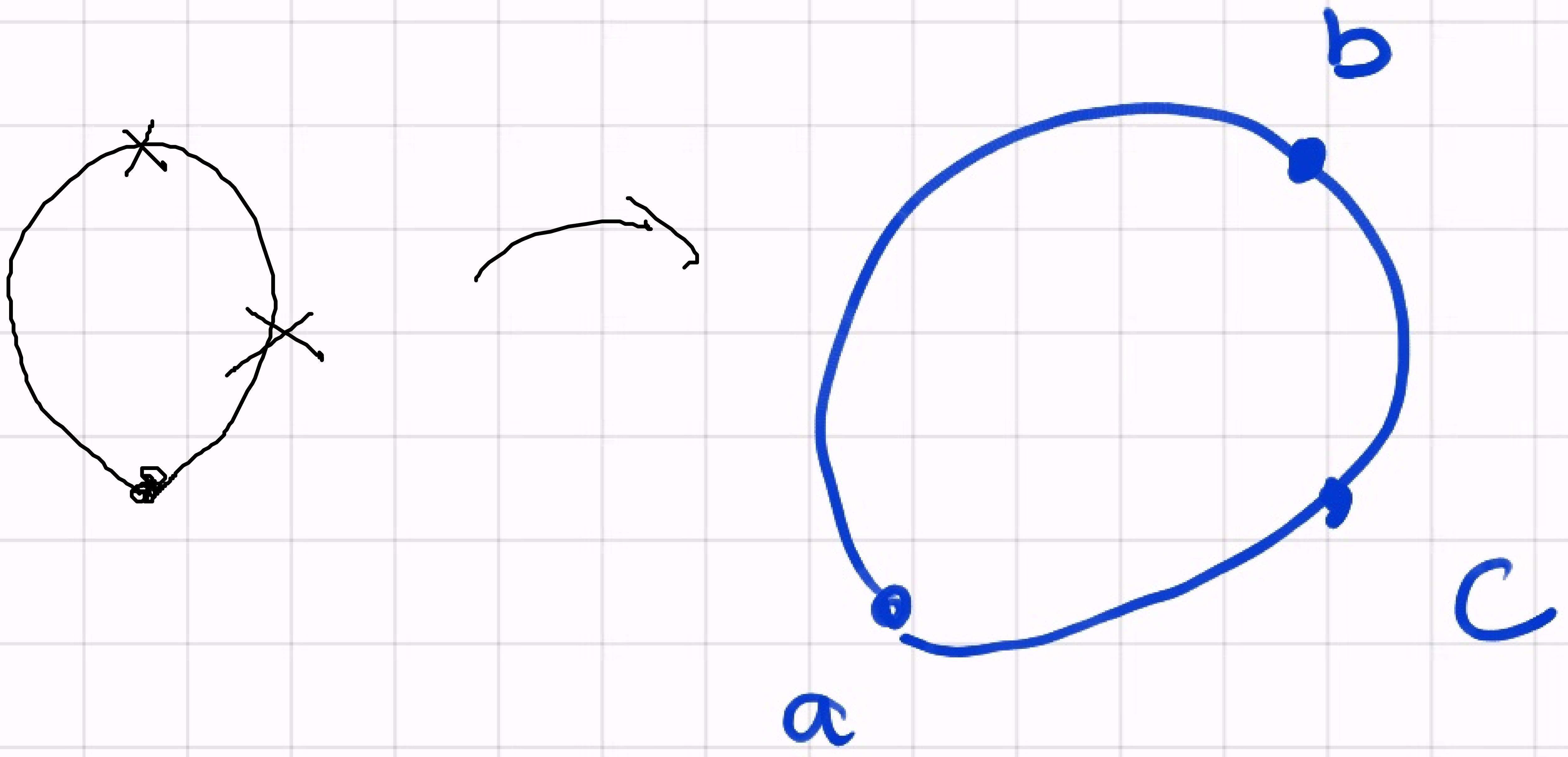
$$\varphi : |G_1| \longrightarrow |G_2|$$

continuous, bijective with continuous inverse.

Planarity depends only on the homeomorphism class

Example

Any graph is homeomorphic to a loopfree graph.

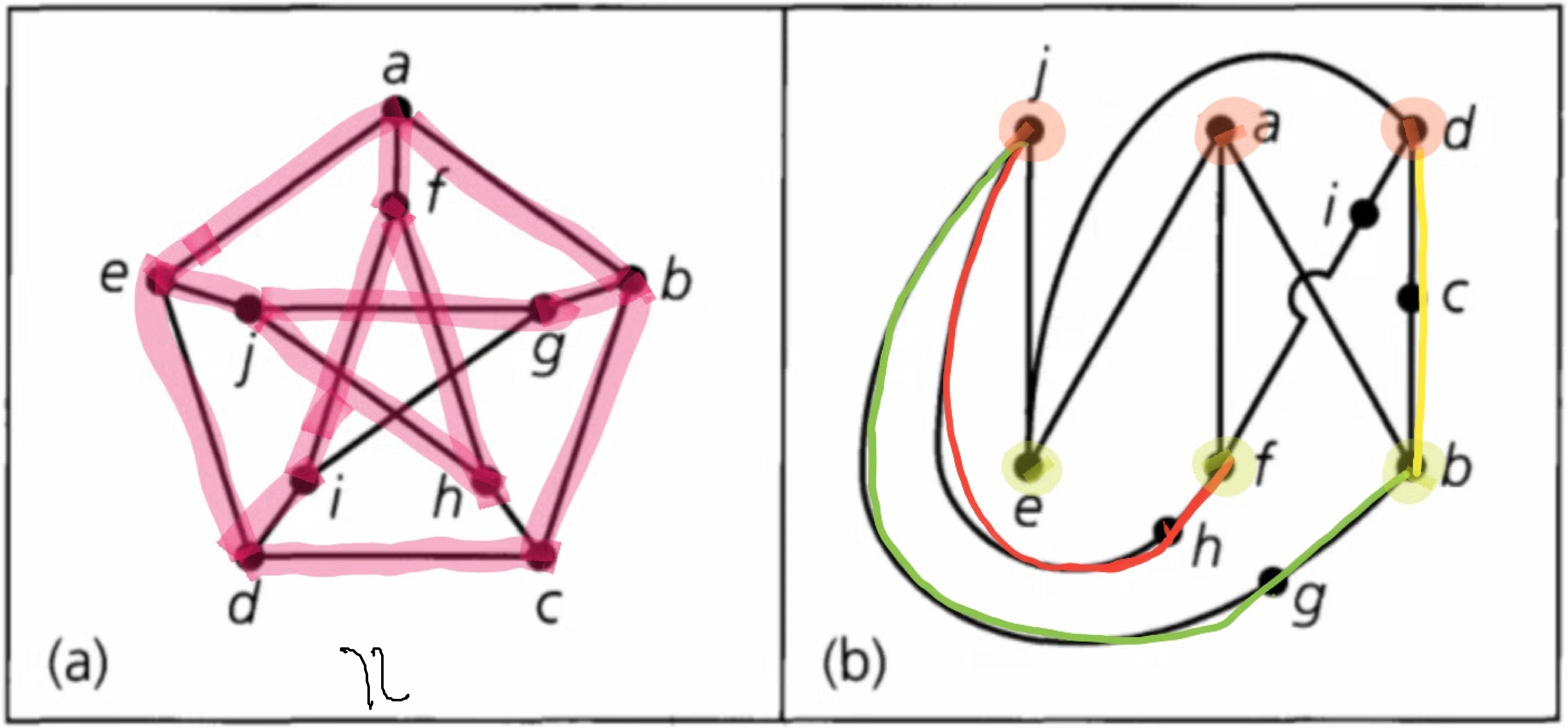


Kuratowsky theorem

A graph $G=(V,E)$ is not planar iff it contains a subgraph homeomorphic to either

K_5 or $K_{3,3}$

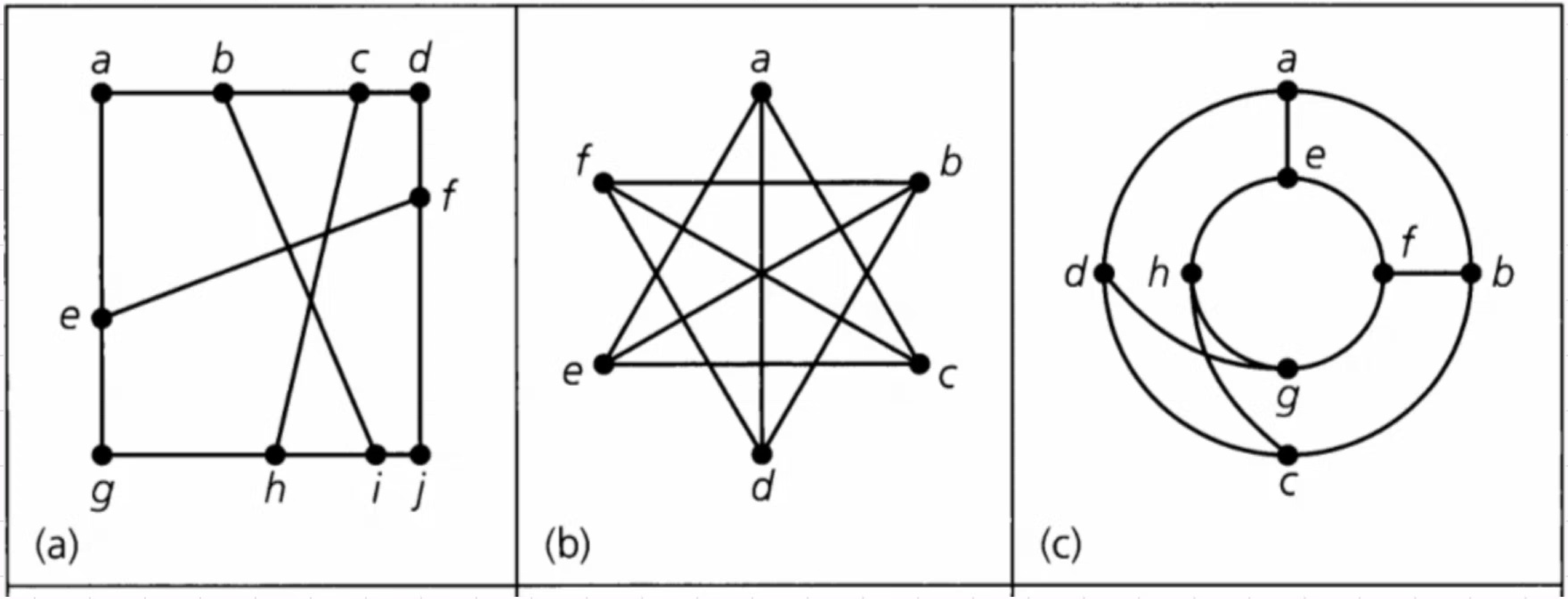
Example Petersen Graph. not plana



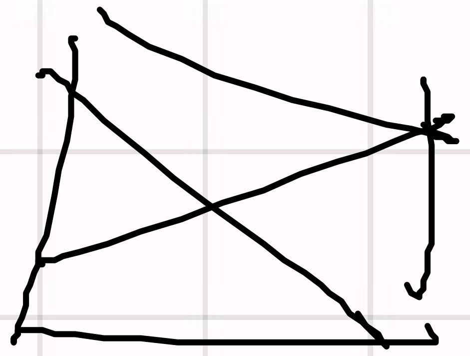
$K_{3,3}$

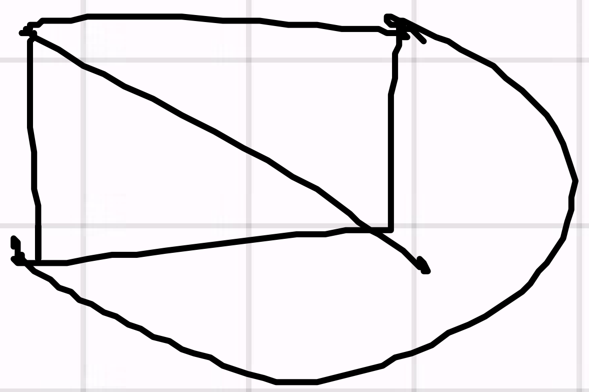
⊓

K_5



Check the numeric 6 3regular





$\frac{18}{2}$ edges 9 try.