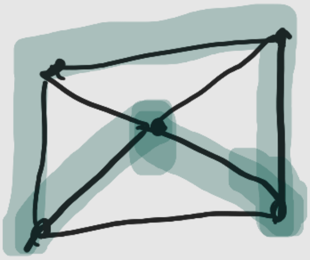
11.5 Exercise 6:

If $n \geq 3$, how many different Hamilton cycles are there in the wheel graph W_n?

$W_4 =$ 

Solution: We start and end at the middle vertex. Then a cycle is determined by the next vertex in the cycle and a direction. There are n choices for the next vertex in the cycle, and from there there are two choices of direction. So in total there are $2n$ cycles.

If we consider that going along a cycle in the reverse direction produces the same cycle, then there are only $\boxed{\frac{2n}{2} = n \text{ cycles.}}$

Exercise 8:

a) For $n \in \mathbb{Z}^+, n \geq 2$ show that the number of distinct Hamilton cycles in the graph $K_{n,n}$ is $\frac{1}{2}(n-1)!\,n!$

b) How many different Hamilton paths in $K_{n,n}$, $n \geq 1$?

Solution:

a) We choose a first vertex, and then has to alternate between the two sides of the graph (since the graph is bipartite).

Moreover the graph is complete bipartite, so from any vertex we can reach every vertex on the other side.

Therefore a cycle is uniquely determined by the order in which the vertices are visited. On the side we start with any of the $(n-1)!$ permutations are possible, and on the other side each of the $n!$ permutations are possible, so we have $\boxed{\frac{1}{2}(n-1)!\,n!}$ (we divide by 2 since going in the reverse direction gives the same cycle).

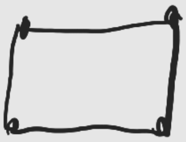
b) The start vertex can be any of the $2n$ vertices, and from there we make a path by any choice of alternating between the two sides. By the same argument as in the previous case we get

$$2n \cdot \frac{1}{2} \cdot (n-1)! \cdot n! = \boxed{(n!)^2} .$$

Exercise 12:

Prove that for $n \geq 2$, the hypercube Q_n has a Hamilton cycle.

Solution: The vertices are binary strings of length n.

$Q_2 =$  it has a Hamilton cycle.

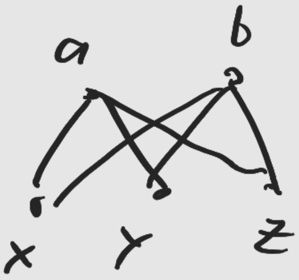
We prove by induction that if Q_n has a Hamilton cycle, so does Q_{n+1}:

Let v_1, v_{2^n} be two adjacent vertices in a Hamilton cycle in Q_n. By removing the edge between v_1, v_{2^n} we get a Hamilton path. Now Q_{n+1} is obtained by taking two copies of Q_n and joining by an edge the corresponding vertices.

Write $(0, v_i)$ and $(1, v_i)$ the vertices in each copy. Then a cycle in Q_{n+1} is obtained by first following the Hamilton path in the first copy of Q_n, then moving from

$(0, v_{2^n})$ to $(1, v_{2^n})$, following the path in the reverse direction in the second copy of Q_n, and returning from $(1, v_1)$ to $(0, v_1)$.

11.6 Exercise 6: $K_{2,3}$



a) Consider the graph $K_{2,3}$ and $\lambda \in \mathbb{Z}^+$ denote the number of available colors.

i) How many proper colorings of $K_{2,3}$ have vertices a, b colored the same?

ii) How many have vertices a, b colored differently.

b) What is the chromatic polynomial for $K_{2,3}$? What is $\chi(K_{2,3})$

c) For $n \in \mathbb{Z}^+$, what is the chromatic polynomial for $K_{2,n}$? What is $\chi(K_{2,n})$?

Solution:

a) i) If a and b are colored the same, then they must use one of the λ colors, and then each of the vertices x, y, z can use any of the $\lambda - 1$ remaining colors. So the number of proper colorings is $\boxed{\lambda \cdot (\lambda-1)^3}$.

ii) If a and b have different colors, then there are $\lambda(\lambda-1)$ choices for how to colour them, and then each of x, y, z can be colored with any of the $\lambda - 2$ remaining colors, so there are

$\boxed{\lambda(\lambda-1) \cdot (\lambda-2)^3}$ proper colorings

b) We notice that a proper coloring of $K_{2,3}$ either has a, b with the same color or with a different one, so $P(K_{2,3}; \lambda)$ is obtained by summing the two contributions from a).

$$P(K_{2,3};\lambda) = \lambda\cdot(\lambda-1)^3 + \lambda(\lambda-1)\cdot(\lambda-2)^3$$

$$= \boxed{\lambda(\lambda-1)\left((\lambda-1)^2+(\lambda-2)^3\right)}.$$

The smallest value of λ for with $P(K_{2,3};\lambda)\neq 0$ is $\lambda=2$: $P(K_{2,3};2)=$

$$2\cdot 1\left(1^2+0^3\right) = 2$$

$$\chi(K_{2,3}) = 2.$$

c) By the same argument, we get

$$P(K_{2,n};\lambda) = \boxed{\lambda\cdot(\lambda-1)^n + \lambda(\lambda-1)(\lambda-2)^n}$$

and the chromatic number $\boxed{\chi(K_{2,n}) = 2}$

Exercise 8:

If G is a loop-free undirected graph with at least one edge, then G is bipartite if and only if $\chi(G)=2$.

Solution:

$\Rightarrow$: If G is bipartite we can separate the vertices into two subsets A, B such that there is no edge between two vertices of A and no edge between two vertices of B. Moreover we know that there is at least one edge, so $\chi(G) > 1$, and there is a 2-coloring by coloring each vertex of A in one color and each vertex of B in the other color.

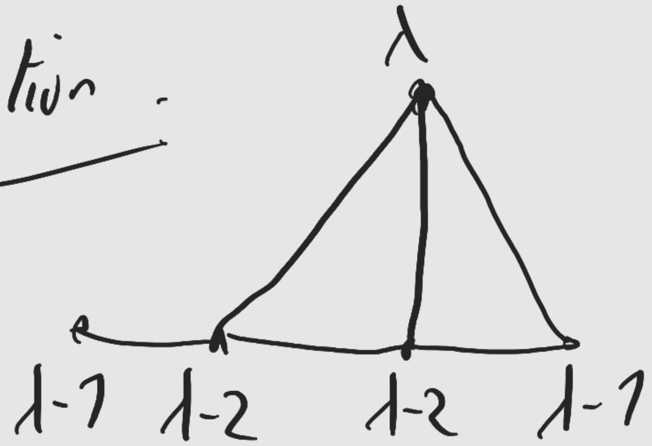
$\Leftarrow$: If $\chi(G) = 2$, any 2-coloring produces a bipartition of the graph.

Exercise 9:

Determine $P(G, \lambda)$; $\chi(G)$ and $P(G, 5)$ for the following graphs G:

Solution.

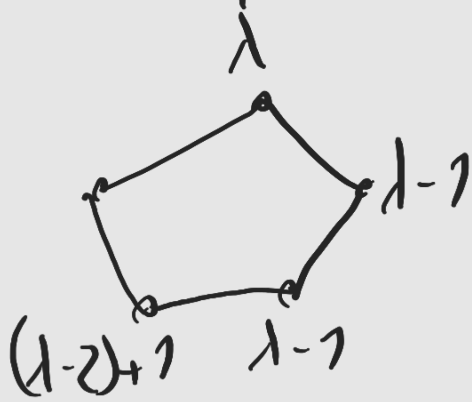
(a)



$\rightsquigarrow P(G,\lambda) = \lambda(\lambda-1)^2(\lambda-2)^2$

$\chi(G) = 3 \qquad P(G,5) = 5\cdot 4^2 \cdot 3^2 = 720$

(b)



We have λ possibilities for the first vertex, and $\lambda-1$ for each of the following two. Then if the next one has the same color as the first one there are $\lambda-1$ choices for the last vertex. Otherwise there are $\lambda-2$ choices. This gives

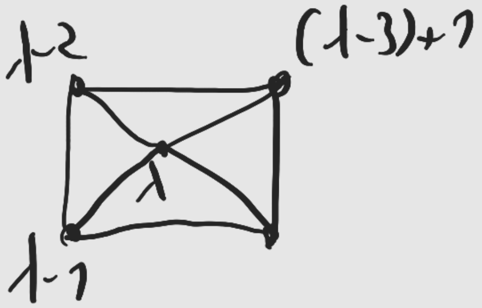
for each of the remaining $\lambda-2$ colors

$$P(G;\lambda) = \lambda(\lambda-1)^2\left[1\cdot(\lambda-1) + (\lambda-2)^2\right]$$

$$= \lambda(\lambda-1)^2\left[\lambda^2 - 3\lambda + 3\right]$$

$\chi(G) = 2, \quad P(G;5) = 5\cdot 4^2(25-15+3) = 80\cdot 13 = 1040$

(c)



The middle vertex can be colored in λ ways, and with a similar argument as in the previous case we obtain

$$P(G,\lambda) = \lambda(\lambda-1)(\lambda-2)\left[1\cdot(\lambda-2)+(\lambda-3)^2\right]$$

$$= \lambda(\lambda-1)(\lambda-2)\left[\lambda^2-5\lambda+7\right]$$

$\chi(G)=3$ $\qquad P(G,5) = 5\cdot 4\cdot 3\cdot 7$
$\qquad\qquad = 420$