MH5023 – Lecture 11

Trees

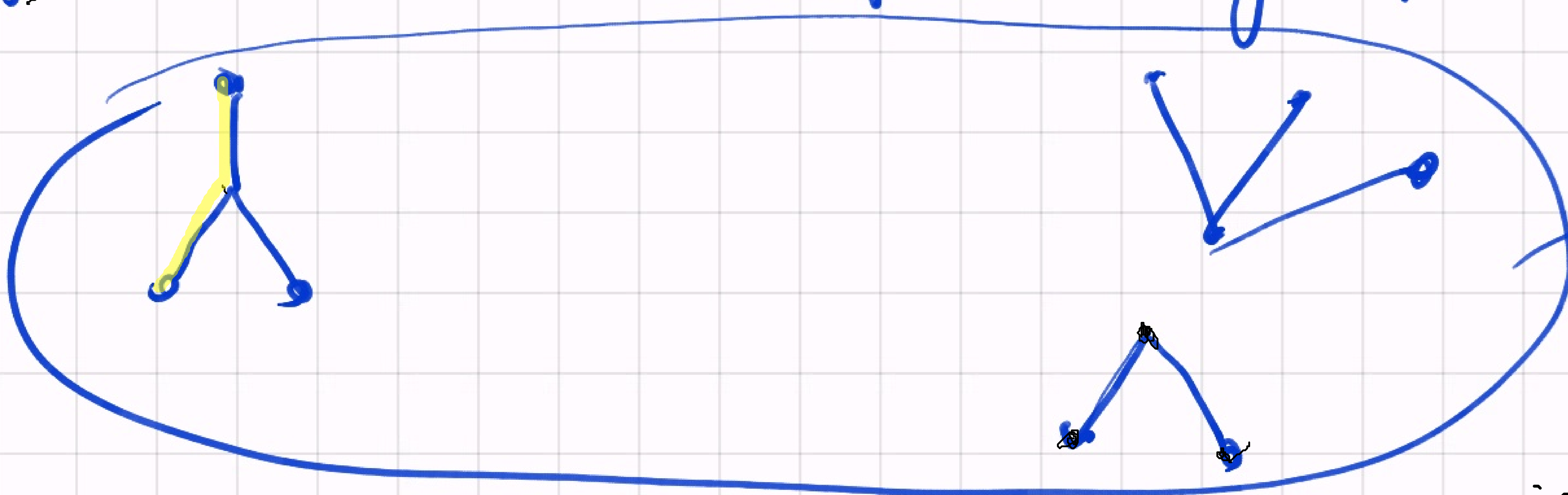
1) Intro

2) Spanning trees
   (Algorithm)

11.1 Intro

A tree is a connected loop-free graph w/out cycles (closed walks with no repeated vertices)

A forest is a loop-free graph w/out cycles

① 

Forest

every connected cp of a forest is a tree.

The mother of all examples

$A_1 \ldots A_n$ finite sets

$$V = \bigcup_{k \le n} A_1 \times \cdots \times A_k =: V$$

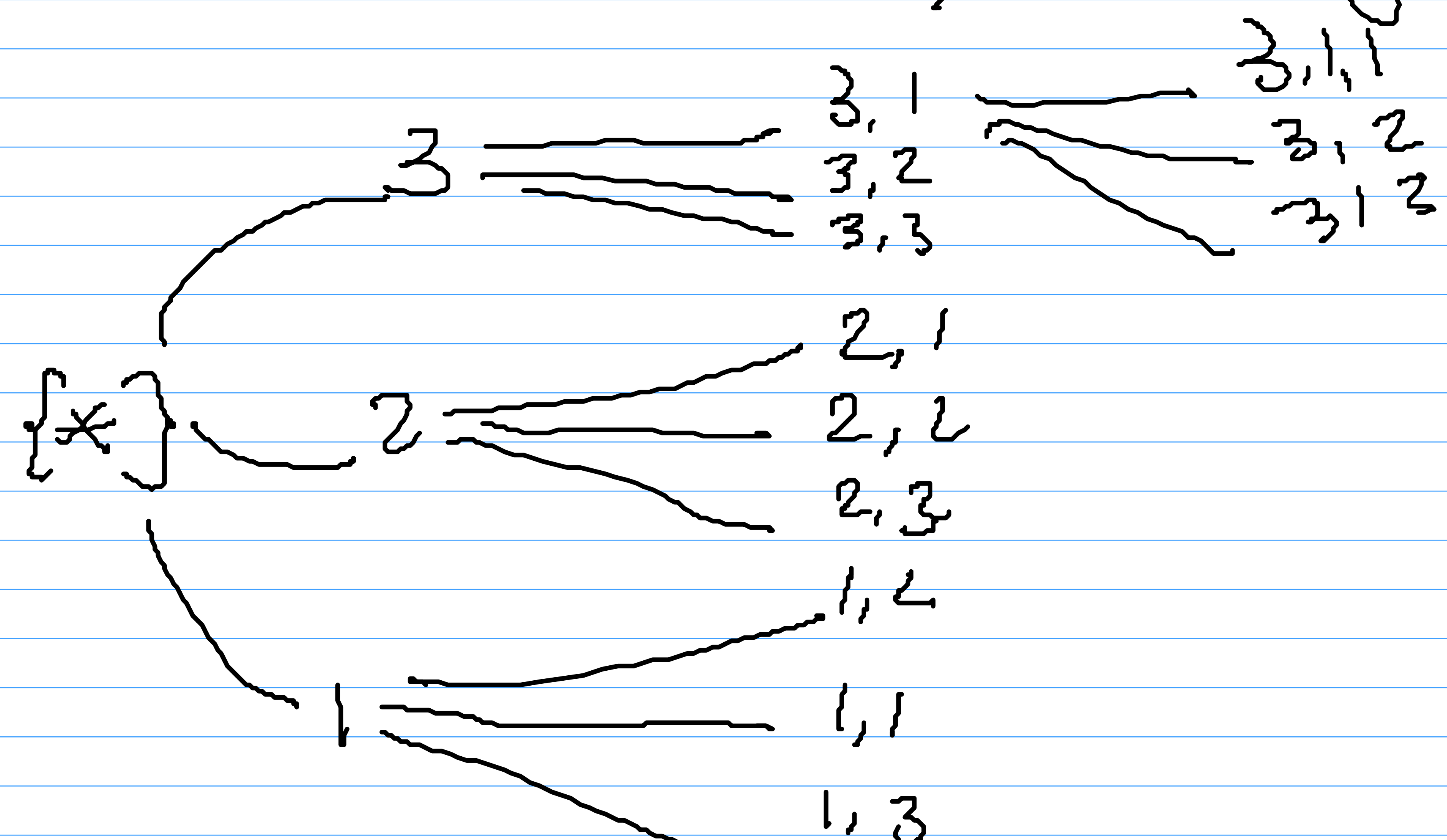
the empty product is $\{*\} \hat{=} \{\emptyset\}$

$$\bigtimes_{k \in \emptyset} A_k = \{*\}$$

$(a_1 \ldots a_k) \sim (a_1 \ldots a_k\ b)$ for all $b \in A_{k+1}$

Example $A_i = \{1, 2, 3\}$

$(a) = (\emptyset, 1)$



Every finite tree is a subgraph of one constructed in this way.

Given a graph $G = (V, E)$ a tree $T \subseteq G$ is called a <u>spanning</u> if $V(T) = V$

Prop: G finite connected $\Rightarrow$ G has a spanning tree

Proof:

subgraph

$$\mathcal{G} := \{ T \subseteq G \mid T \text{ is a tree} \}$$

this is an ordered set with $\subseteq$ (subgraph relation)

Since G is finite

$\mathcal{G}$ has finitely many subgraphs

$\mathcal{G}$ has a maximal element T_0

Claim T_0 is a spanning tree

Suppose by contradiction that $\exists\, v \in V \setminus V(T_0)$

Let $w \in V(T_0)$ since G is connected there is a path in G with

$(v_0 \text{ — } v_n)$

$T_0 \not\ni$ $v_0 = v \in T_0$ and $v_n = w$

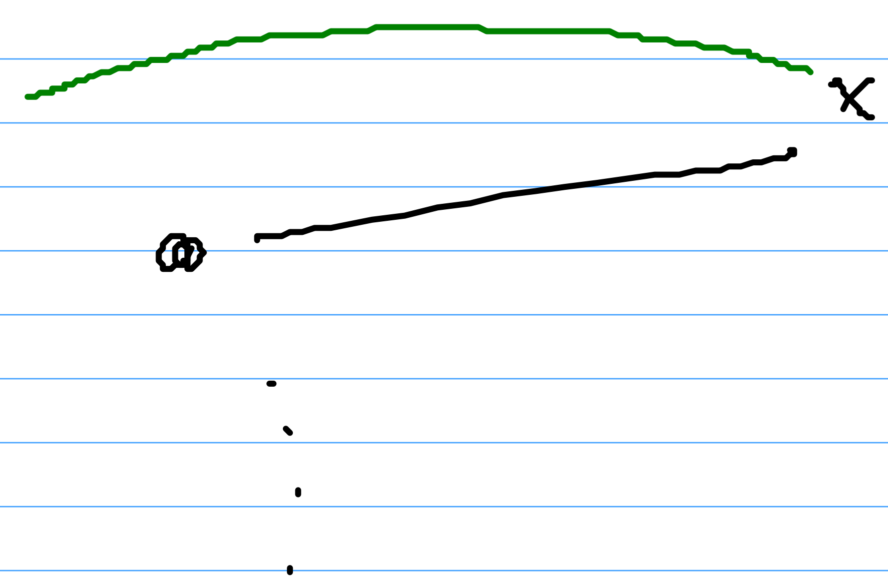
let $0 \le k \le n$: minimal such that $v_k \in V(T_0)$

↳ The first time where the path visits T_0

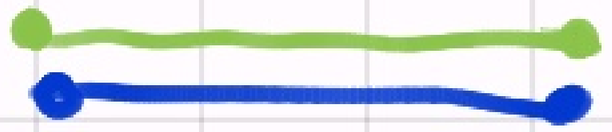
$\notin V(T_0)$

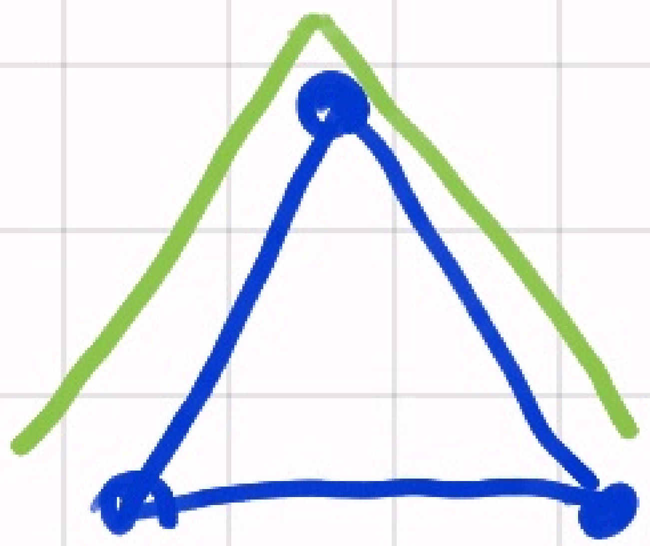
$T_0 + \{v_k, v_{k-1}\}$ is a tree $\supsetneq T_0$

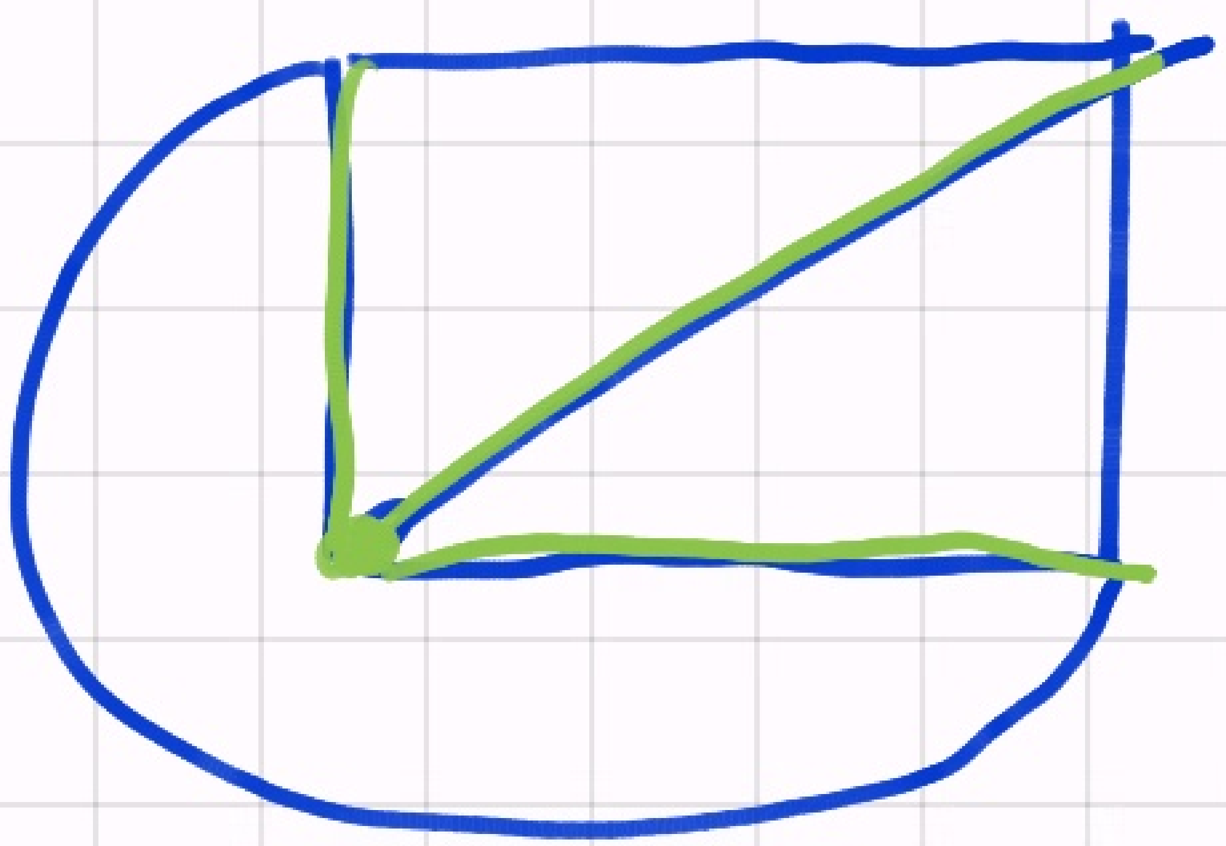
this contradict to maximal #

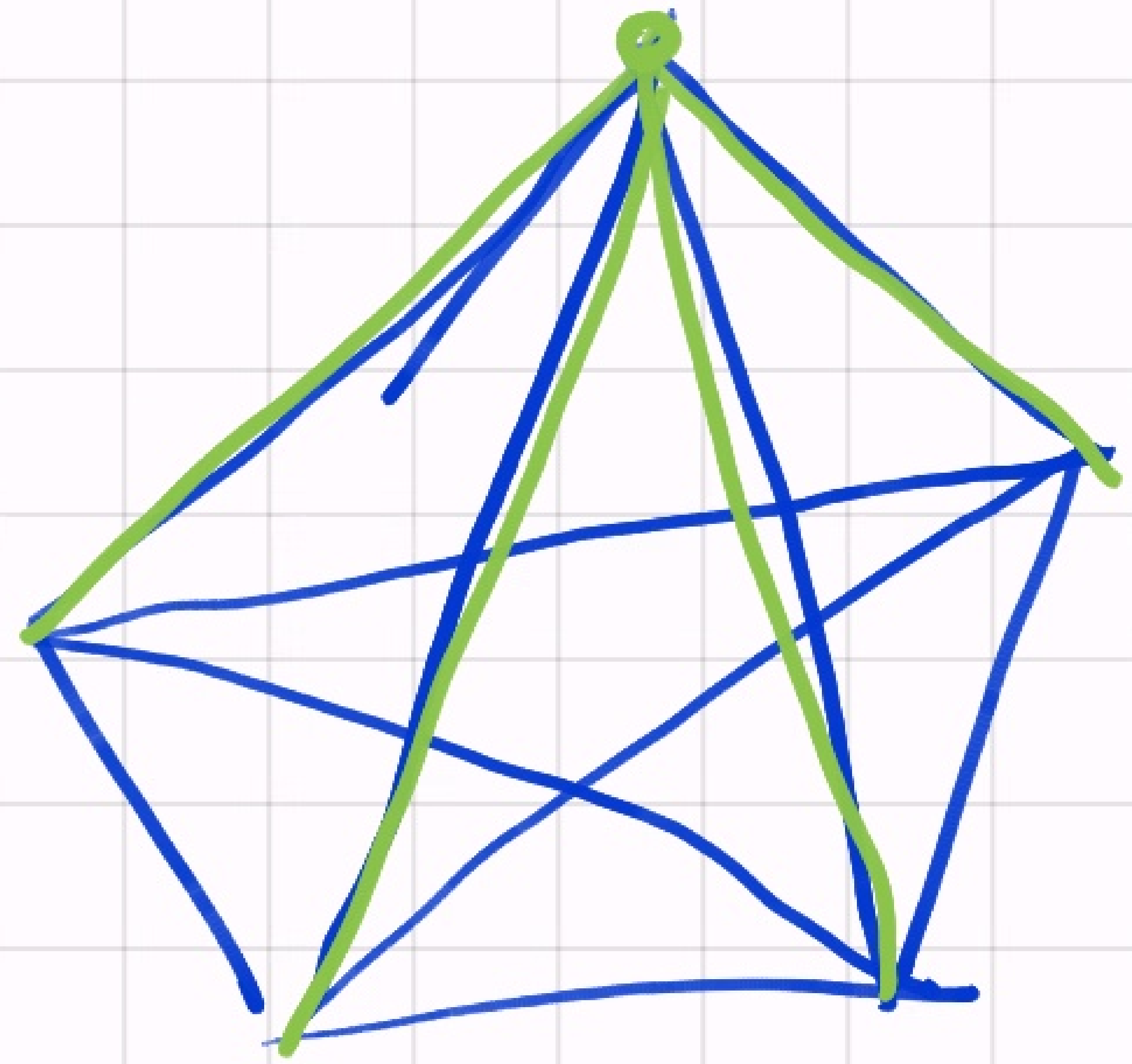


Examples

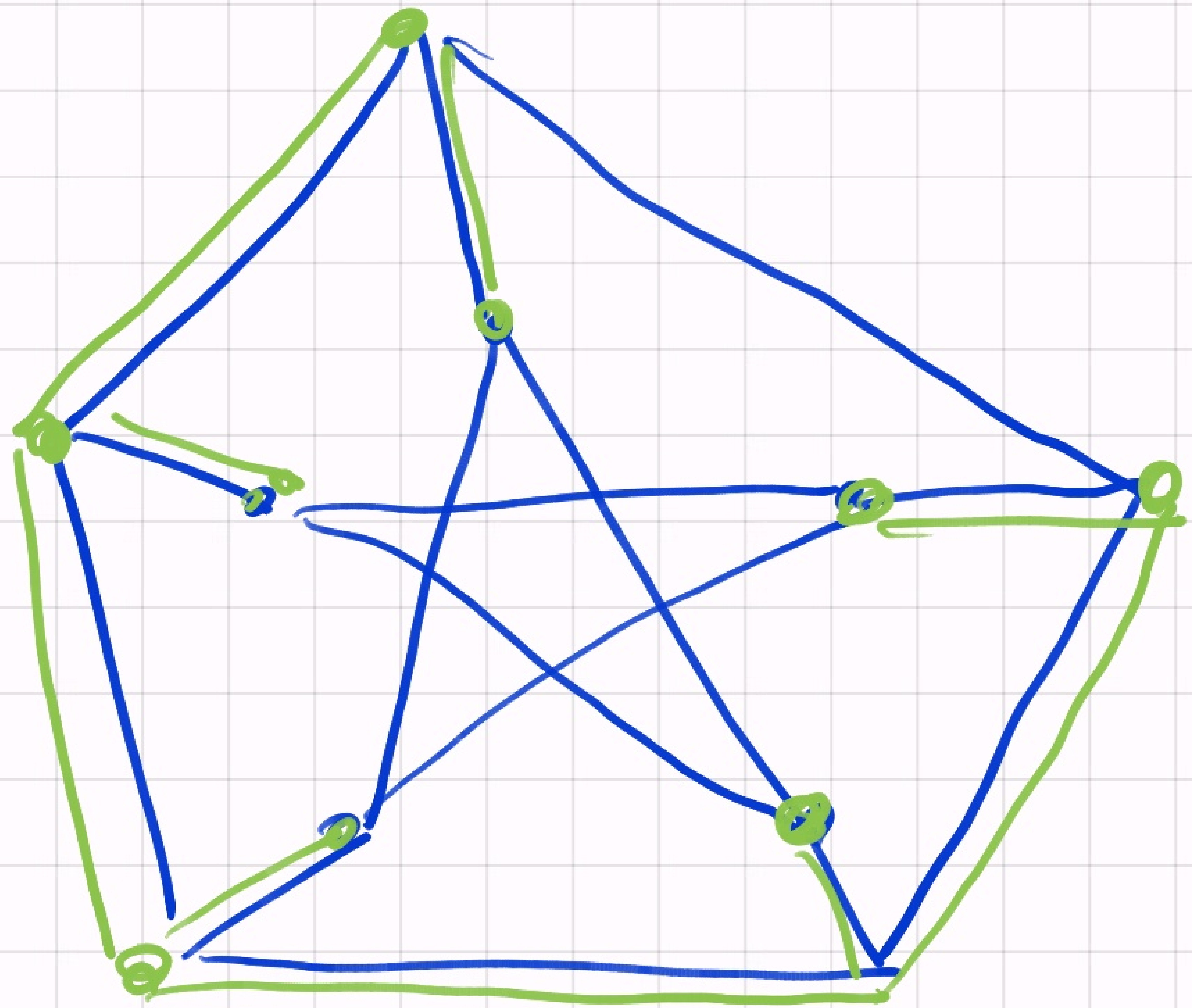








Spanning trees are not uniques



Note: I speak about a spanning tree the order on $\mathcal{G}$ given by $\subseteq$ is not complete

$\rightsquigarrow$ There might be more than one maximal element.

Plan

ⓞ Characterization of trees.

① Rooted tree

→ Spanning tree "starting from a given vertex"

② Algorithms to find spanning trees

DEPTH FIRST | BREADTH FIRST (they need Ques / Stack)

① the output should be what you want

② an algorithm has to STOP

Characterization theorem

Let $G = (V, E)$ a (loop free) graph, then the following conditions are equivalent

① G is a tree

connected

② If $x \neq y \in V$ then there is a unique path from x to y.

If, furthermore, G is finite ① & ② are equiv to

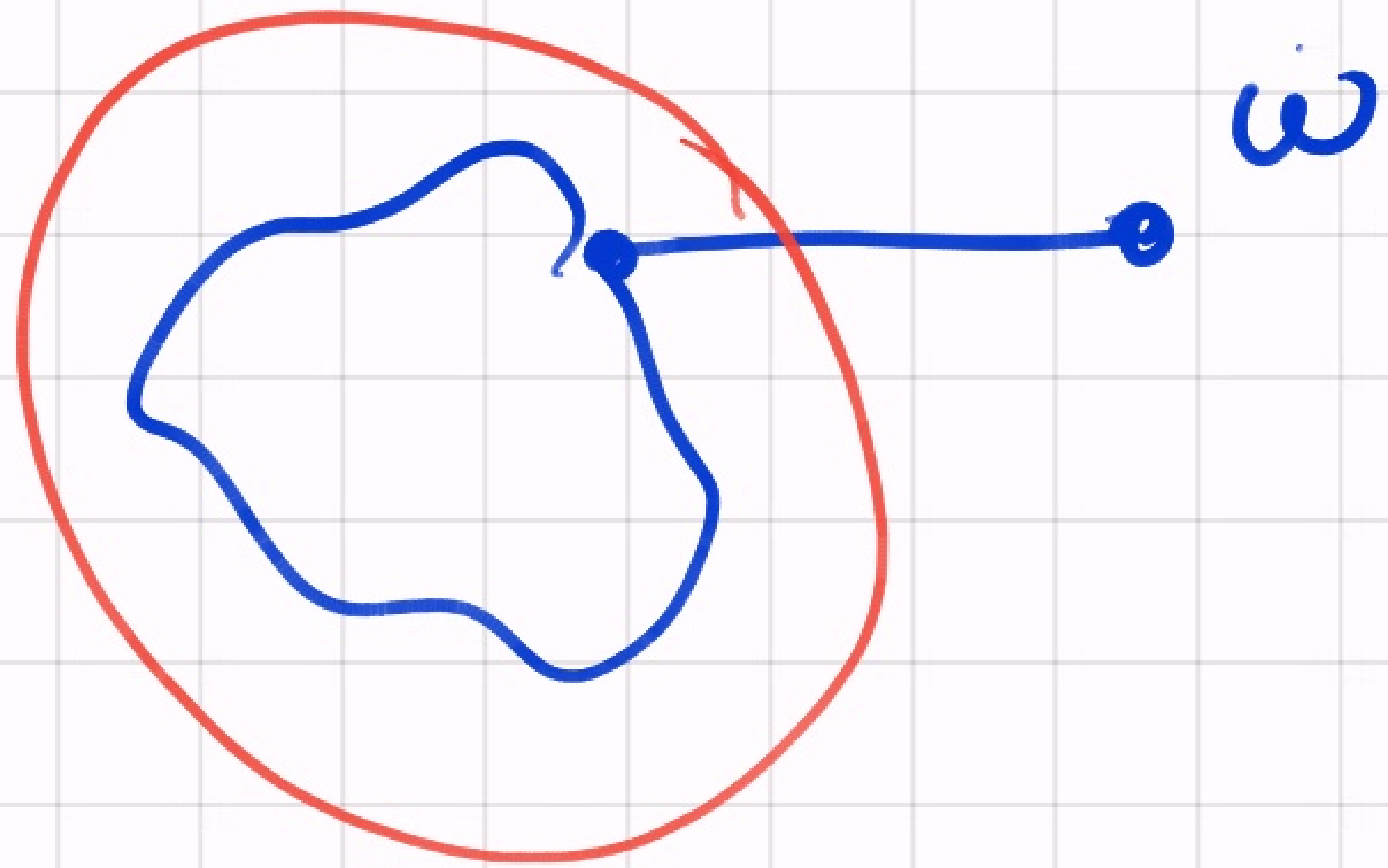
③ G is connected an $\boxed{|V| = |E| + 1}$ stopping condition

$|V(G)| = n+1$

w a terminal vertex

e the edge connecting it to the rest

$G - e$



$|V(G-e)| = n$

$|E(G-e)| = |E(G)| - 1$

$\parallel$ induction

$|V(G-e)| - 1 = n - 1$

$$\left.\begin{aligned} &|E(G)| = n \\ &|V(G)| = |E(G)| + 1 \checkmark \end{aligned}\right.$$

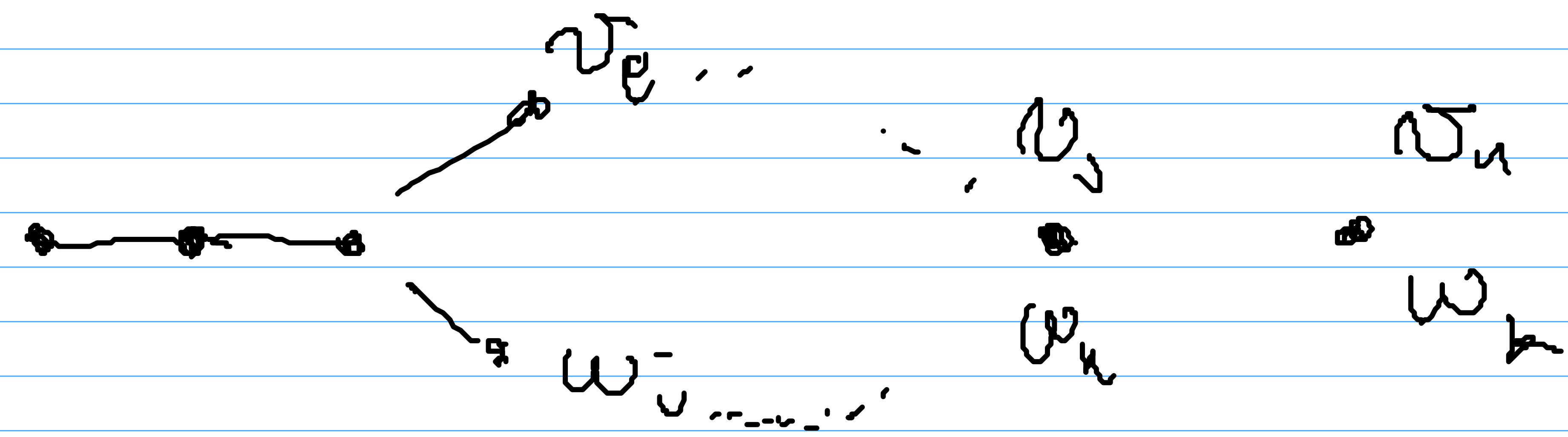
Proof $(1) \Rightarrow (2)$

Suppose that $x \neq y$ and there are two different path from x to y (not (2) $\stackrel{?}{\Rightarrow}$ not (1))

$(v_1 \qquad v_n)$ with $x = w_1 = v_1$

$(w_1 \; \ldots \; w_k)$ $\quad y = w_k = v_n$

$$i = \min \{ j > 1 \mid v_j \neq w_j \}$$



$$I := \{ \min j > i \mid v_j = w_m \text{ for some } m > i \}$$

$(v_{i-1} \; v_{i+1} \; \ldots \; v_j \; w_{m-1} \; \ldots \; w_i \; \underset{w_{i-1}}{\overset{}{v_{i-1}}})$ this is a cycle.

6 mod other

$(2) \Rightarrow (1)$ $\quad (\neg(1) \Rightarrow \neg(2))$

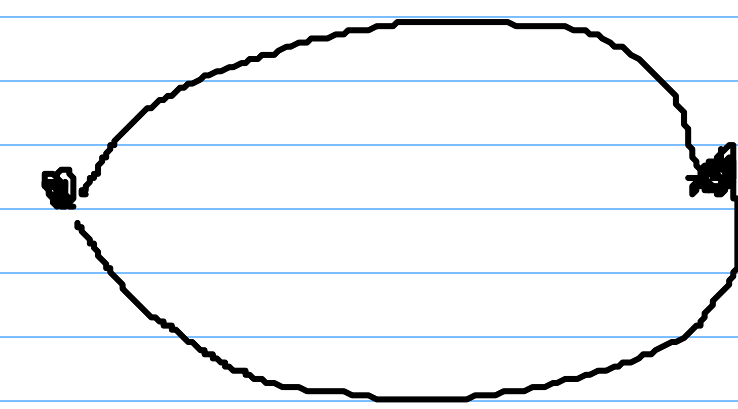
Suppose that G is not a tree.

if G is not connected then there are x, y with no path between them $\Rightarrow \neg(2)$

We can assume G connected. (loop free)

Thus G has a cycle

$(v_0 \ldots v_m = v_0)$



$(v_0\, v_1)$

$(v_0\, v_{m-1} \ldots v_1)$ are two different paths from v_0 to v_1

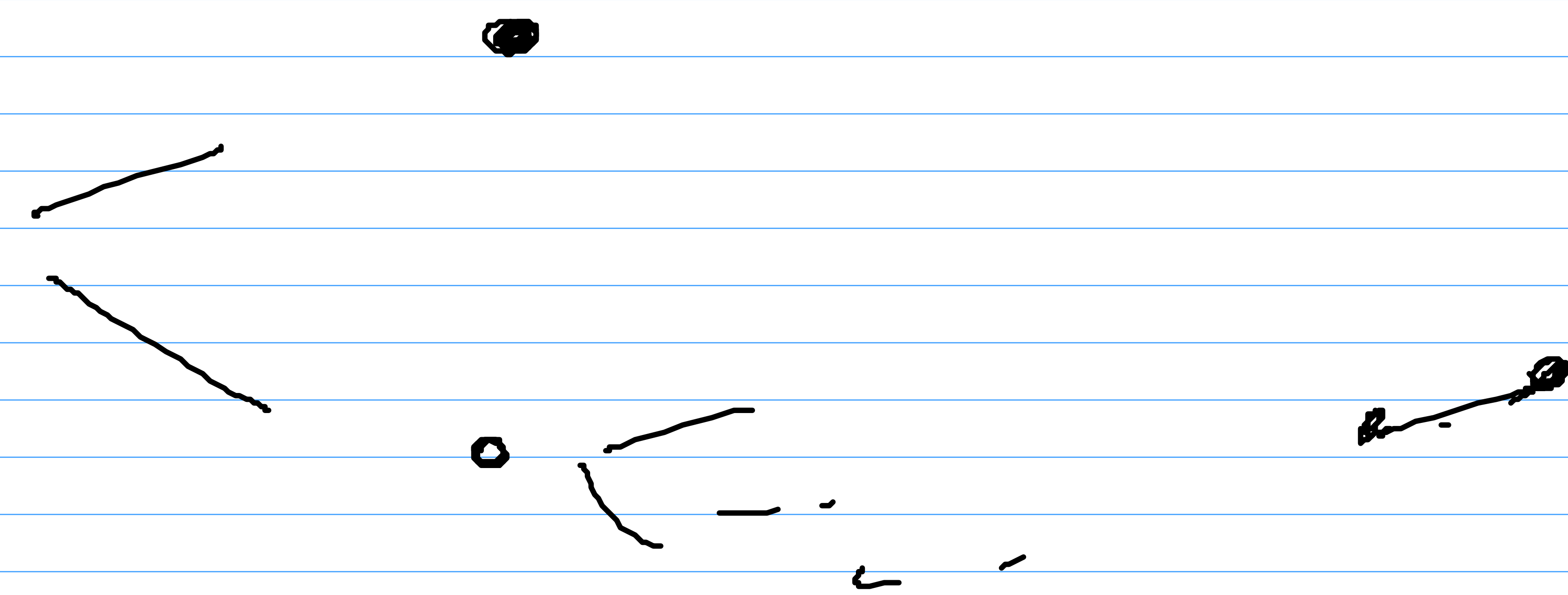
(2) does not hold.

Now we assume G finite. We are going to show

$(1) \Leftrightarrow (3)$

(1) $\Longrightarrow$ (3)

G is a tree.



Either G ..

① is trivial $\qquad |V| = 1 = |E| + 1$ ✓

$|E| = 0$

② is not trivial, by the finiteness G has a terminal vertices

let $(v_1 \;\ldots\; v_n)$ path of maximal length $\therefore v_n \sim v_{n-1}$ but not to any other vertices in V

Or we can make a longer path

Now we reason by ~~ind~~ induction on $|E|$

$|E| = 0$ G is trivial ✓

True for $|E| = k$

suppose that $|E| = k+1$

let $v \in V$ a terminal vertex

$G' = G - v$ is a tree. (still cause loop free no cycle)

$|E(G')| = k$

$|V(G')| = k+1$

$$|V(G)| = |V(G')| + 1 = \underbrace{k+1}+1 = |E(G)| + 1$$ ✓

(3) ⇒ (1) $|V(G)| = |E(G)| + 1$

G is connected ⇒ it has a spanning tree T we have $E(T) \subseteq E(G)$

$$|E(T)| = |V(G)| - 1 = |V(G)| - 1 = |E(G)|$$

$$\Rightarrow E(T) = E(G)$$

$$T = G$$

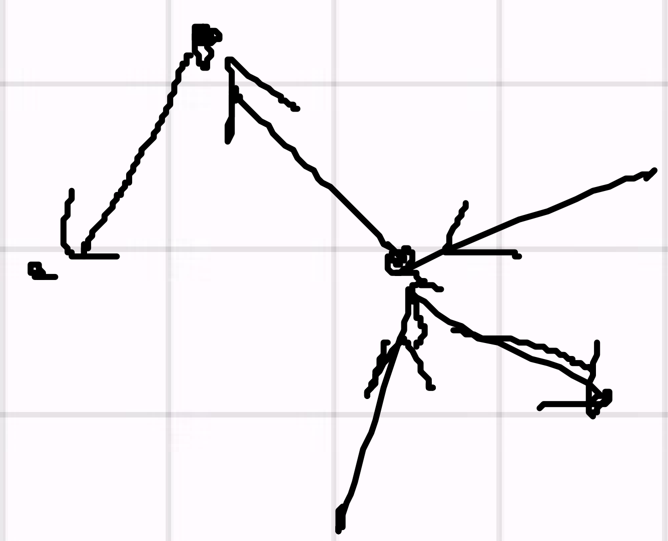
#

11.3 $\boxed{\text{Rooted trees}}$

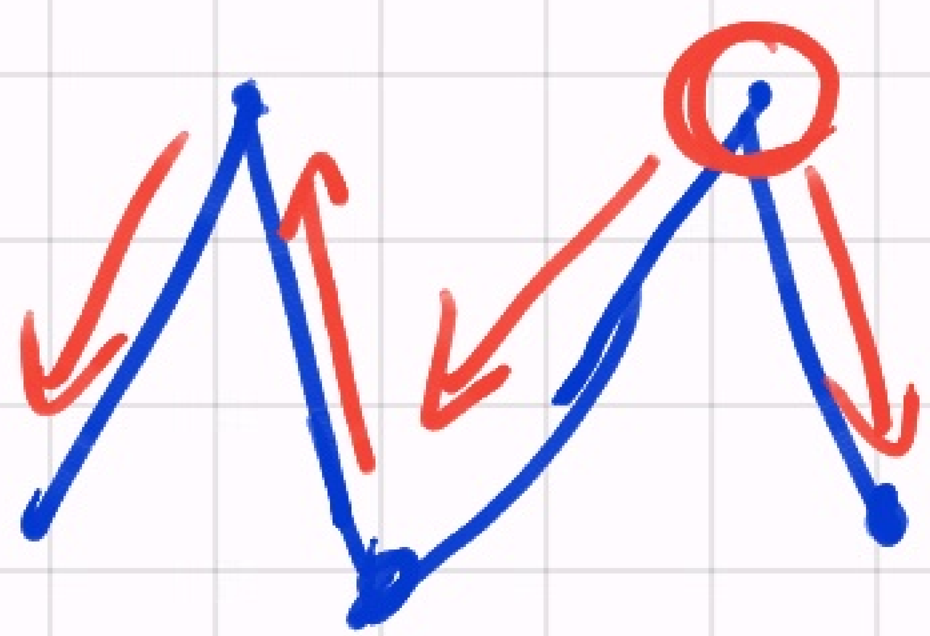
A rooted tree is just (T, v_0) with T a tree and $v_0 \in V(T)$.

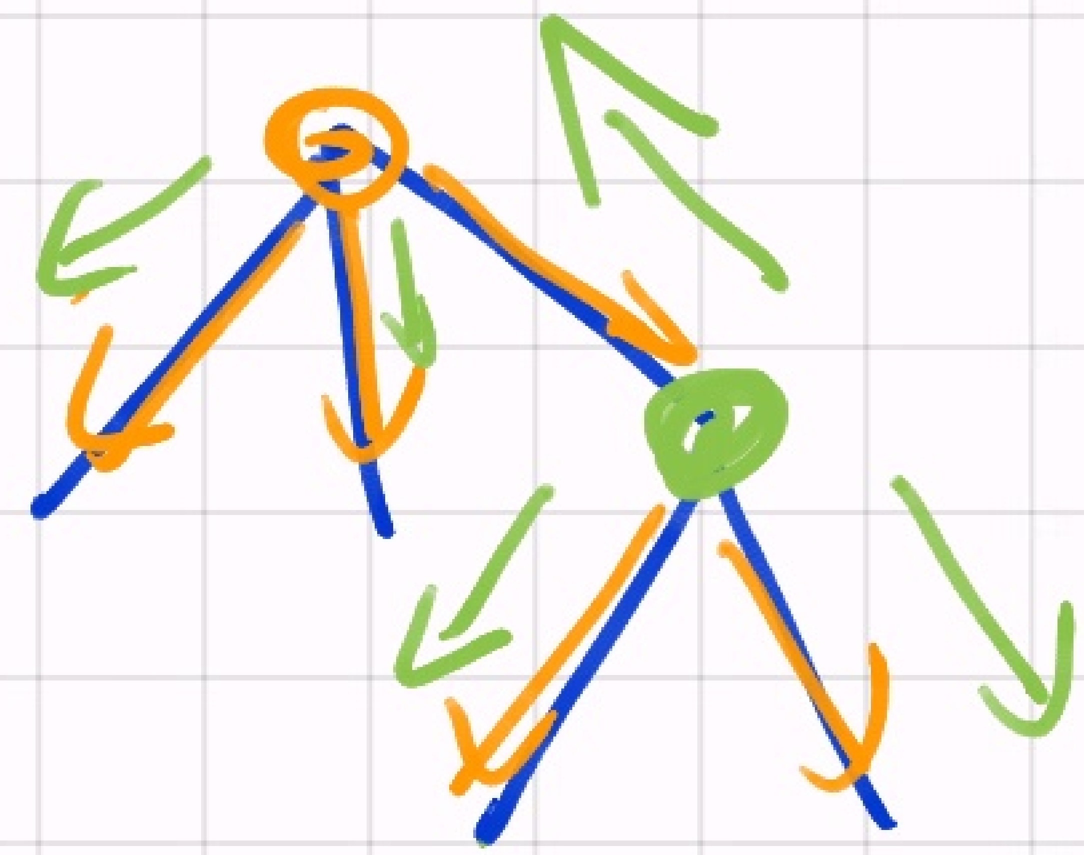
Tree with a distinguished vertex

A directed tree is a directed graph G whose underlying undirected graph is a tree.



Prop if (T, v_0) is a rooted tree then there is a unique orientation in $E(T)$ such that the incoming deg of v_0 is 0 & all the other have incoming degree 1





Proof (T finite)

$(T\ r)$ rooted tree $r \in V(T)$ the root

$\{v, w\} \in E(T)$

there are unique paths

$(r, v_2 \ldots v_n = v)$ joining r and v

$(r\ w_2 \quad w_k = w)$ —— r and w

either $v_{n-1} = w$ we orient $\{v, w\}$ as $(w\ v)$

or $w_{k-1} = v$ —— $(v\ w)$

either $\{v, w\}$ is walked by the first path or not

$v_{n-1} = w$

$w_i = v_i \quad i \leq n-1$

the path does not touch w

the second path uses $\{v, w$

This gives us a well defined orientation.

No edge goes into r (because every path starts from r)

By the unicity of the path connectin r and v there is only one edge going into v

(1) Suppose that there are two different orientatio

there is an edge $\{v, w\}$ which is orientated different,

take the unique path from r to v

(∴ in one of the orientation.

it is oriented pathwise

and in one is not

We can choose a path $(r \quad v_n)$ of minimal length such that

$\{v_n, v\}$ is oriented path wise in one and not the other.

Since for both orientation
in deg $r = 0$
the first edge has the same orientation in both

$$(r = v_1 \;,\; v_2 \quad v_3 \qquad \cdots v_n)$$

$n \geq 3$

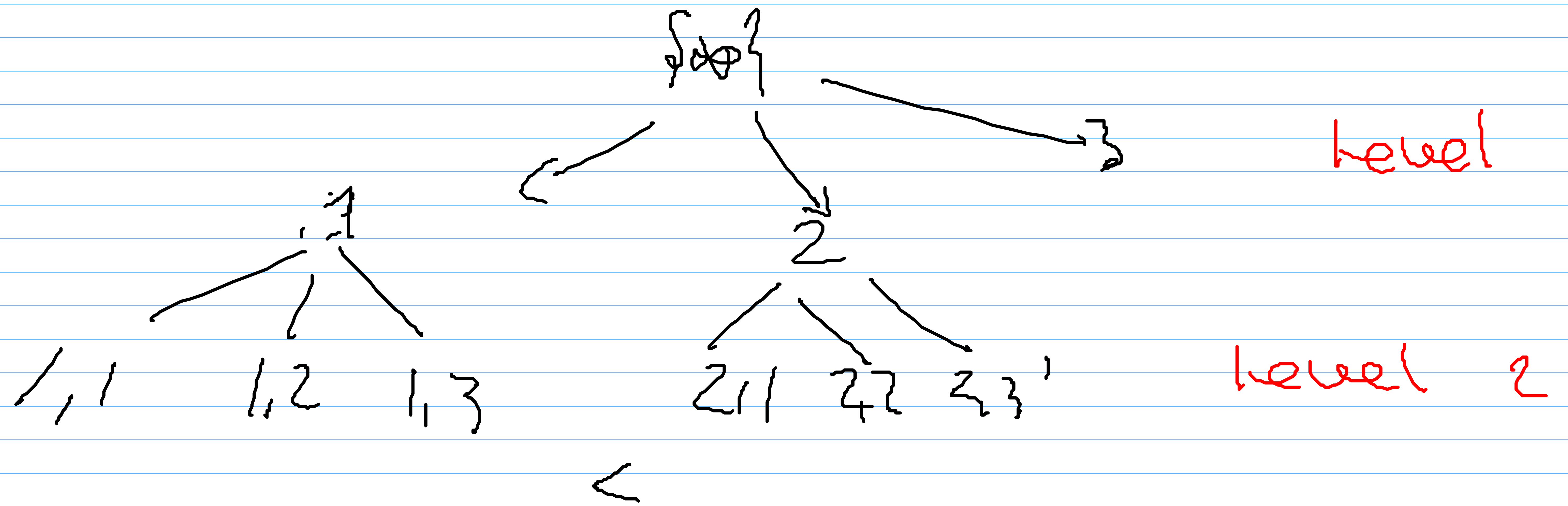
The incoming deg of v_{n-1} is 2 in the orientation "not path wise"
contradicting the assumption. #

This allows us to induce some kind of orders in the vertices

With the mother of all examples.

$\{\emptyset\}$ root

$\{(a), (a,b)\} \longrightarrow ((a), (a,b))$



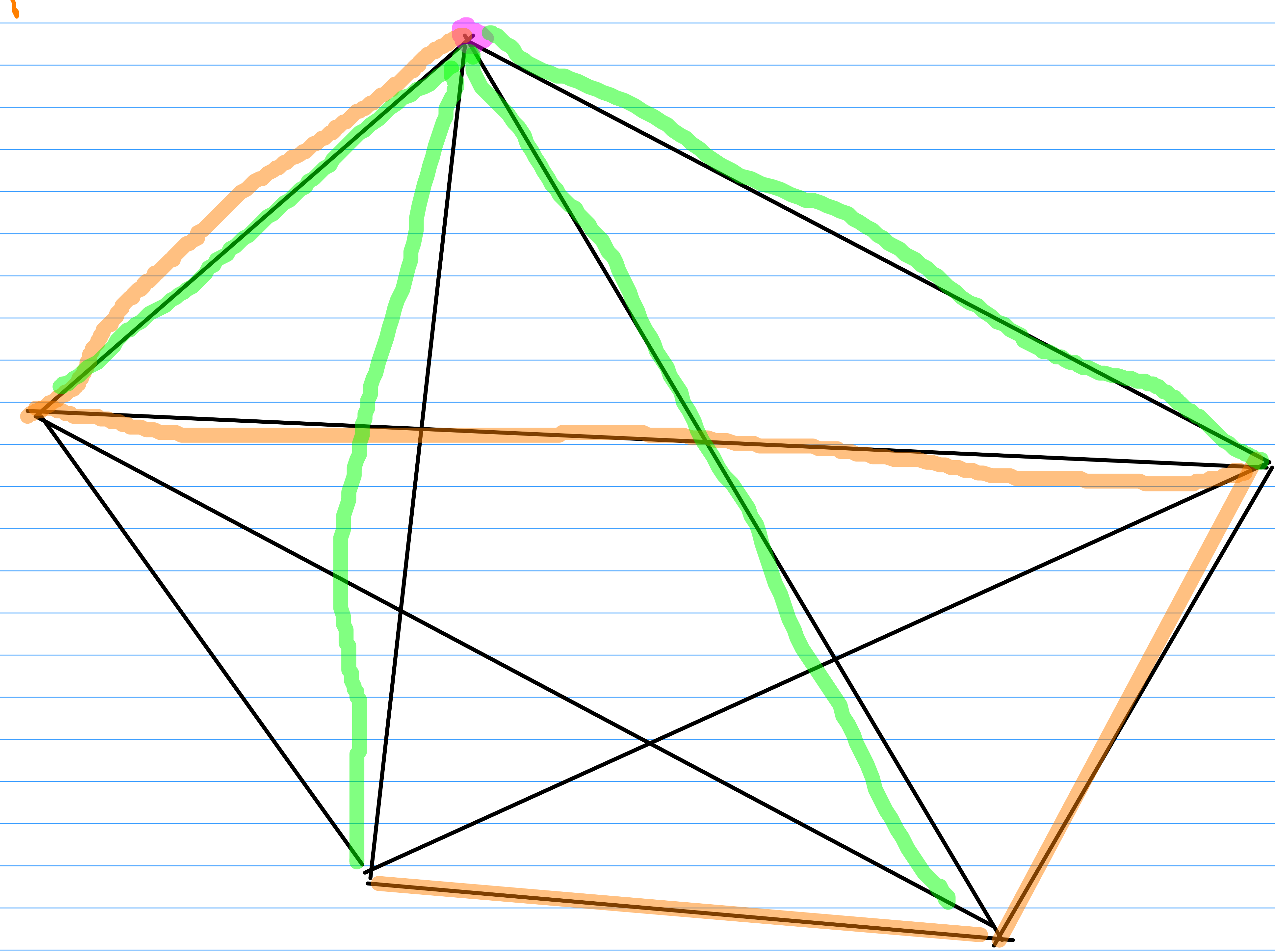
Lexographical order.

There is some kind of lexograpical order in every ~~gree~~ tree

Input G $v \in V(G)$

Output Rooted tree (T, v) with $V(T) = V(G)$

Depth first



Breadth first

Depth first

(V, E) $\qquad V = \{v_1, \dots, v_n\}$

want (T, v_1) spanning tree.

$$\left.\begin{array}{l} T \leftarrow (\{v_1\}, \emptyset) \\ v \leftarrow v_1 \\ i = 1 \end{array}\right\} \text{ initialization.}$$

For $i = 2 \dots n$

if $\exists \{v\ v_i\} \in E(G) \setminus E(T)$

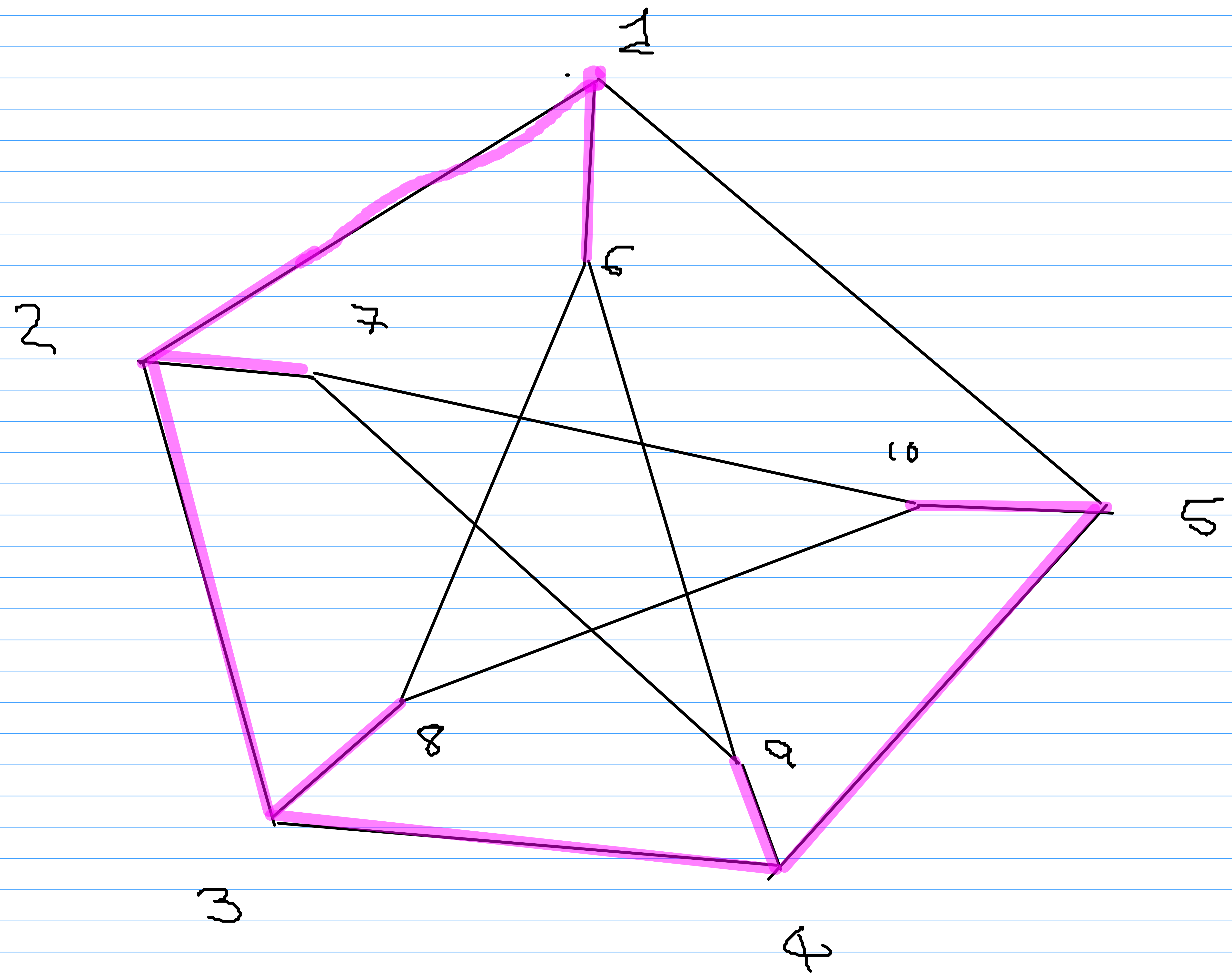
$T \leftarrow T + \{v, v_i\}$

$v \leftarrow v_i$

$i \leftarrow i + 1$

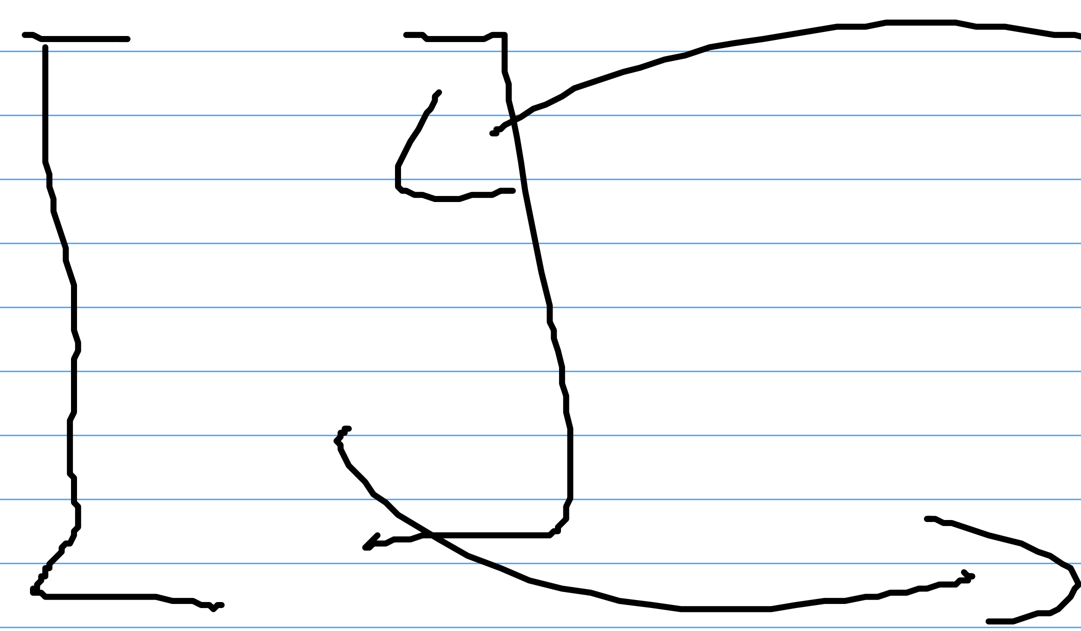
otherwise I replace v with is parent

STOP when $v = v_1$


1
2
7
6
10
5
8
9
3
4

Breadth first

Need a Queue



put things in the rea

use things in the front

$G = \{\{v_1 \ldots v_n\}, E\}$

v_1

every vertices as a label $l \begin{cases} 0 \\ 1 \end{cases}$

$T \leftarrow (v_1, \emptyset)$

$Q = (v_1)$

$l(v_1) = 1$

If v is at the front of the Q

$Q \Leftarrow Q - v$

for $w \sim v$ if $l(w) = 0$

$Q \Leftarrow wQ$ (put w in the rear of the Q)

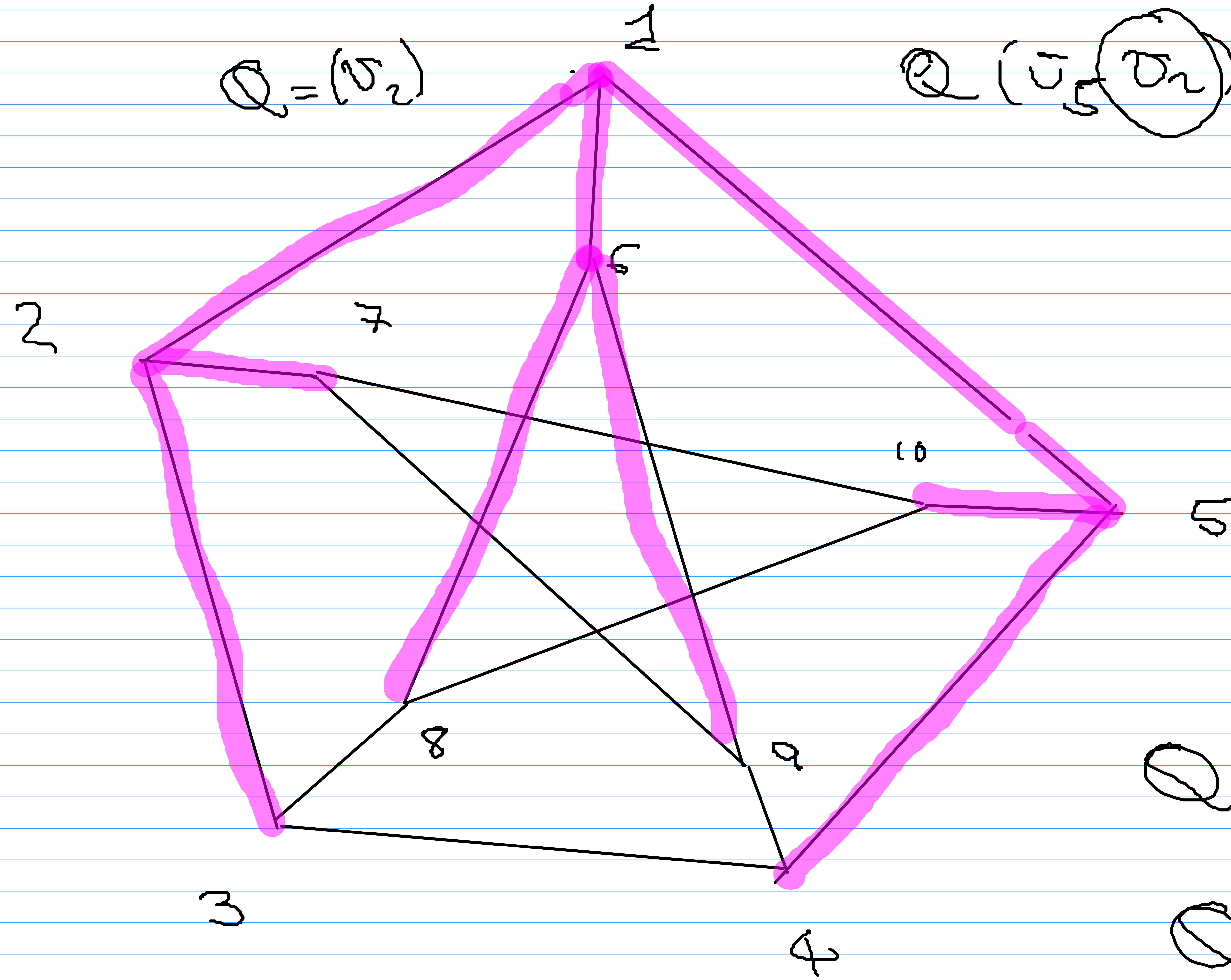
$T \Leftarrow T + \{v, w\}$

$l(w) = 1$

STOP $l(v) = 1$ for all v

$Q = (v_2)$

$Q\ (v_5\ v_2)$



$Q\ (v_6\ v_8\ v_7)$

$Q\ (v_3\ v_6\ v_8)$

$Q\ (v_7\ v_6\ v_8)$

$Q = (v_4\ v_7\ v_6)$

(T, r) rooted tree

r is the root

$v \in V(T)$ the level of v

is $lv(v)$ = length of the unique path from r to v

if $v \sim w$

- $lv(v) < lv(w)$ we say that v is a parent of w

w ——— child of v

if $lv(v) < lv(w)$ and the unique path from r to w passes through v

We say that v is an Ancestor of w

w is a descendants of v

External vertices are called LEAVES