11.3 **Exercise 1:**

Determine $|V|$ for the following graphs or multigraphs G:

a) G has nine edges and all vertices have degree 3.

b) G is regular with 15 edges

c) G has 10 edges with two vertices of degree 4 and all others of degree 3.

Solution:

$$2 \cdot |E| = \sum_{v \in V} \deg(v)$$

a) $2 \cdot |E| = 18 = \sum_{v} \deg(v) = 3 \cdot |V|$

$$\rightsquigarrow \boxed{|V| = 6}$$

b) Since G is regular, every vertex has degree $|V|-1$, so $2 \cdot |E| = 30$

$$= |V| \cdot (|V|-1) \rightsquigarrow \boxed{|V| = 6}$$

c) $2 \cdot |E| = 20 = 2 \cdot 4 + (|V|-2) \cdot 3$

$$(|V|-2) \cdot 3 = 12 \rightsquigarrow \boxed{|V| = 6}.$$

Exercise 8:

a) Find the number of edges in Q_8

b) Find the maximum distance between pairs of vertices

c) Find the length of a longest path in Q_8.

Solution:

a) Q_8 has 2^8 vertices, and each vertex has degree 8. Therefore, $2 \cdot |E| = 8 \cdot 2^8$

$$= 2048 \rightsquigarrow \boxed{|E| = 1024}$$

b) The maximal distance is 8. The vertices are labeled by binary strings of length 8, and two vertices are adjacent if and only if they differ in exactly one position. Therefore any pair of vertices is at distance at most 8, since we can always choose a path in which every step changes one of the unequal positions.

Moreover, the vertices 00000000 and 11111111 are at distance 8 apart, since every edge flips only one position at a time, so we need at least 8 edges in a path between them.

c) A path uses each vertex at most once, so it has length at most $|V| - 1 = 255$.

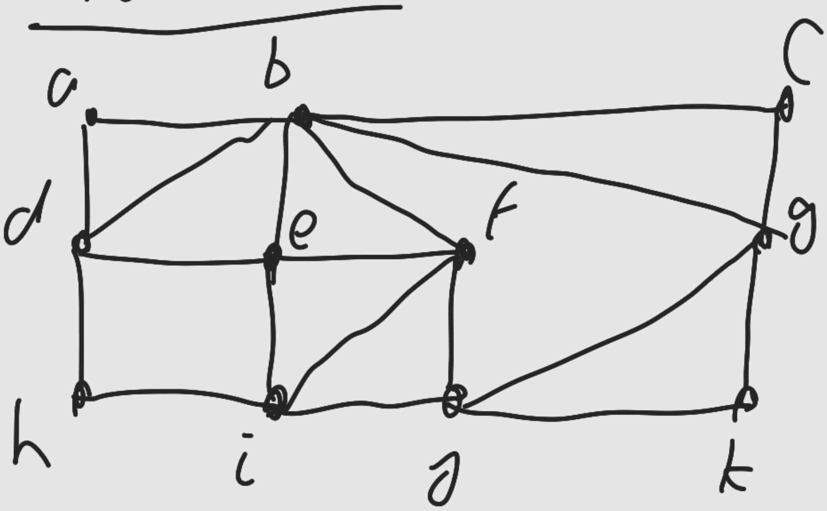
There actually exists a Hamiltonian path in Q_8. We prove that by induction:

In Q_1 =  we have a Hamiltonian path clearly.

If there is a Hamiltonian path in Q_n, then Q_{n+1} is obtained by having two copies of Q_n and connecting the corresponding vertices, so we find a Hamiltonian path in Q_{n+1} by first following the Hamiltonian path in the first copy, then moving to the corresponding vertex in the second copy, and from there following the (same) Hamiltonian path in reverse in the second copy. Therefore by induction there is a Hamiltonian path in Q_8.

(Note that we can complete the Hamiltonian path obtained into a Hamiltonian circuit)

<u>Exercise 20</u>:

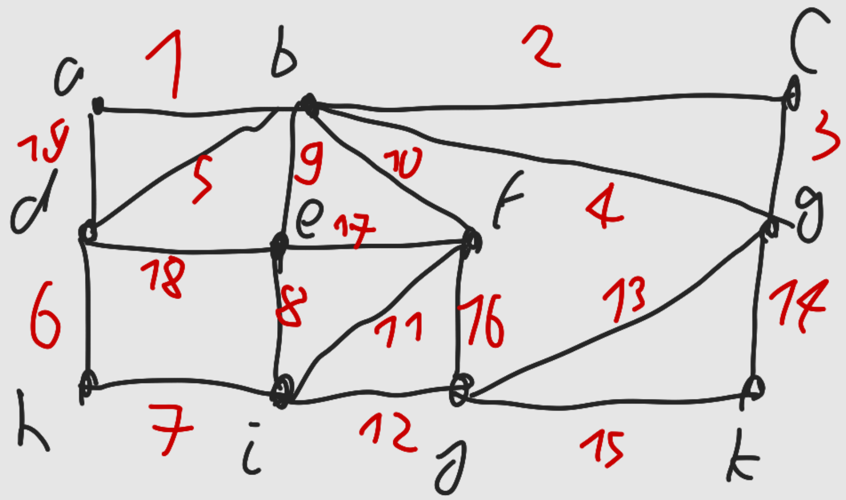


a) Find a Euler circuit

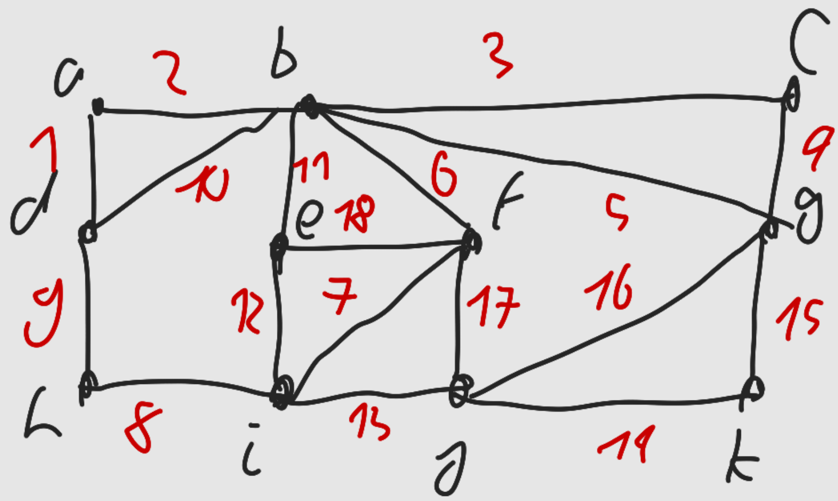
b) If the edge $\{d,e\}$ is removed, find an Euler trail.

<u>Solution</u>:

a) We start from a:

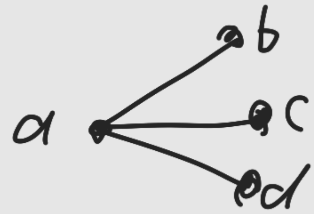


b) We know that we will have to start at d and end at e (or conversely):



11.4 Exercise 6:

Let $n \in \mathbb{Z}^+$ with $n \geq 4$. How many subgraphs of K_n are isomorphic to $K_{1,3}$?

Solution: $K_{1,3} =$ 

A $K_{1,3}$ subgraph is obtained by first picking the vertex a, and then choosing its three neighbors b, c, d. In K_n all vertices are neighbours of each other, so we have n possibilities to choose a and then $\binom{n-1}{3}$ for picking the remaining 3 vertices. In total this gives

$$n \cdot \binom{n-1}{3}$$

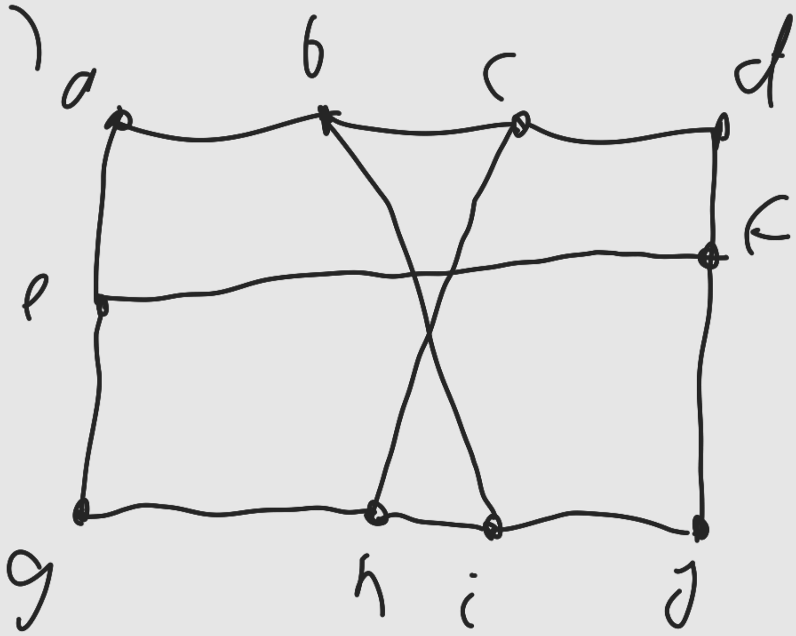
subgraphs isomorphic to $K_{1,3}$.

Exercise 14:

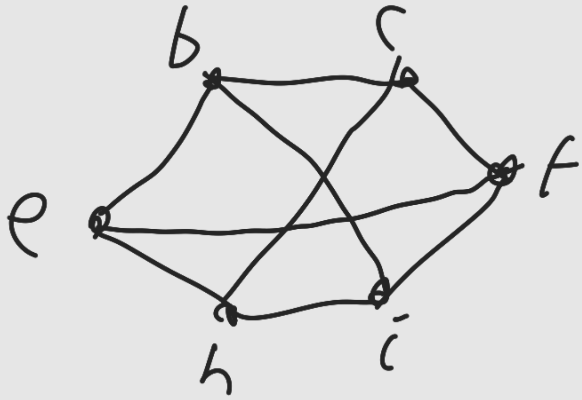
Determine which of the following graphs are planar. If a graph is planar redraw it with no edges overlapping. If it is nonplanar, find an embedded K_5 or $K_{3,3}$.

Solution:

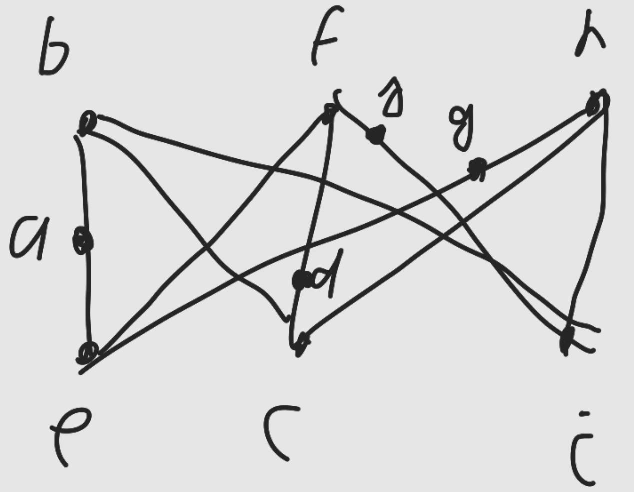
a)



the bivalent vertices do not matter, so forgetting gives us

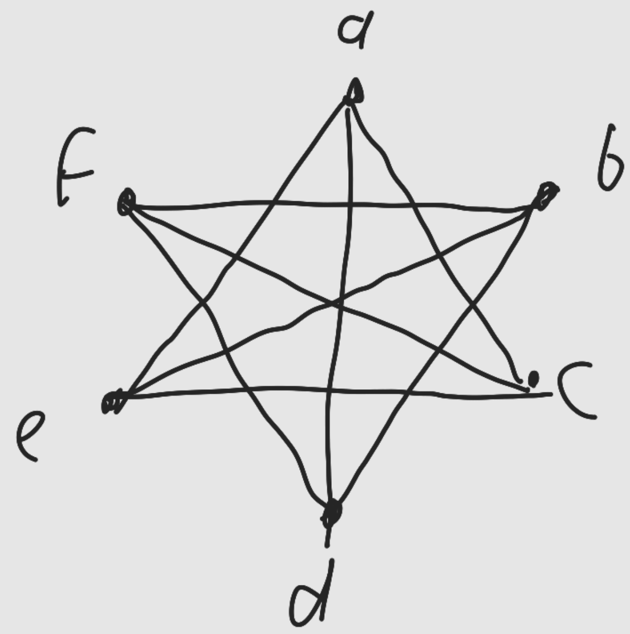


reordering the vertices gives us a $K_{3,3}$:



Non planar

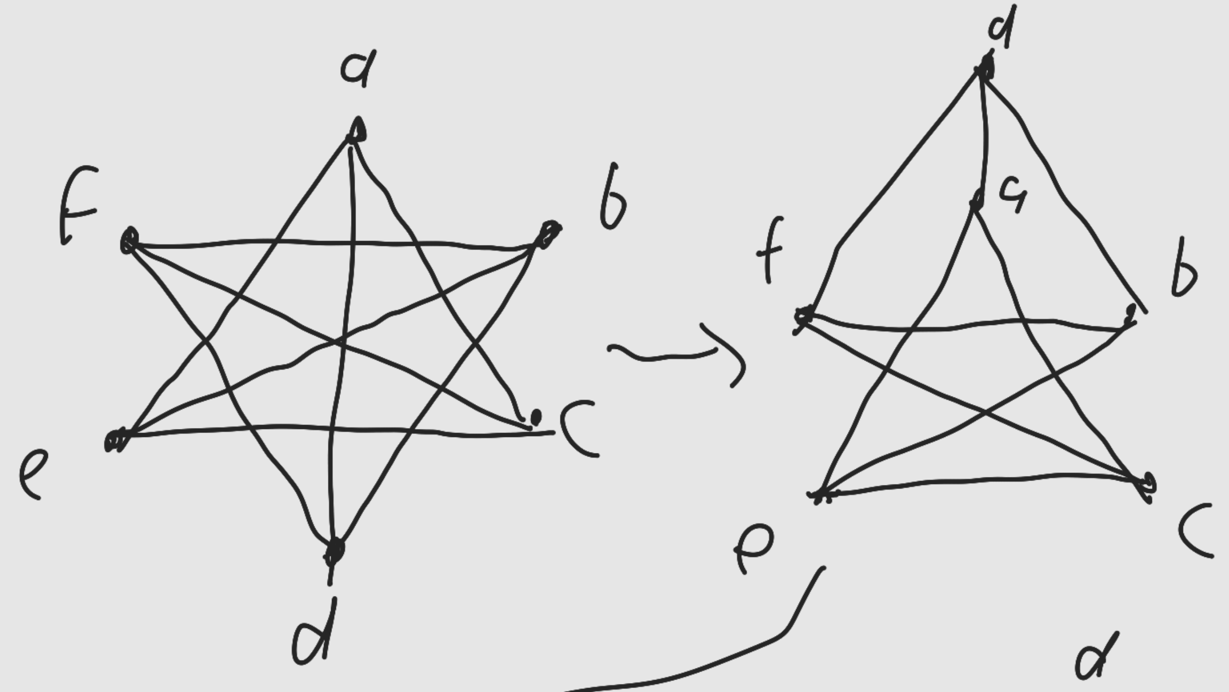
b)

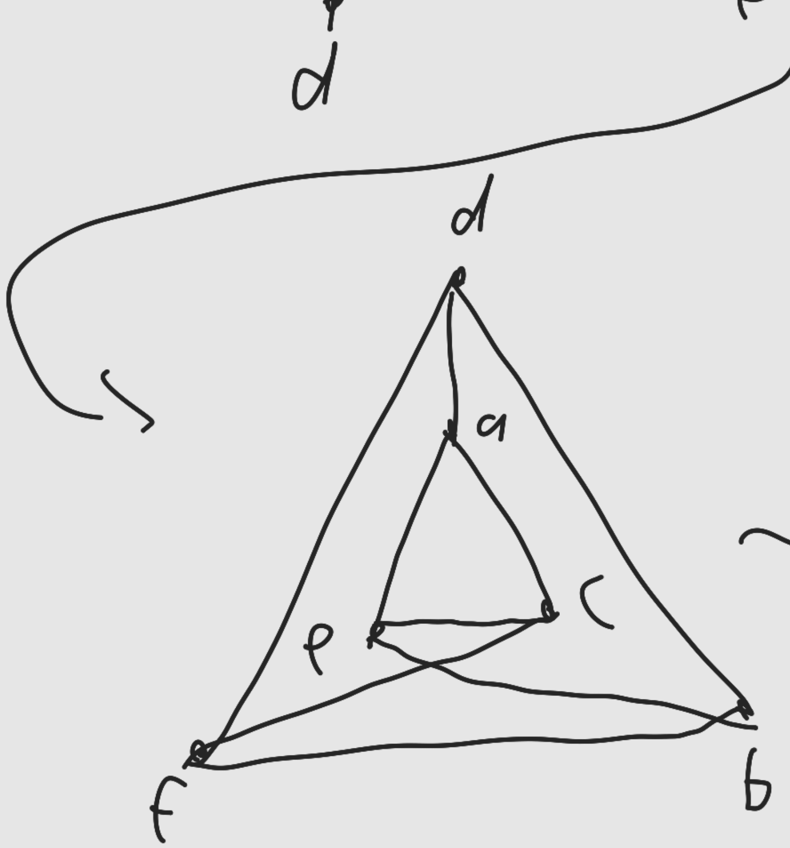


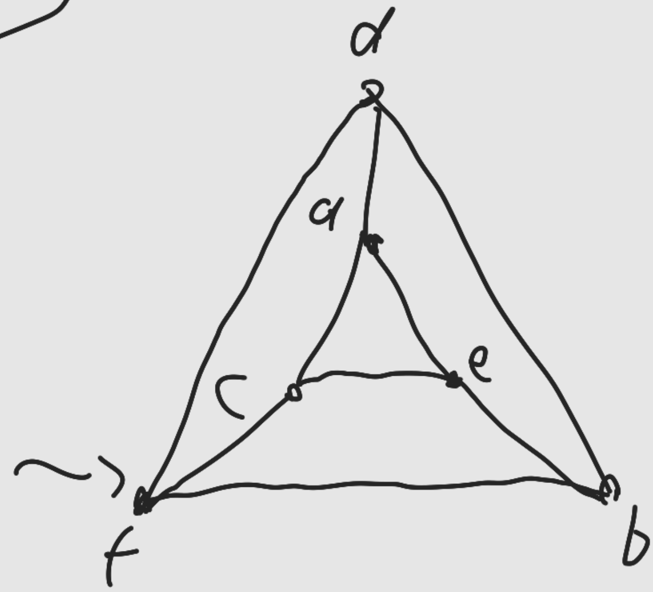
We see that the vertices have degree 3, so there cannot be an embedded K_5.

Also we have 6 vertices and 9 edges, so the only possible embedded $K_{3,3}$ would be the whole graph. However the graph is not bipartite since we have a cycle of odd length $a-c-e-a$.

So the graph should be planar. We will deform the graph until it is planar:

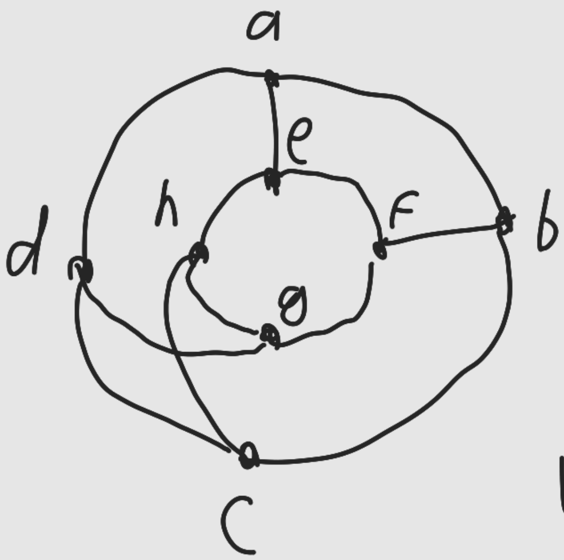






Planar

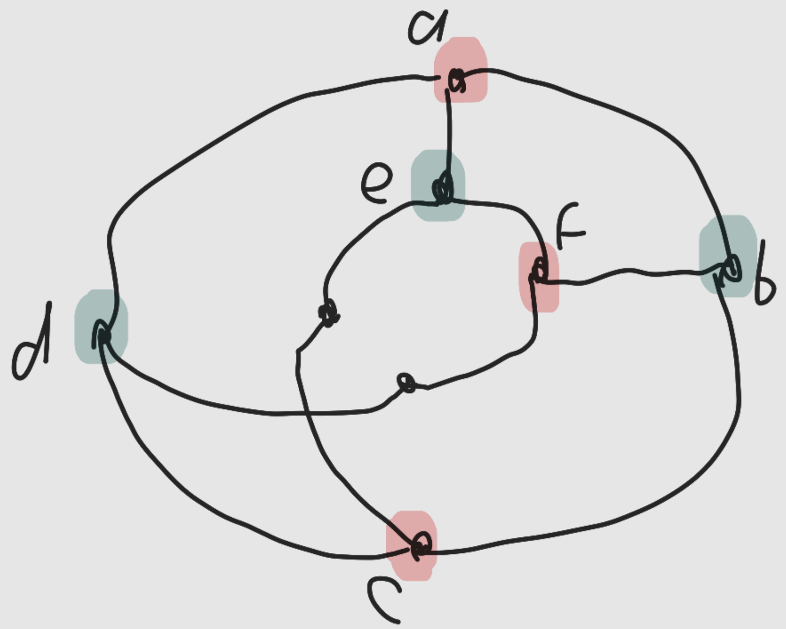
c)



All vertices have degree 3, so there cannot be an embedded K_5.

We try to find an embedded $K_{3,3}$:

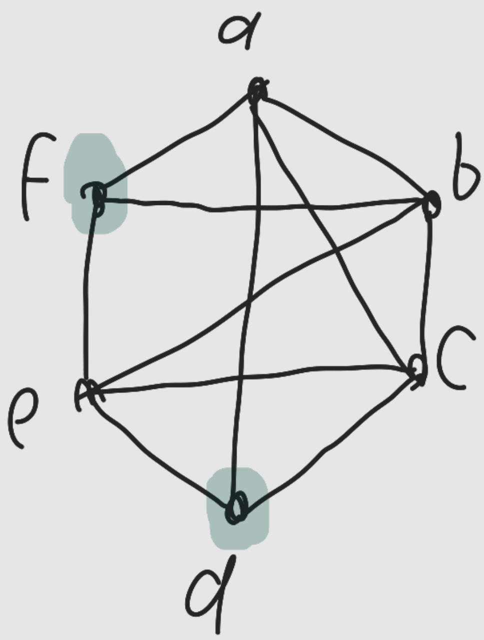
Start with a, it has three neighbors b, d, e. We try to find two other vertices to complete the $K_{3,3}$. We can pick f and c. So we forget g and h:



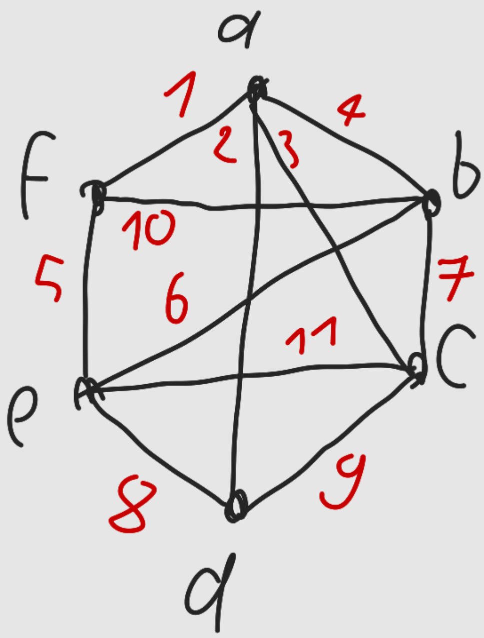
is an embedded $K_{3,3}$

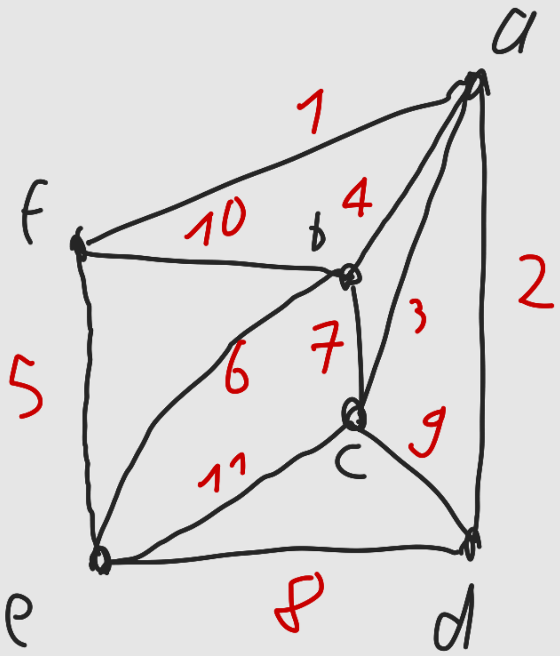
<u>Non-planar</u>

d)



Only four edges have degree 4, so there cannot be an embedded K_5. An embedded $K_{3,3}$ should use up all the vertices. In particular the vertex d should belong to one side and its three neighbors a, c, e should belong to the other side. But then f should also neighbor with a, c, e, but it is not adjacent to c so there is no embedded $K_{3,3}$ and the graph is planar.





Planar

e) is planar

f) has an embedded K_5 so is non planar