Week 7 - lectures 13 - 14

Combinatorial Optimization II

Max flow min-cut theorem

- Finite Geometry
  - A bit of Algebra
  - Latin Squares
  - Affine planes

Combinatorial Optimization

II

Transport network

Definition 13.1 Let $N = (V, E)$ be a loop-free connected directed graph. Then N is called a *network*, or *transport network*, if the following conditions are satisfied:

a) There exists a unique vertex $a \in V$ with $id(a)$, the in degree of a, equal to 0. This vertex a is called the *source*.

b) There is a unique vertex $z \in V$, called the *sink*, where $od(z)$, the out degree of z, equals 0.

c) The graph N is weighted, so there is a function from E to the set of nonnegative integers that assigns to each edge $e = (v, w) \in E$ a *capacity*, denoted by $c(e) = c(v, w)$.

Example

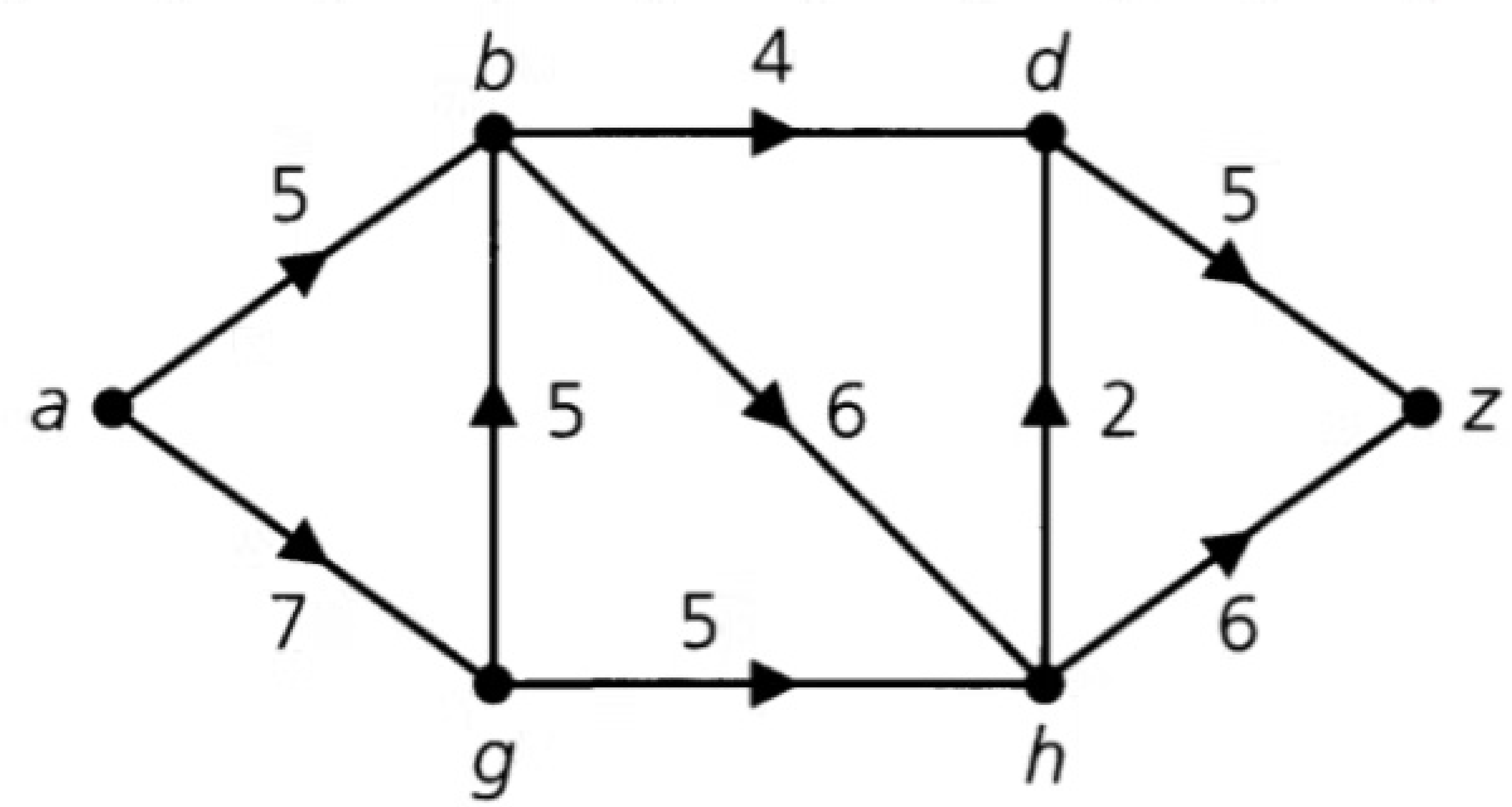


Figure 13.9

c for capacity

Definition 13.2 If $N = (V, E)$ is a transport network, a function f from E to the nonnegative integers is called a *flow* for N if

a) $f(e) \le c(e)$ for each edge $e \in E$; and

b) for each $v \in V$, other than the source a or the sink z, $\sum_{w \in V} f(w, v) = \sum_{w \in V} f(v, w)$. (If there is no edge (v, w), then $f(v, w) = 0$.)

material is generated or lost at the source / sink.

not a flow

Flow

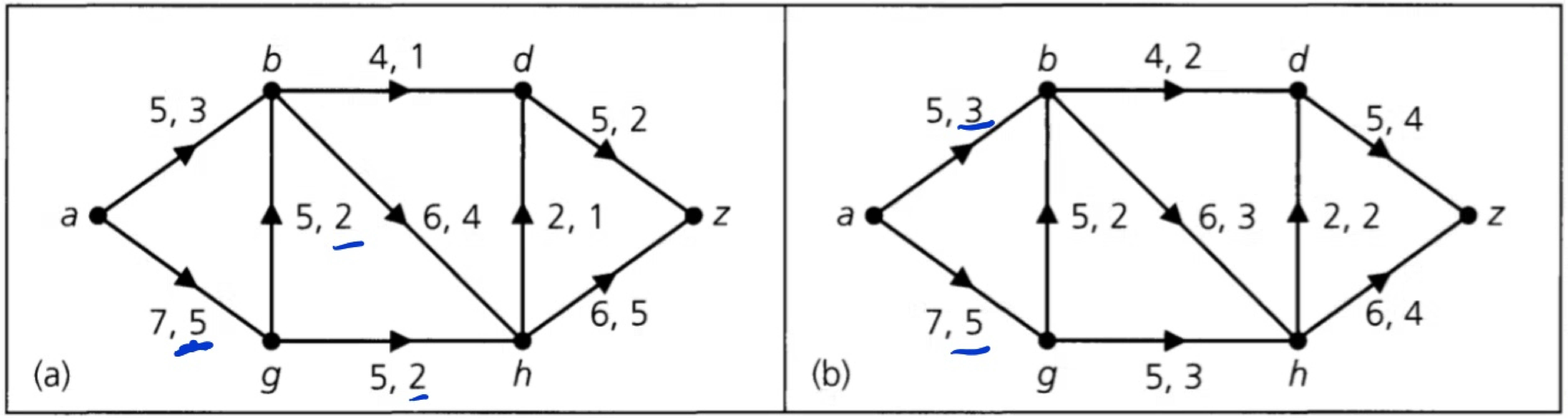


Figure 13.10

$Val(f) = 8$

An edge e is saturated if $c(e) = f(e)$ otherwise It is unsaturated

a source $Val(f) = \sum_{v \in V} f(a, v)$ is the value of the flow

Goal Optimize the flow !

Definition A cut of a (directed) graph is $V(G) = P \sqcup P^c$. If $N = (G, c)$ is a TN a cut of N is a cut of G such that $a \in P$ and $z \in P^c$ $\left(P^c = V(G) \setminus P\right)$

$$c(P, P^c) := \sum_{\substack{v \in P \\ w \in P^c}} c(v, w)$$

$\hookrightarrow$ capacity

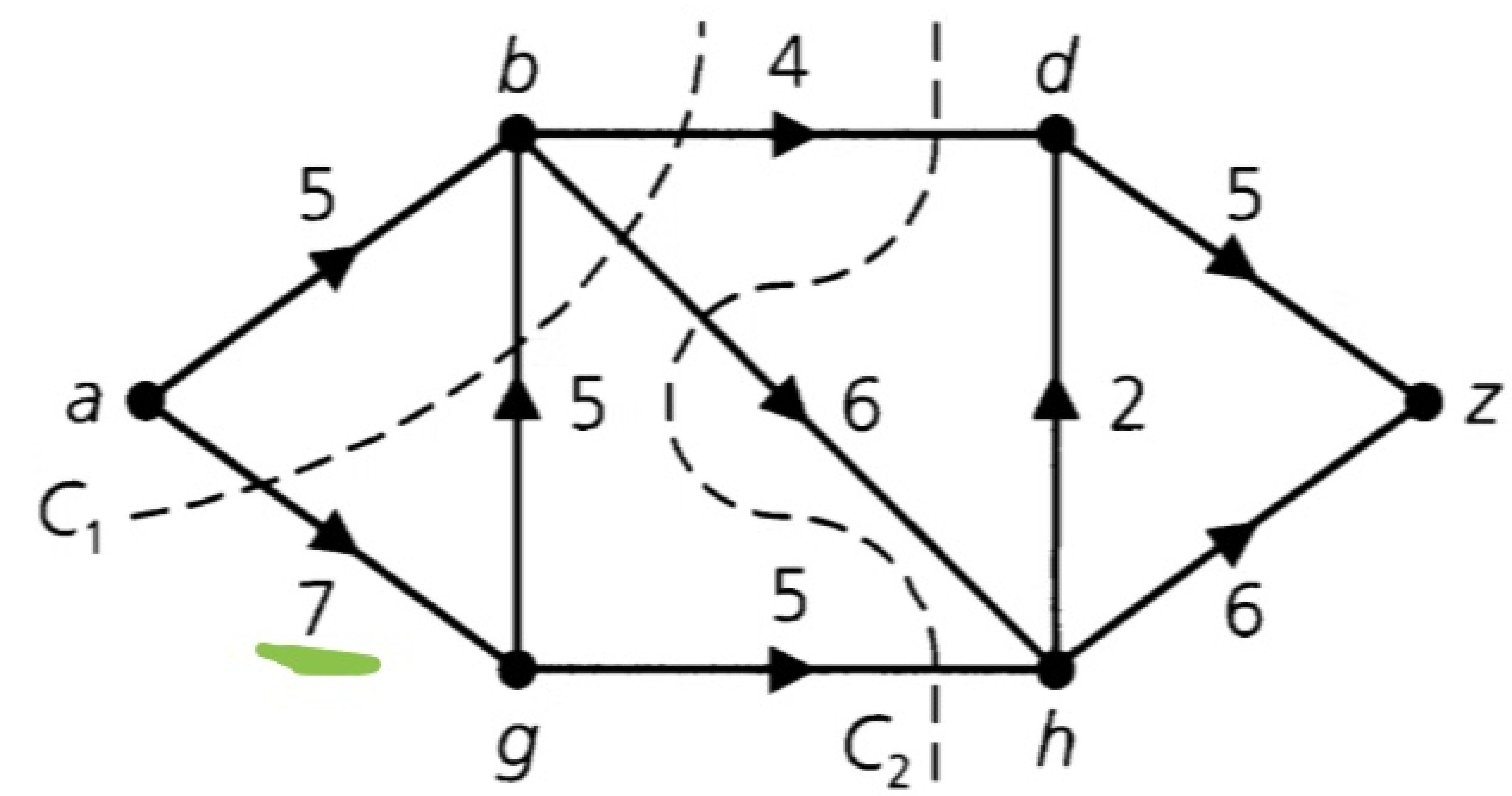


Figure 13.11

$c(P, P^c) = 17$ $P = \{a, b\}$ $P^c = \{g, d, h, z\}$

Theorem N a TN f a flow, for any cut $N=(P,P^c)$ we have that

$$\boxed{val(f) \leq c(P,P^c)}$$

Lemma $val(f) = \sum_{\substack{u\in P\\ w\in P^c}} f(u,w) - f(w,u)$

Proof

$$Val(f) = \sum_{v\in V} f(a,v) \quad =$$

$id(a) = 0 \qquad f(w,a) = 0 \;\; \forall w$

$$\mathrm{Val}(f) = \sum_{v \in V} f(a,v) - f(v,a)$$

$$= \sum_{v \in V} f(a,v) - f(v,a) + \underbrace{\sum_{\substack{x \in P \\ x \neq a}} \left(f(x,v) - f(v,x) \right)}_{\substack{\text{balancing condition} \\ \text{this is } 0\,!}}$$

$(z \notin P)$

$$\sum_{\substack{x \in P \\ v \in V = P \cup P^c}} f(x,v) - f(v,x)$$

Proof of the theorem.

$$\mathrm{Val}(f) = \sum_{\substack{x\in P \\ v\in P^c}} f(xv) - \sum_{\substack{x\in P \\ v\in P^c}} f(vx)$$

$$\leq \sum_{\substack{x\in P \\ v\in P^c}} \underline{f(xv)} \leq \sum_{\substack{x\in P \\ v\in P^c}} c\underline{(xv)}$$

$$= c(P, P^c)$$

#

Remark : $val(f) = c(P, P^c)$

$\iff$ 1) $\displaystyle\sum_{\substack{x \in P \\ v \in P^c}} f(v\,x) = 0$

$\iff f(v, x) = 0 \quad \forall\ x \in P,\ v \in P^c$

edges going from P^c to P have 0 flow

2) $f(x\,v) = c(x\,v) \quad \forall\ x \in P,\ v \in P^c$

When you go toward max ≥ the flow is ~~cut~~ capacity

$$= \cancel{\sum_{\substack{x \in P \\ v \in P}} f(xv)} + \sum_{\substack{x \in P \\ v \in P^c}} f(xv) - \cancel{\sum_{\substack{x \in P \\ v \in P}} f(vx)}$$

$$- \sum_{\substack{x \in P \\ v \in P^c}} f(vx)$$

$$= \sum_{\substack{x \in P \\ v \in P^c}} f(xv) - \sum_{\substack{x \in P \\ v \in P^c}} f(vx)$$

Max Flow min Cut Theorem

Given a transport network N the max value of a flow is the min value of the capacity of its cuts.

Constructive gives us a way to construct the max flow.

Lemma $\mathrm{val}(f) = c(P, P^c) \iff$

1) $f(e) = c(e)$ for each edge (x,y) with $x \in P$ and $y \in P^c$

2) $f(e) = 0$ $\forall$ edge $(x\,y)$ with $y \in P$ and x P^c

under these circumstance f is a maximum flow and (P, P^c) a minimum cut.

Proof: See the remark before

A semipath in a network $N = (G, c)$ is a <u>undirected path</u> ($v_1 = a$ v_2 ... $v_n = z$)

If $e = (v_i\ v_{i+1}) \in E(G)$ we say that this is a forward edge. Otherwise if $(v_{i+1}, v_i) \in E(G)$ it is a backward edge.

A semipath is f-augmenting if for every edge

$f(e) < c(e)$ for e forward.

$f(e) > 0$ for e backward.

Given p an f-augmenting path in a network and e an edge of p

$$\Delta_e := \begin{cases} c(e) - f(e) > 0 & \text{if } e \text{ forward} \\ f(e) > 0 & \text{if } e \text{ backward} \end{cases}$$

Is the tolerance of e

$$\boxed{\Delta_p = \min_{e \in p} \{ \Delta_e \}}$$

$\vee$

0

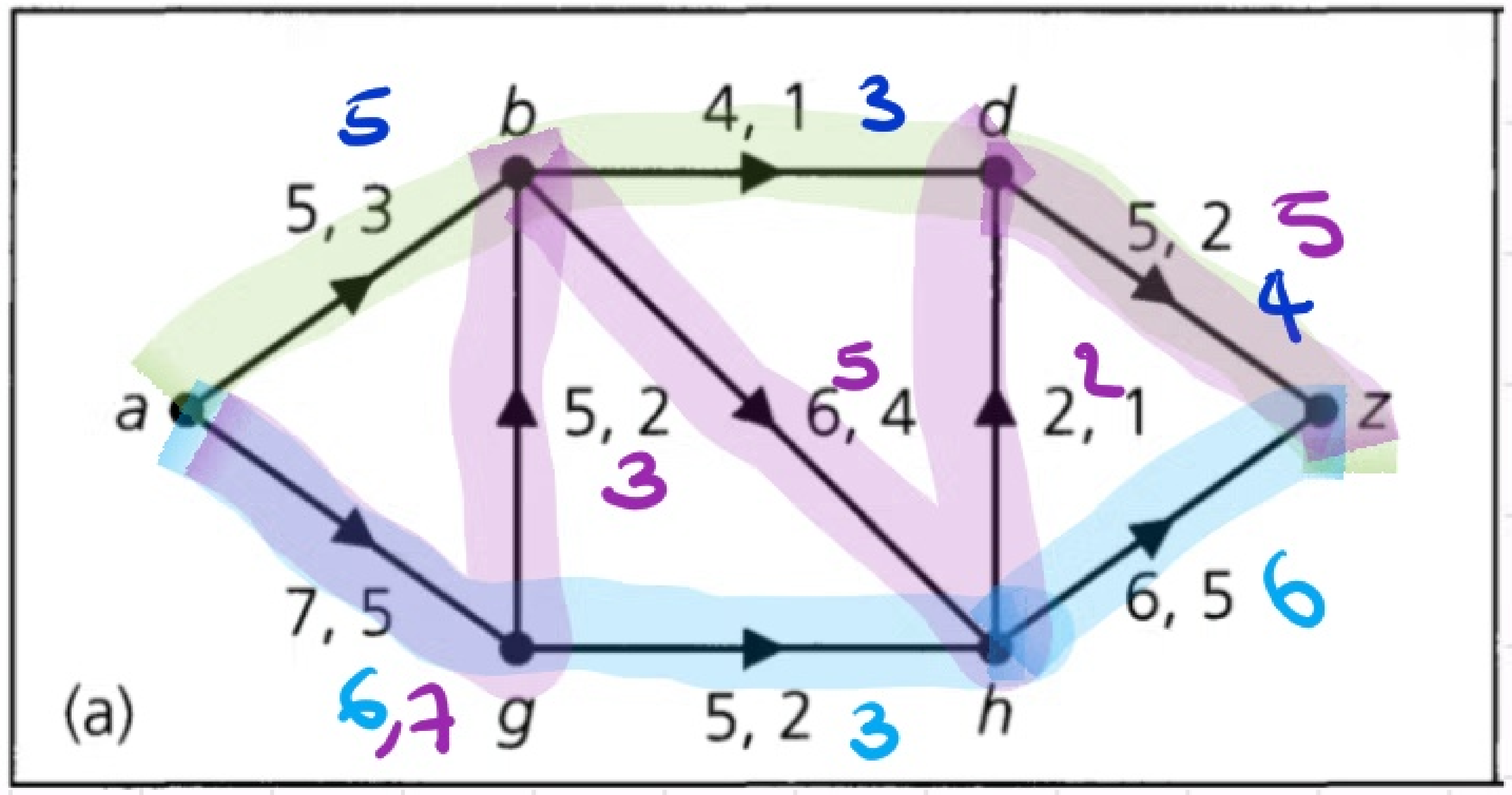
How much we can modify the flow in each edge.

THEOREM 13.4 Let f be a flow in a transport network $N = (V, E)$ and let p be an f-augmenting path in N with $\Delta_p = \min_{e \in p}\{\Delta_e\}$. Define $f_1 \colon E \to \mathbf{N}$ by

$$f_1(e) = \begin{cases} f(e) + \Delta_p, & e \in p,\ e \text{ a forward edge} \\ f(e) - \Delta_p, & e \in p,\ e \text{ a backward edge} \\ f(e), & e \in E,\ e \notin p. \end{cases}$$

Then f_1 is a flow in N with $\text{val}(f_1) = \text{val}(f) + \Delta_p$. $> \text{val}(f)$

THEOREM 13.5 Let $N = (V, E)$ be a transport network with flow f. The flow f is a maximum flow in N if and only if there exists no f-augmenting path in N.



Goal maximise the flow

- look for f-augmenting path

- tweek the flow
- If there are no f-augmenting path we are done

$$\Delta p = \min\{5-3, 4-1, 5-2\} = 2$$

$$\Delta p = \min\{7-5, 5-2, 6-5\} = 1$$

$\Delta_P = 1$

We reached the end : if I leave a
I have to travel either $\{ab\}$ or $\{ag\}$
both forward. An they are at full
capacity $\Rightarrow$ no f-augmenting path

max val of flow $7+5 = 12$.

How you find the "min cut"

Take all the partial - f augmenting paths from a (they do not reach z)

P = {vertices you touch with these path}

$P = \{a\}$ $\qquad$ $P^c = \{$ all the other $\}$

The Edmonds-Karp Algorithm

Step 1: Place the source a into set P (thus initializing the set of processed vertices.) Assign the label $(\ , 1)$ to a and set the counter $i = 2$.

$\to f(e) < c(e)$ f
or $f(e) > 0$
if b.

Step 2: While the sink z is not in P

If there is a usable edge in N

Let $e = \{v, w\}$ be usable with labeled vertex v having the smallest counter assignment

If w is unlabeled

Label w with (v, i)

Place w in P

Increase the counter i by 1.

Else

Return the minimum cut $(P, \overline{P})$.

Step 3: If z is in P, start with z and backtrack to a using the first component of the vertex labels. (This provides an f-augmenting path p with the smallest number of edges.)

$\hookrightarrow$ Gives an f-augmenting path.

The Ford-Fulkerson Algorithm

Step 1: Define the initial flow f on the edges of N by $f(e) = 0$ for each $e \in E$.

Step 2: Repeat

Apply the Edmonds-Karp algorithm to determine
an f-augmenting path p.
Let $\Delta_p = \min_{e \in p}\{\Delta_e\}$.
For each $e \in p$
If e is a forward edge
$f(e) := f(e) + \Delta_p$
Else (e is a backward edge)
$f(e) := f(e) - \Delta_p$
Until no f-augmenting path p can be found in N.
Return the maximum flow f.

Step 3: Return the minimum cut $(P, \overline{P})$ (from the last application of the Edmonds-Karp algorithm, where no further f-augmenting path could be constructed).

Finite Geometry

Finite Fields

Let $(F, +, \cdot, 1, 0)$ be a field. If $|F| < +\infty$

Then F has characteristic p, with p a pirime. That is

$$p = \min \{ n \in \mathbb{N}, n > 0 \mid \underbrace{1 + \quad + 1}_{n\text{-times}} = 0 \}$$

Then it can be proven that $|F| = p^n$ for some positive integer n and $F \simeq \mathbb{Z}/p\mathbb{Z}[x] \big/ (p(x))$

Irreducible poly of degree n.

if F and F' are two finite fields and

$$|F| = |F'|$$

then $F \cong F'$ as fields

Notation

if $q = p^n$

$\mathbb{F}_q$ = field with q - elements

Latin Squares

Definition 17.9 An $n \times n$ *Latin square* is a square array of symbols, usually $1, 2, 3, \ldots, n$, where each symbol appears exactly once in each row and each column of the array.

Table 17.3

Auto	Day			
	Mon	Tues	Wed	Thurs
A	1	2	3	4
B	2	1	4	3
C	3	4	1	2
D	4	3	2	1

Table 17.4

Auto	Day			
	Mon	Tues	Wed	Thurs
A	1	2	3	4
B	3	4	1	2
C	4	3	2	1
D	2	1	4	3

Orthogonal latin squares

Definition 17.10 Let $L_1 = (a_{ij})$, $L_2 = (b_{ij})$ be two $n \times n$ Latin squares, where $1 \leq i, j \leq n$ and each a_{ij}, $b_{ij} \in \{1, 2, 3, \ldots, n\}$. If the n^2 ordered pairs (a_{ij}, b_{ij}), $1 \leq i, j \leq n$, are distinct, then L_1, L_2 are called a *pair of orthogonal Latin squares*.

Table 17.5

	Day			
Auto	Mon	Tues	Wed	Thurs
A	(1, 1)	(2, 2)	(3, 3)	(4, 4)
B	(2, 3)	(1, 4)	(4, 1)	(3, 2)
C	(3, 4)	(4, 3)	(1, 2)	(2, 1)
D	(4, 2)	(3, 1)	(2, 4)	(1, 3)

Definition 17.11 If L is an $n \times n$ Latin square, then L is said to be in *standard form* if its first row is $1 \quad 2 \quad 3 \quad \cdots \quad n$.

THEOREM 17.14 Let L_1, L_2 be an orthogonal pair of $n \times n$ Latin squares. If L_1, L_2 are standardized as L_1^*, L_2^*, then L_1^*, L_2^* are orthogonal.

Theorem for any $n \in \mathbb{N}$ there are at most $n-1$ of mutually orthogonal latin squares in standard form

If $n = p^t$ for some t then there are exactly $n-1$ mutually orthogonal latin squares in standard form.

Proof $L_1 \dots L_k$ are distinguished latin squares in standard form which are pairwise athogonal. We will show that $k \leq n-1$

$$L_m = \left(a_{ij}^{(m)} \right)$$

$$\begin{array}{|ccccc} 1 & 2 & 3 & \dots & n \\ \vdots & & & & \\ & & & & \\ \hline \end{array}$$

Std $\Rightarrow$ $a_{1j}^{(m)} = j \quad \forall \; j = 1 \dots n \quad \forall \; m = 1 \dots k$

$$a_{21}^{(m)} \neq 1$$

Mutually athogonal $\Rightarrow$ $a_{21}^{(l)} \neq a_{21}^{(m)} \quad \forall \; l \neq m$

If $k = n-1$ then you have only one choice left for $a_{21}^{(k)}$. If $k > n-1$ you have no choice but to have $a_{21}^{(k)} = a_{21}^{(l)}$ for some $l < n-1$ $\Rightarrow$ The square cannot be orthogonal.

Now suppose that $n = p^t$ with p prime $t \in \mathbb{N}$ $t \geq 0$

$$\mathbb{F}_n = \{ \underset{\underset{1}{\shortparallel}}{f_1}, \ldots, \underset{\underset{0}{\shortparallel}}{f_n} \}$$

$$\{1 \dots n\} \hookrightarrow \{f_1 \dots f_n\}$$

consider these simbols

$m = 1 \dots n-1$ $\quad i, j : 1 \dots n$

$$a_{ij}^{(m)} = f_m \cdot f_i + f_j \quad \in \mathbb{F}_n$$

It is one of the symbols $f_1 \dots f_n$.

We need to show

1) $L_m = \left(a_{ij}^{(m)}\right)$ is a Latin square.

2) L_m & L_k are orthogonal if $k \neq m$.

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1) $a_{ij}^{(m)} = a_{jk}^{(m)} \overset{?}{\Longrightarrow} j = k$

$a_{ij}^{(m)} = a_{lj}^{(m)} \overset{?}{\Longrightarrow} i = l$

$$a_{ij}^{(m)} = \cancel{f_m} \cdot f_i + f_j = \cancel{f_m} \cdot f_i + f_k = a_{ik}^{(m)}$$

$$-f_m f_i \qquad\qquad -f_m f_i$$

$$f_j = f_k \implies j = k.$$

$$a_{ij}^{(m)} = f_m \cdot f_i + \cancel{f_j} = f_m \cdot f_l + \cancel{f_j} = a_{lj}^{(m)}$$

$$\cancel{f_m^{-1}} f_m \cdot f_j = \cancel{f_m^{-1}} f_m \cdot f_i$$

$\mathbb{F}_n$ field $f_m \neq 0 \implies$ it is a unit $\exists\, f_m^{-1}$

$$f_i = f_j \implies i = j$$

They are orthogonal.

$$a_{ij}^{(k)} = a_{rs}^{(k)} \quad , \quad a_{ij}^{(m)} = a_{rs}^{(m)} \overset{?}{\Longrightarrow} k=m!$$

$$\begin{cases} f_k f_i + f_j = f_k f_r + f_s \\ f_m f_i + f_j = f_m f_r + f_s \end{cases}$$

we subtract one from the other

$$(f_m - f_k) f_i = (f_m - f_k) f_r$$

if $k \neq m$ $\quad f_m - f_k \neq 0 \Rightarrow$ is a unit.

$$f_i = f_r \qquad \Rightarrow i = r. \quad \Rightarrow j = j \ \#$$

Take away from the proof

1) Generate Latin squares with p^t elements

$u \neq 0$ in $\mathbb{F}_q$ $\quad i, j \in \mathbb{F}_q$

$a_{ij} = u \cdot i + j$ is a latin square

2) Generate orthogonal latin squares

$u_1, u_2 \neq 0$ in $\mathbb{F}_q$

$a_{ij} = u_1 i + j \qquad b_{ij} = u_2 i + j$

Example $n=3$ $\mathbb{F}_3 = \{1, 2, 0\}$ $1, 2$ are unit

$$a_{ij} \equiv 1\cdot i + j \mod 3$$

$i=0$

$i=1$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

only two possible!

$$b_{ij} \equiv 2\cdot i + j \mod 3$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

Try to do $\mathbb{F}_5 \cong \mathbb{Z}/5\mathbb{Z}$

Finite Geometry

Definition 17.12 Let $\mathcal{P}$ be a finite set of points, and let $\mathcal{L}$ be a set of subsets of $\mathcal{P}$, called lines. A (*finite*) *affine plane* on the sets $\mathcal{P}$ and $\mathcal{L}$ is a finite structure satisfying the following conditions.

A1) Two distinct points of $\mathcal{P}$ are (simultaneously) in only one element of $\mathcal{L}$; that is, they are on only one line.

Euclide's V

A2) For each $\ell \in \mathcal{L}$, and each $P \in \mathcal{P}$ with $P \notin \ell$, there exists a unique element $\ell' \in \mathcal{L}$ where $P \in \ell'$ and ℓ, ℓ' have no point in common.

A3) There are four points in $\mathcal{P}$, no three of which are collinear (that is, no three of these four points are in any one of the subsets $\ell \in \mathcal{L}$).

Figure 17.1

The reason for condition (A3) is to avoid uninteresting situations like the one shown in Fig. 17.1. If only conditions (A1) and (A2) were considered, then this system would be an affine plane.

Example

$\mathbb{A}^2(\mathbb{F}_q) = \{(a,b) \ / \ a,b \in \mathbb{F}_q\} = \mathcal{P}, \quad |\mathcal{P}| = n^2$

Lines :

$x = a \quad a \in \mathbb{F}_q$ $\qquad\qquad y = mx + b \qquad\qquad m, b \in \mathbb{F}_q$

vertical lines $\qquad\qquad$ all the other.

$|\mathcal{L}| = n(n+1)$

How many points in the vertical lines? q

and in any other line? q

$(a, y) \qquad y \in \mathbb{F}_q \rightsquigarrow q$

How many lines through a given point ?

vertical + a line for every $m \in \mathbb{F}_q$

$q+1$

Two points determine a line (A1)

$P_1 = (a, b)$ $P_2 = (c, d)$ $P_1 \neq P_2 \in \mathcal{P}$

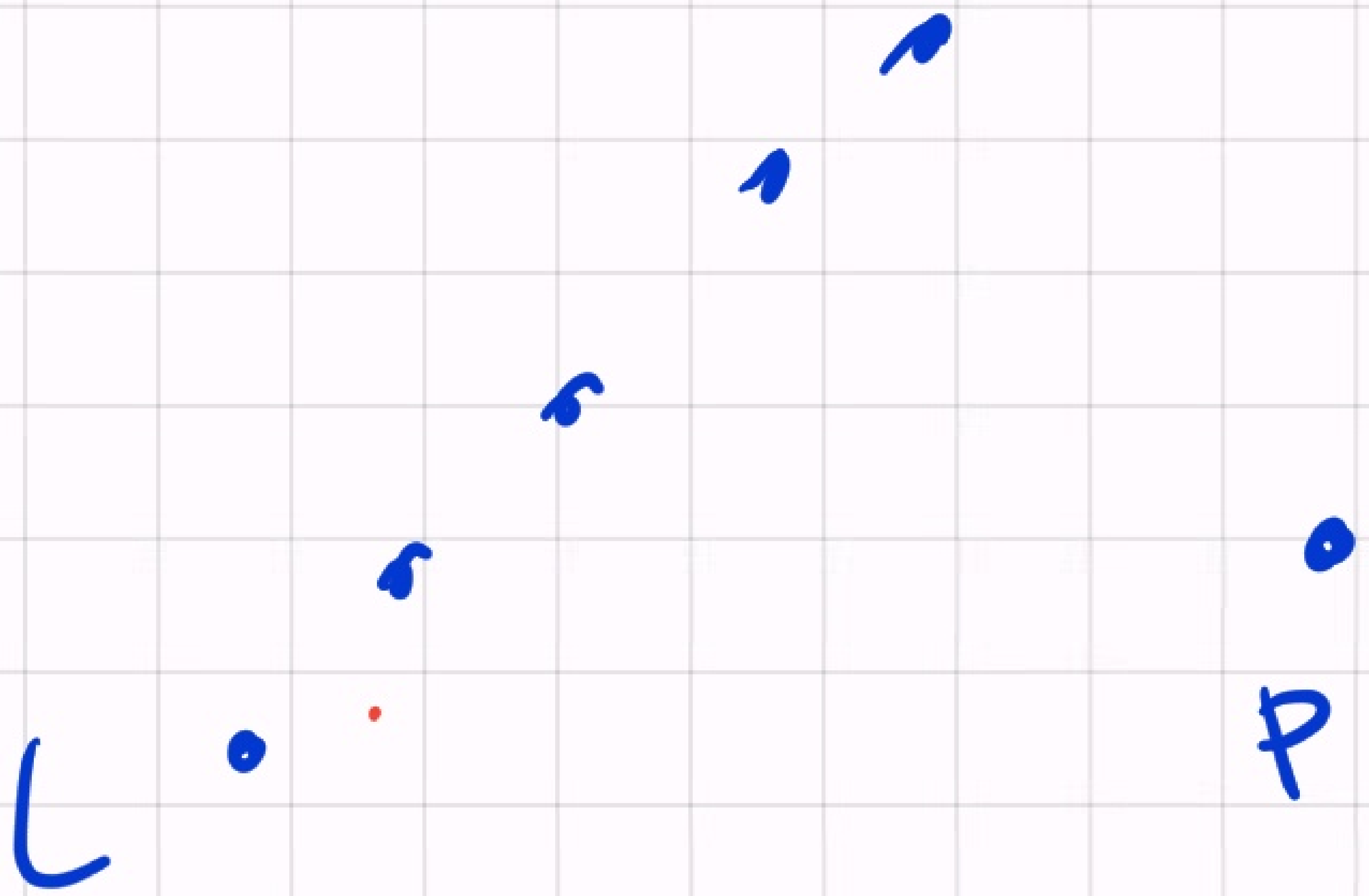
- $a = c$ $(b \neq d)$ $\mathcal{L}_{ab}$: $x = a$

- $a \neq c$ $m = (a-c)^{-1}(b-d) = b-d / a-c$

  $\hookrightarrow$ $a-c$ is a unit in $\mathbb{F}_q$!

$\mathcal{L}_{P_1P_2}$: $y = m(x-a) + b$

Euclides's V



The line L has q points for every $P_i \in L$ we can construct $\mathcal{L}_{P_i P}$ $\rightsquigarrow$ q lines through P

There is an additional line through P that does not meet L.

$\mathbb{F}_q \ni 1, 0$

$\mathbb{A}^2(\mathbb{F}_q) \ni (0,0) \quad (1,0) \quad (1\ 1) \quad (0,1)$

Four points. And not three are on a line.

$y=0$ is the unique line through $(0,0)$ & $(1,0$

$(1,1) \notin \{y=0\}$

EXAMPLE 17.18

For $F = (\mathbf{Z}_2, +, \cdot)$, we have $n = |F| = 2$. The affine plane in Fig. 17.3 has $n^2 = 4$ points and $n^2 + n = 6$ lines. For example, the line $\ell_4 = \{(1, 0), (1, 1)\}$, and ℓ_4 contains no other points that the figure might suggest. Furthermore, ℓ_5 and ℓ_6 are parallel lines in this finite geometry because they do not intersect.

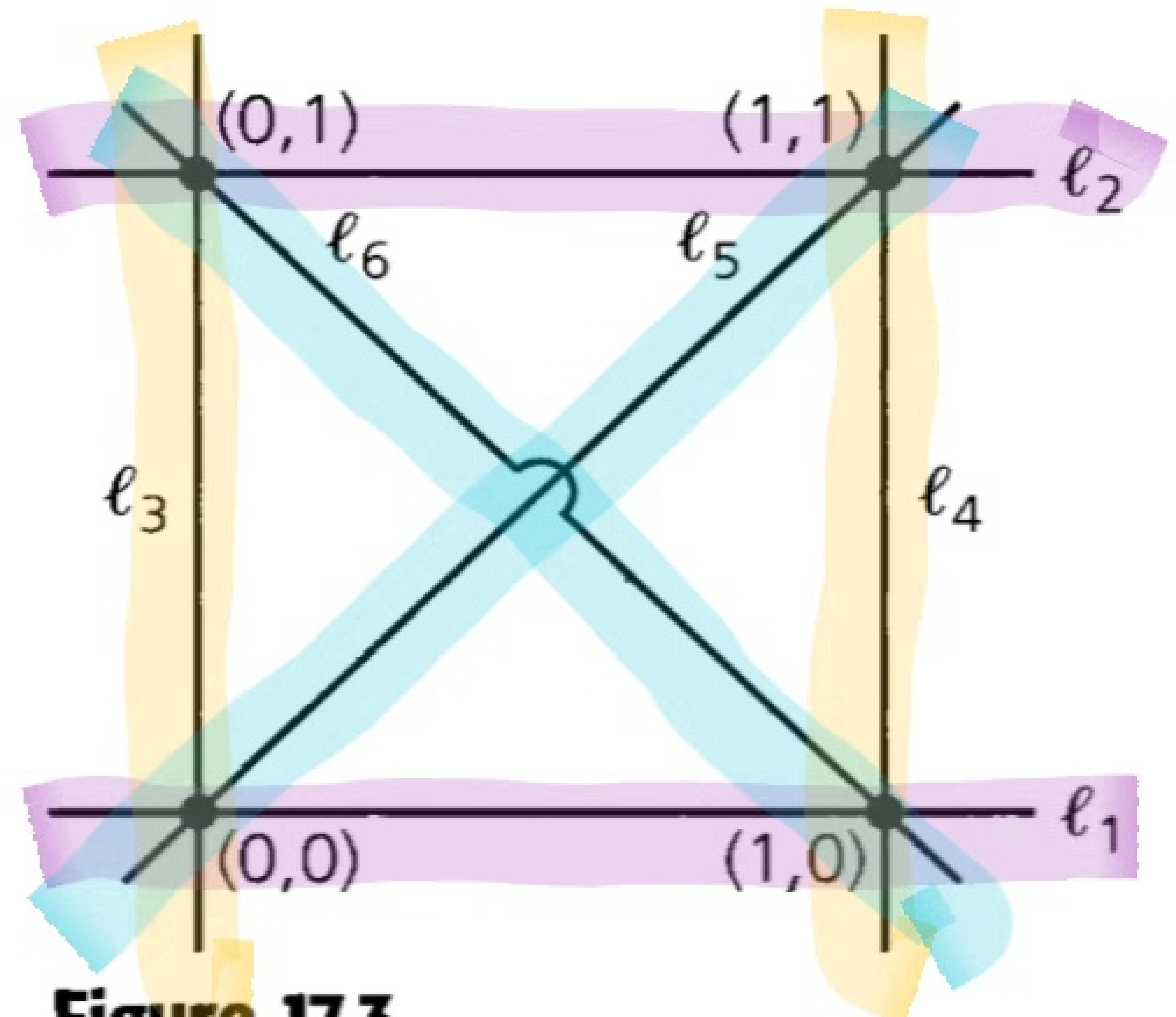


Figure 17.3

Def we say that 2 lines L_1 & L_2 in an affine plane $(\mathcal{P}, \mathcal{L})$ are parallel if

$$L_1 = L_2 \quad \text{or} \quad L_1 \cap L_2 = \emptyset$$

This is an equivalence relation.

$\rightsquigarrow$ equivalence classes are called **parallelity classes**.

How many? $\quad q+1$

Mutually orthogonal latin squares & parallelity class that are not vertical or horizontal

$$L = \{ mx + f_1 \ , \ \underline{mx + f_2} \ \ldots \ , \ mx + f_q \}$$

$$m \in \mathbb{F}_q \setminus \{0\} \qquad f_i \in \mathbb{F}_q$$