# Mm5023 lecture 10

# Graphs III

## Plan

- Hamilton cycles
- Coloring ( coloring number and chromatic polynomials)

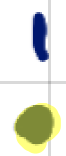
# Hamilton cycles

Recall: a cycle is closed walk that passes throug any vertex at most once

**Definition:** Given a graph $G=(V,E)$ an Hamilton cycle is a cycle which visits every vertex

! It seems similar to Euler circuit but
- it is more elusive with open conjectures.

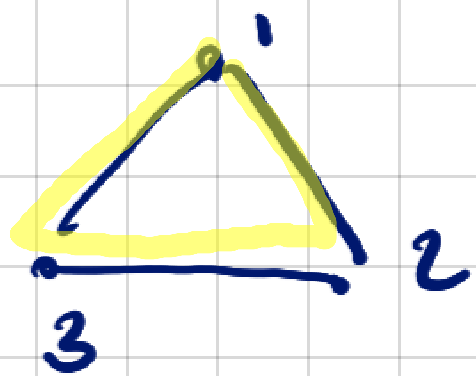
# Examples

1) 

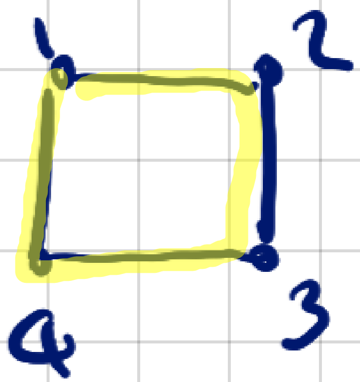

$(1)$

2) 


$(1\ 2\ 1)$

3) 


$(1\ 2\ 3\ 1)$

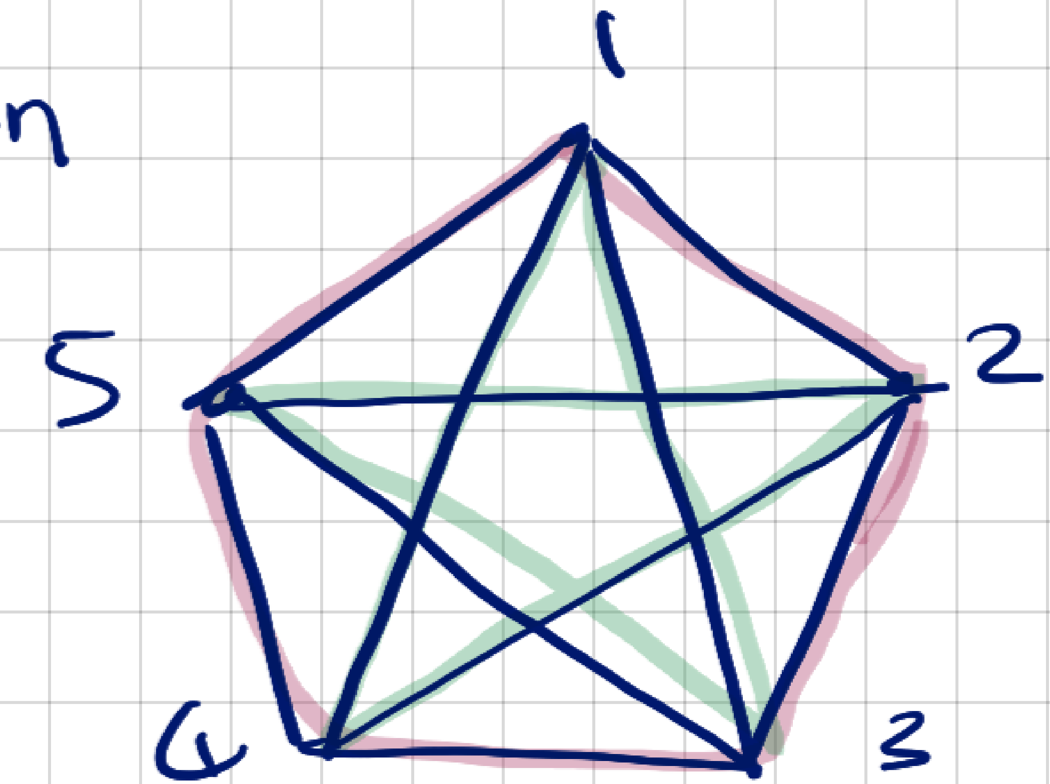
4) 


$(1\ 2\ 3\ 4\ 1)$

$C_n$


$(1\ 2\ 3 \ldots m\ 1)$

$K_n$




$(1\ 2\ 3 \ldots n)$

$(1\ 4\ 2\ 5\ 3\ 1)$

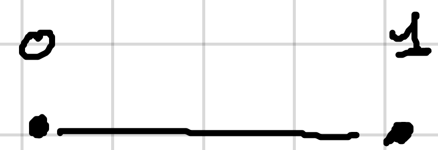
# The hypercube

$Q_n$ $\qquad V = \{0,1\}^n$

$(a_1 \dots a_n) \sim (b_1 \dots b_n) \iff$

$\sum_i |a_i - b_i| = 1$ (they differ just in one coordinate)

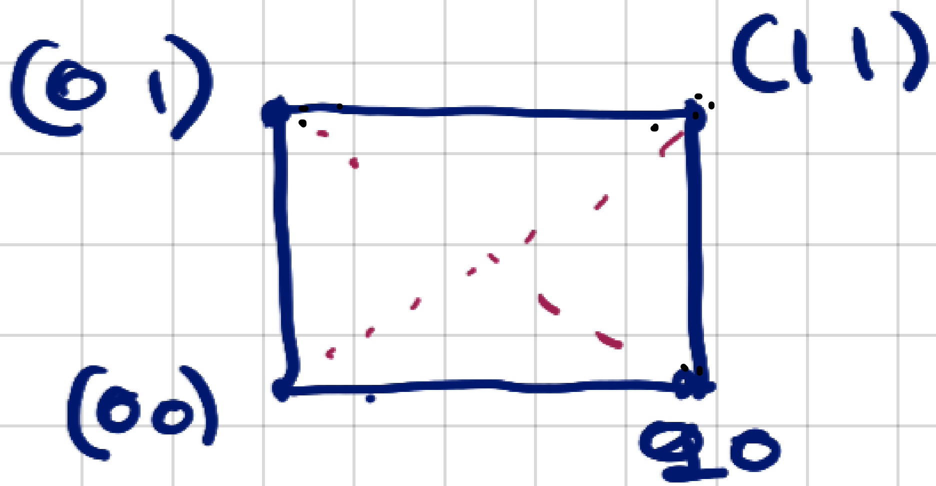
$n=1$ $\qquad V = \{0,1\}$



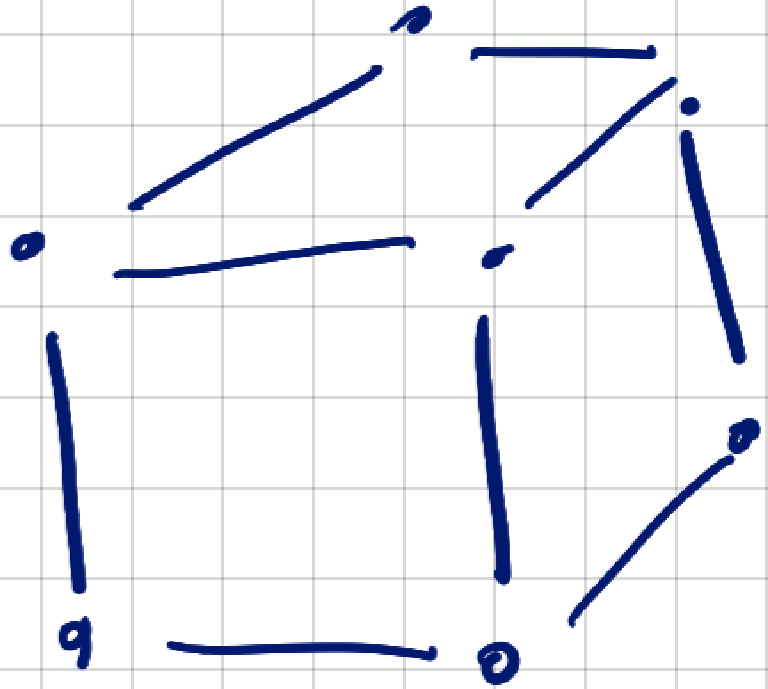
$m = 0$ $\qquad \{1, 0\}^0 = \{*\}$

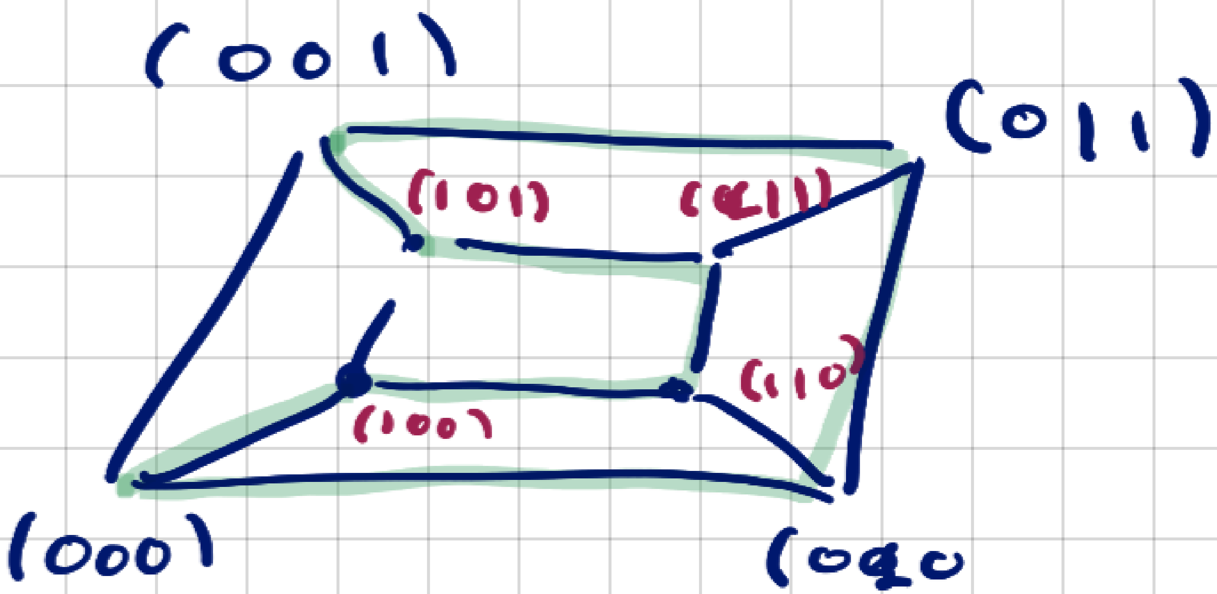
$m = 1$ $\qquad \{1\ 0\}^1$

$n = 2$ $\qquad \{(10)\ (00)\ (01)\ (11)\}$



$n = 3$



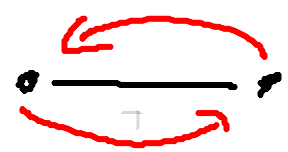
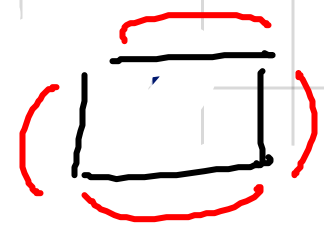



Hypercube graph

# Lemma

$Q_n$ has an Hamilton Cycle for every $n \geq 2$.

Proof by induction on $n$

$n = 1$  $n = 2$ 

We assume the result know for $n = k$
we prove it for $n = k+1$

$(v_1 \ldots \ldots v_{2^k}, v_1)$ HC in $Q_k$

$v_i \in \{0\ 1\}^k$ $\quad (0, v_i) \in \{0,1\}^{k+1}$

$(1, v_i) \in \{0,1\}^{k+1}$

$\big( (0 v_1), (0 v_2), (0 v_3) \ldots (0 v_{2^k}) \; (1 v_{2^k}), (1 v_{2^k-1})$

differ of 1 cordins

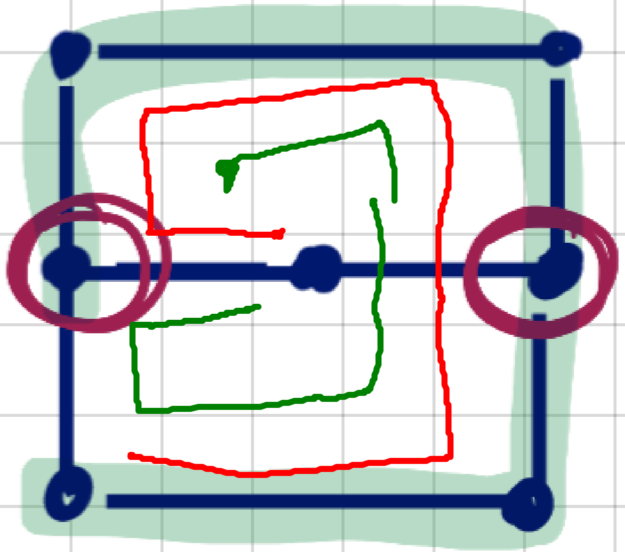
$\ldots \; (1 v_1) \; (1, v_0) \; (0 v_0) \big)$

∴ this is a cycle of in $Q_n$

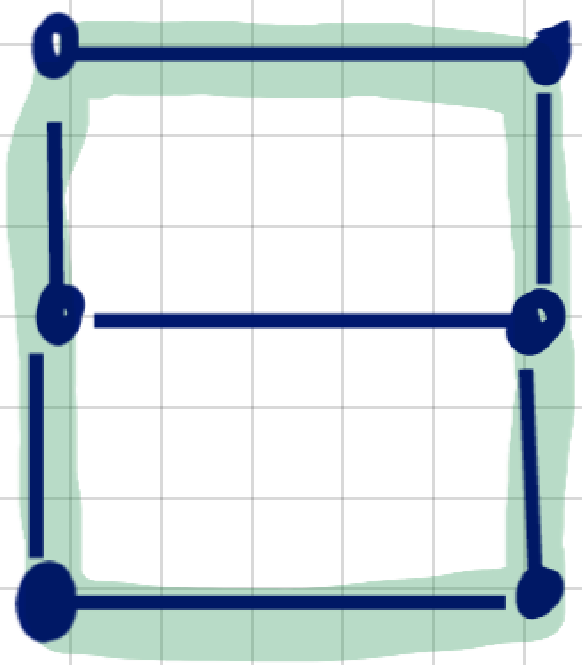
the length is $2 \cdot 2^k = 2^{k+1} = |V(G)|$

$\Rightarrow$ It is an HC. #

# Example

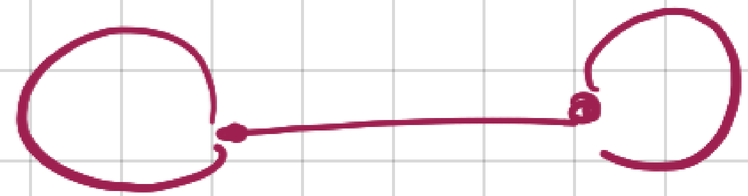


This has not an HC
(Brute force)



This has an
Hamilton cycle.

# Theorem

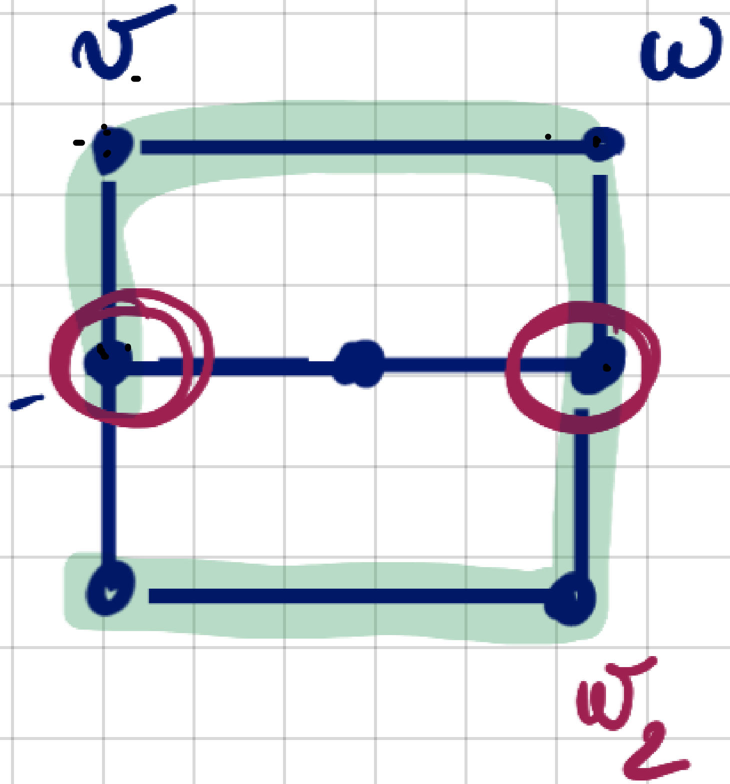


Let $G = (V, E)$ a loop-free with

$$\deg v + \deg w \geq |V| - 1$$

for all $v \neq w$

$\Rightarrow$ $G$ has an Hamilton Cycle




$3+3 = 6 < 7$ # edges.

$2+2 = 4 < 7-1$

Theorem

Let $G = (V, E)$ a loop-free graph with $|V| \geq 3$. Suppose that

$$\deg(v) + \deg(w) \geq |V|$$

for all $v \neq w$

$\Rightarrow$ $G$ has an Hamilton Cycles

! it goes only in one direction if the hypotesis is not satisfied then you do not know anything

We are proving the contrapositive
we assume $G$ has no HC
we find $v \not\sim w$
$\deg_G(v) + \deg_G(w) < n$

Proof: $G$ loop free & connected
$G$ is a subgraph of $K_n$ with $n = |V|$
We have seen that $K_n$ has an Hamilton cycle.
$G \subseteq \underset{\subseteq K_n}{G'}$ maximal without an Hamilton cycle.

Assume that you know the theorem for $G'$
$v \not\sim w$ in $G'$ than $v \not\sim w$ in $G$

$$\deg_{G'}(v) + \underset{\geq}{\deg_{G'}}(w) < n$$

$$\deg_G(v) + \deg_G(w)$$

We have to prove the theorem for $G'$ maximal without an HC

let $v, w$ not adjacent

$e = \{v, w\} \in K_n$ is not in $G'$

I consider $\underset{\underset{G'}{\cup}}{G'+e} = (V(G), E(G) \cup \{e\})$

We know that $G'+e$ has an HC by the max of $G'$

this HC walks | uses the edge $e$

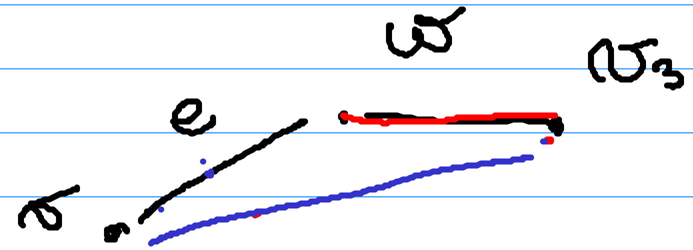
$\{v_1 \dots v_n\ v_1\}$ we can assume up to cycling the cycle that $v = v_1$

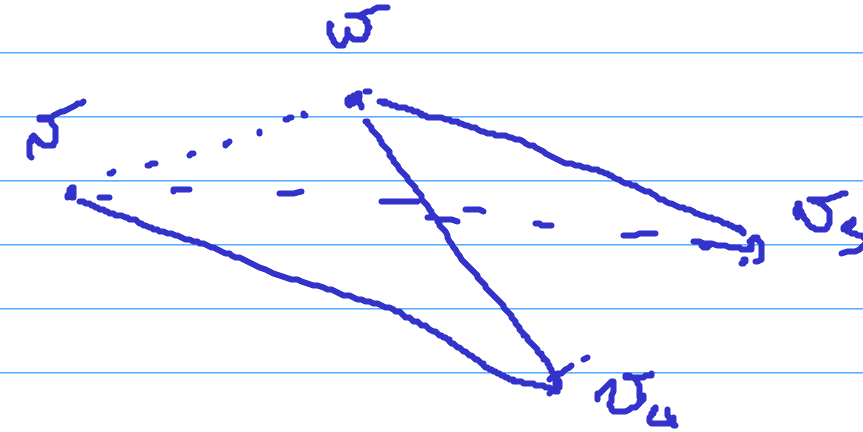
$w = v_2$

For $3 \le i \le n$ we have that $(n \geq 3)$

$\{v\, v_{i+1}\} \notin E(G')$ or.

$\{w\, v_i\} \notin E(G')$




$(w\, v_i \dots v_{n-1}\, v_n\, v\, v_i \dots v_3)$ is an HC

$\deg_{G'}(v) + \deg_{G'}(w) \le m-2+1 \le n-1 < n$

$\{v, v_n\}$

for every $i = 3 \dots n$

I have one edge

# Corollary

Let $G$ be a loop-free graph with

$$|E(G)| \geq \binom{|V|-1}{2} + 2$$

$\Rightarrow$ $G$ has an Hamilton Cycle

Proof $\quad v w$ not adjacent

$G' = G - v - w \qquad |V(G')| = |V(G)| - 2$

$|E(G')| = E(G) - \deg v - \deg w \leq \binom{|V(G)|-2}{2}$

$\uparrow$
# edges in
$K_{|V(G)|-2}$

$$\binom{|V(G)|-1}{2} + 2 \leq |E(G)| \leq \binom{|V(G)|-2}{2}$$

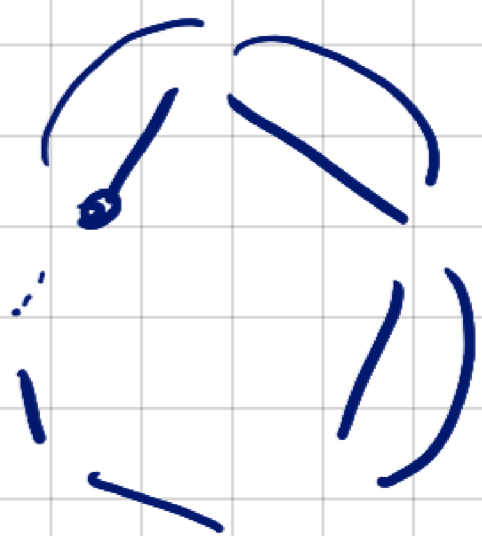
$$\deg v + \deg w \leq \binom{|V(G)|-1}{2} - \binom{|V(G)|-2}{2} + 2$$

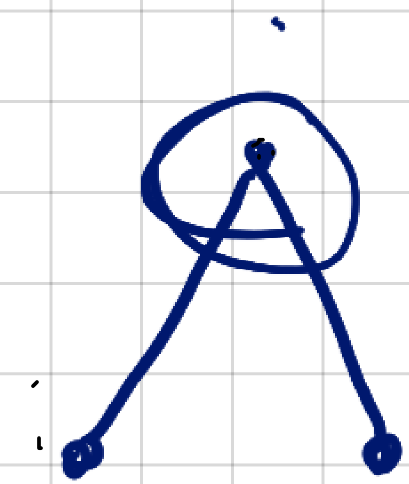
$$= |V(G)| - 2 + 2 = |V(G)|$$

# An open problem

A graph $G$ is said to be **vertex transitive** iff for all $v \neq w \in V$ there is a graph isomorphism $f: G \to G$ such that $f(v) = w$

$C_n$





not vertex transitive

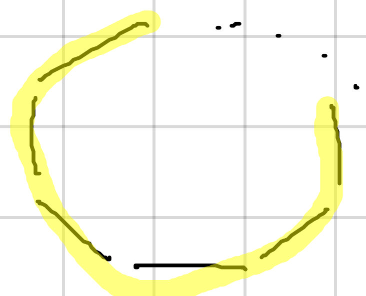
$\Rightarrow$ all vertex have to have the same deg.

# Conjecture (Lovász)

Every vertex transitive graph has a Hamilton Cycle.

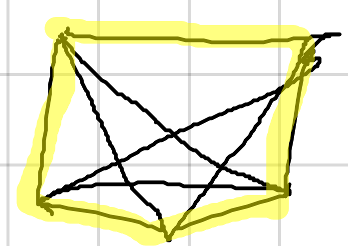
Examples

$C_n$ is vertex transitive



$K_n$ is vertex transitive.

$K_5$



If a graph $G$ contains a spanning subgraph with an HC $\Rightarrow$ $G$ has an HC.

# Example

Give conditions for $K_{n,m}$ to have an Euler circuit or an Hamilton cycle.

deg m

deg n

deg m + deg m $\geq$ n + m

deg n + deg n $\geq$ n + m

# Coloring (Loop free)

Given a graph $G=(V,E)$ and $n \in \mathbb{N}$ an **n-coloring** of $G$ is a function

$$f : V \longrightarrow \{1 \dots n\}$$

such that $f(v) \neq f(w)$ whenever $v \sim w$

The **coloring number** of $G$ is

$$\chi(G) := \min\{n \,/\, G \text{ has an } n\text{-coloring}\}$$

# The 4 colors theorem

If $G = (V, E)$ is a planar graph then

$$\chi(G) \leq 4$$

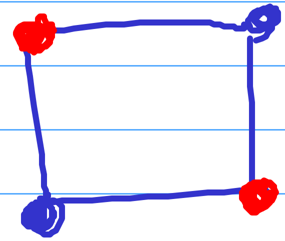
You need just 4 colors to color a map.

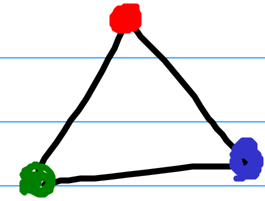
Example

$K_n$ $\quad \chi(K_n) = n$

need to choose $n$ different colors since every vertex is adjacent to all the other

$C_n$





$$\chi(G) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

If $n$ is even

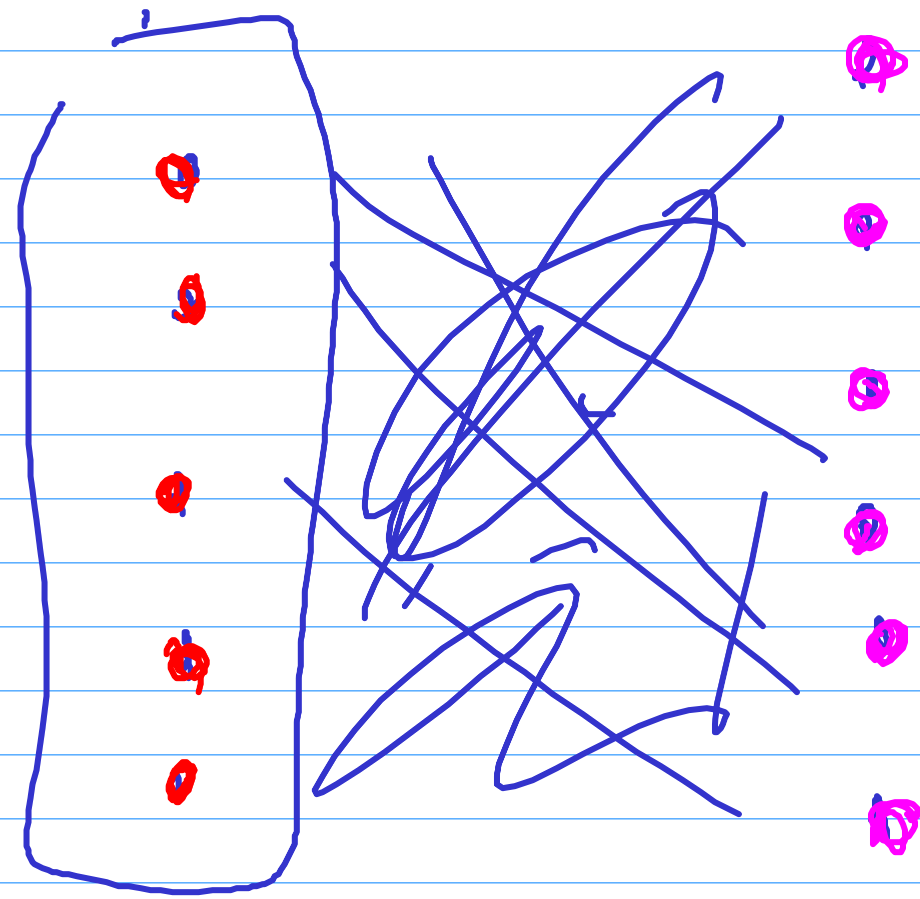
$$f : V \longrightarrow \mathbb{Z}/2\mathbb{Z}$$
$$j \longmapsto j \bmod 2$$

$$j \sim j+1$$
$$\phantom{j \sim{}} j-1$$

$$\underbrace{\dot{n} \sim \dot{1} \sim \ddot{2}}_{\text{same parity.}}$$

$K_{n,m}$

NOT ADJACENT



# Proposition

A graph $G$ has an $n$-coloring $\Leftrightarrow$ it admits a graph-homomorphism

$$\varphi : G \longrightarrow K_n$$

: Skip for now

# Proposition

We have that $\chi(G) \leq 2 \iff$

$G$ is bipartite

<u>Proof</u> If $G$ is bipartite $V = V_1 \sqcup V_2$

$$f : V \longrightarrow \{1, 2\}$$

$$f(v) = \begin{cases} 1 & \text{if } v \in V_1 \\ 2 & \text{if } v \in V_2 \end{cases}$$

Conversely suppose that $\chi(G) \leq 2$ If $\chi(G) = 1$ no two vertices are adjacent but any $V = V_1 \sqcup V_2$ will make $G$ bipartite

$\chi(G) = 2$

$f: V \longrightarrow \{1, 2\}$ proper colo

$V_1 = f^{-1}(1)$

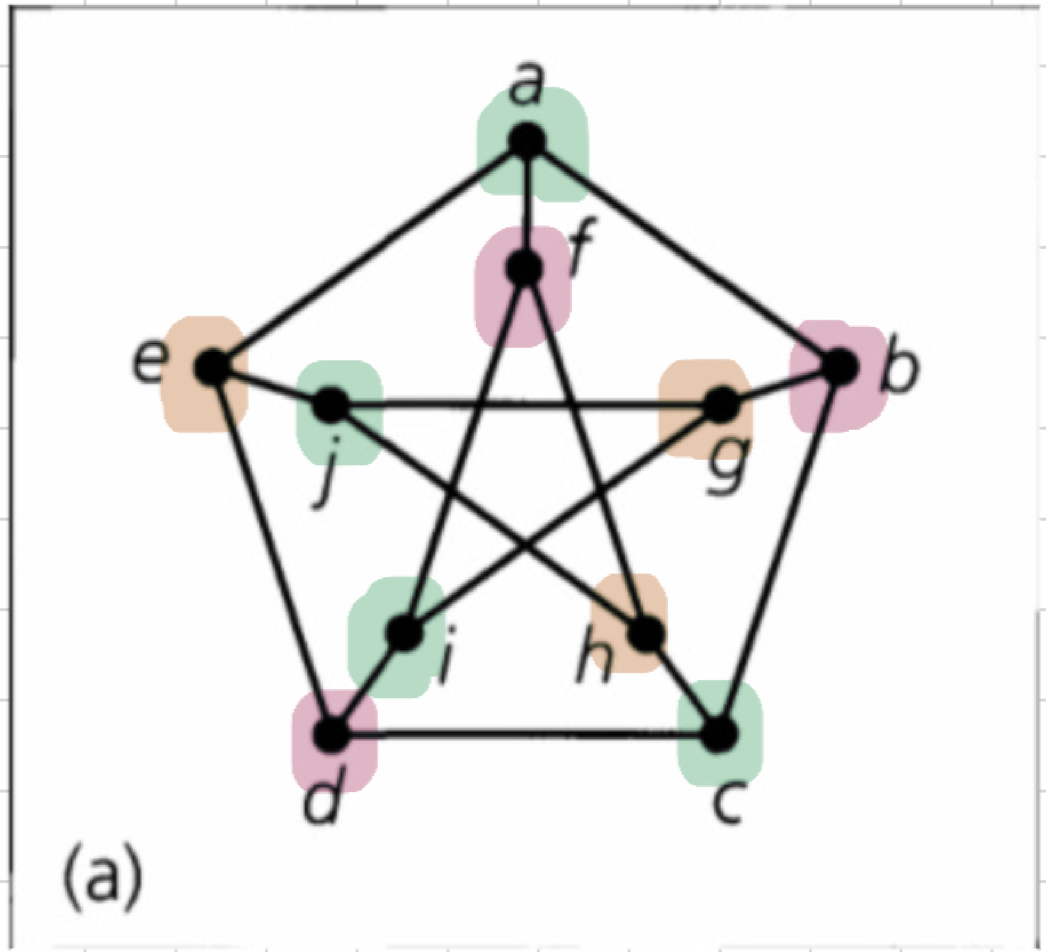
$V_2 = f^{-1}(2)$

$V = V_1 \sqcup V_2$

no vertex in $V_i$ is adjacent to any other vertex in $V_i$

$\Rightarrow V$ is bipartite

# Example




$\chi(G) = 3$ not bipartite

$\chi(G) \geq 3$ since $\supseteq C_5$

but you can provide a 3 coloring.

# N coloring number

The n-coloring number of a graph $G$

is $\chi_G(n) := \left| \{ \text{n-colorings of } G \} \right|$

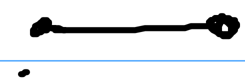
$= \left| \{ f : G \to K_n \text{ homomorphism} \} \right|$

# of ways you can color $V(G)$ using $n$ colors such that $v \sim w$ are painted in $\neq$ colors.

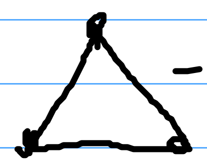
Example

$\chi_G(n) = n$

$\chi_G(n) = n^2$

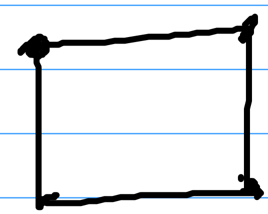


$\chi_G(n) = \binom{n}{2} = \frac{1}{2} n(n+1)$



$\chi_G(n) = \binom{n}{3}$

polynomials in $n$



$\binom{n}{4} + \binom{n}{2}$

# Theorem

Given a graph $G$ there exist a unique polynomial $P(G,\lambda)$ (called the chromatic polynomial of $G$) such that

$$P(G,n) = \chi_G(n)$$

for all $n \in \mathbb{N}$

The proof will be by induction on $|V|+|E|$

To perform the induction step we need to introduce a new operation on graphs:

collapsing an edge

$G = (V, E)$ $e = \{v, w\}$

$G'_e := G/_{v=w}$ (If loops are created they are removed.)

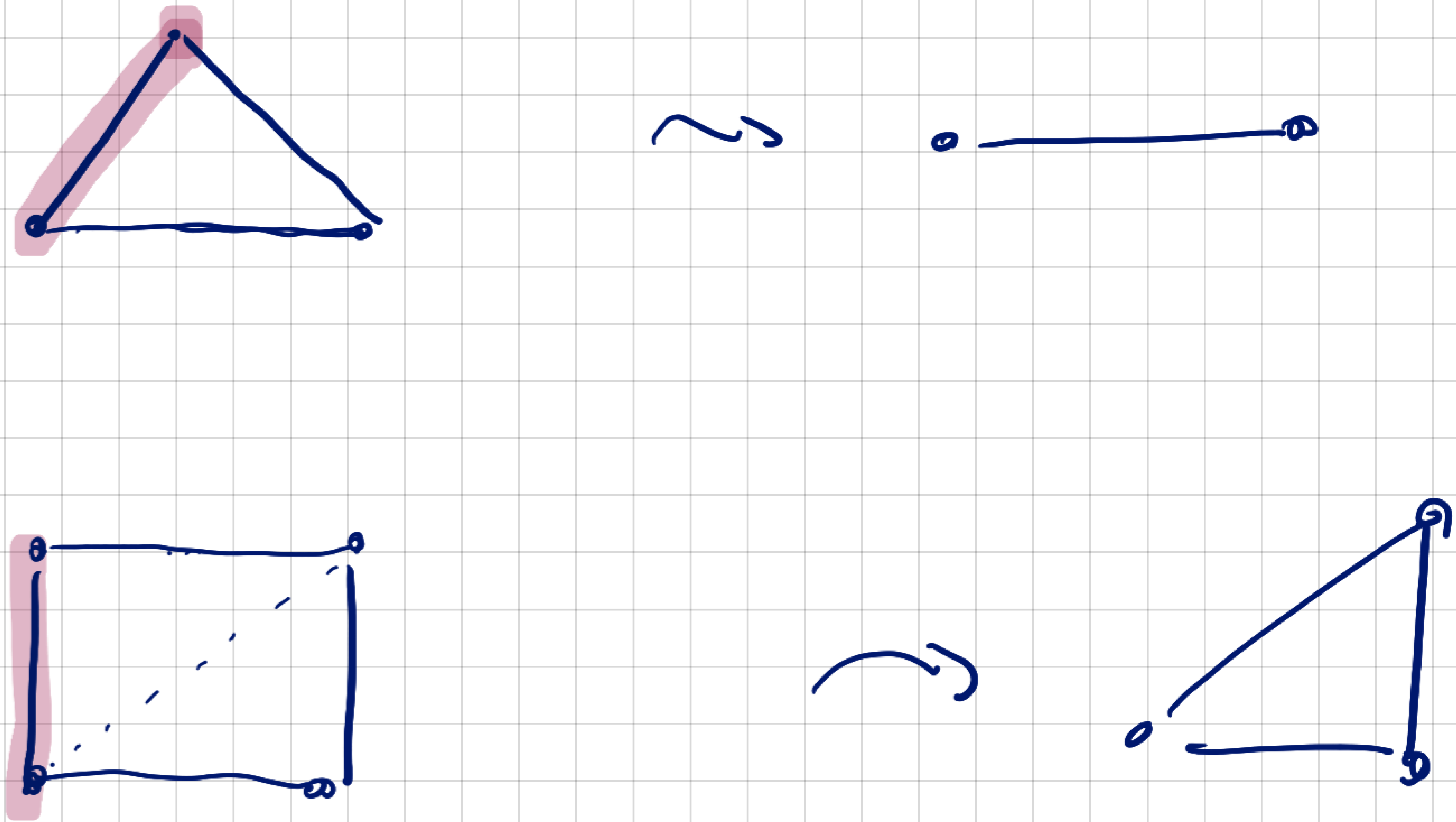
__Lem__ $G = G_1 \cup \quad \cup G_n$ connected componen

$$\chi_G(n) = \pi \chi_{G_i}(n)$$

__Proof__ the coloring are independent

__Rmk__ $=$ We can assume that $G$ is connected in the proof of the thm

# Examples



**Theorem**

$$G = G_1 \cup G_2$$

$$G_1 \cap G_2 = K_n$$

$$\boxed{P(G, \lambda) = \frac{P(G_1, \lambda)\, P(G_2, \lambda)}{P(K_n, \lambda)}}$$

Proof of thm

1) $p(x)$ $q(x)$ are two polynomials such that

$$p(n) = \chi_G(n)$$

$$q(n) = \chi_G(n)$$

$$(p-q)(x) \equiv 0 \qquad \text{(every natural \# is a root)}$$

$\exists$) Induction on $k = |V(G)| + |E(G)|$ $G$ connected

If $k=1$ $\bullet$

$$\chi_G(n) = n$$

$$P(G, \lambda) = \lambda$$

If $G$ has a loop no admissible coloring is possible then

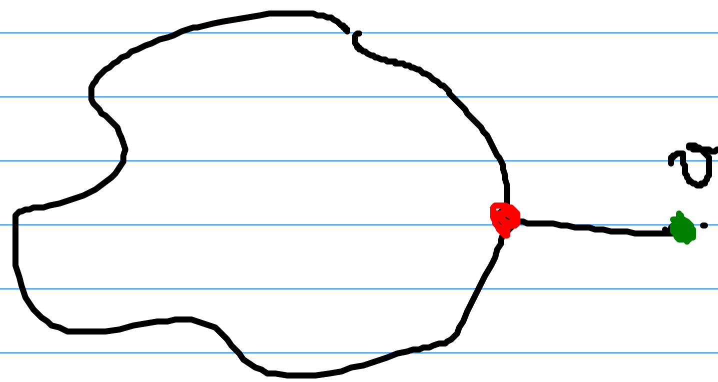
$$\chi_G(n) = 0 \qquad P(G,\lambda) \equiv 0$$

Assume $G$ loop free.

We assume the statement holds for $k \le p$

An we prove it for $k = p+1$

CASE 1 G has a terminal vertex

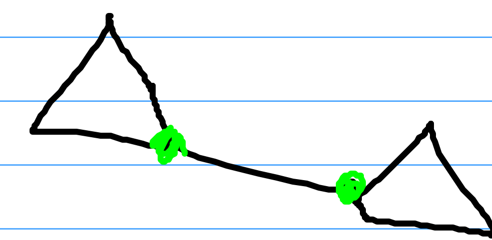



$\chi_G(n) = (n-1) \cdot \chi_{G-v}(n)$ ∃ $P(G-v, \lambda)$

$\chi_{G-v}(n) = P(G-v, n)$

$P(G, \lambda) = (\lambda - 1) \cdot P(G-v, \lambda)$

Case 2 G has no terminal vertex



let $e = \{v, w\} \in E(G)$

$f : V(G-e) \longrightarrow \{1 \dots n\}$ proper coloring

a) $f(v) = f(w)$ $\longrightarrow$ proper coloring of $G_e$

b) $f(v) \neq f(w)$ $\longrightarrow$ proper coloring of $G$

$$\chi_{G-e}(n) = \chi_G(n) + \chi_{G_e}(n)$$

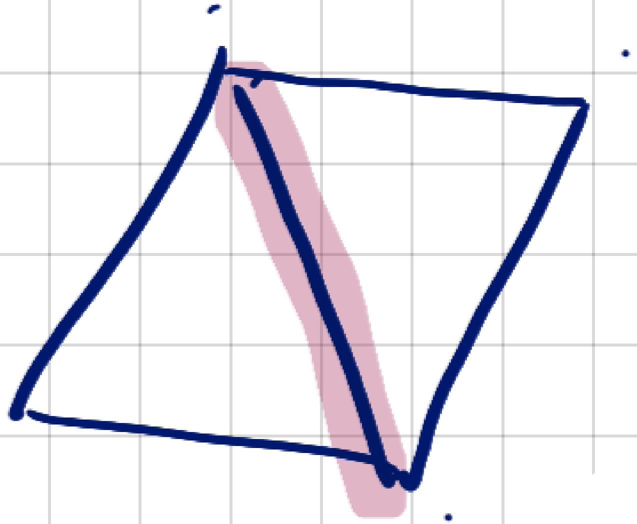
$$\chi_G(n) = \chi_{G-e}(n) - \chi_{G_e}(n)$$

$$= P(G-e, \lambda) - P(G_e, \lambda)$$

$\uparrow$

vert + eg $p+1-1$ ✓ $\leq p$

$$\boxed{\begin{aligned} P(G,\lambda) &= P(G-e, \lambda) \\ &\quad - P(G_e, \lambda) \end{aligned}}$$



$P(6\lambda) = ?$