Mm5023 lecture 13

Transport Networks

Plan

- Networks and flows
- The max flow problem ← important one
- The min cut problem
- The max flow min cut theorem (how to solve the max flow problem)
- Applications

Definition of transport network

A transport network is a connected directed graph with no loop $G = (V, E)$ together with a "capacity" function

$$c : V \times V \longrightarrow \mathbb{N}$$ such that

1) There are unique vertices $S, T \in V$ such that

   $\text{indeg } S = 0$ — S is called the source

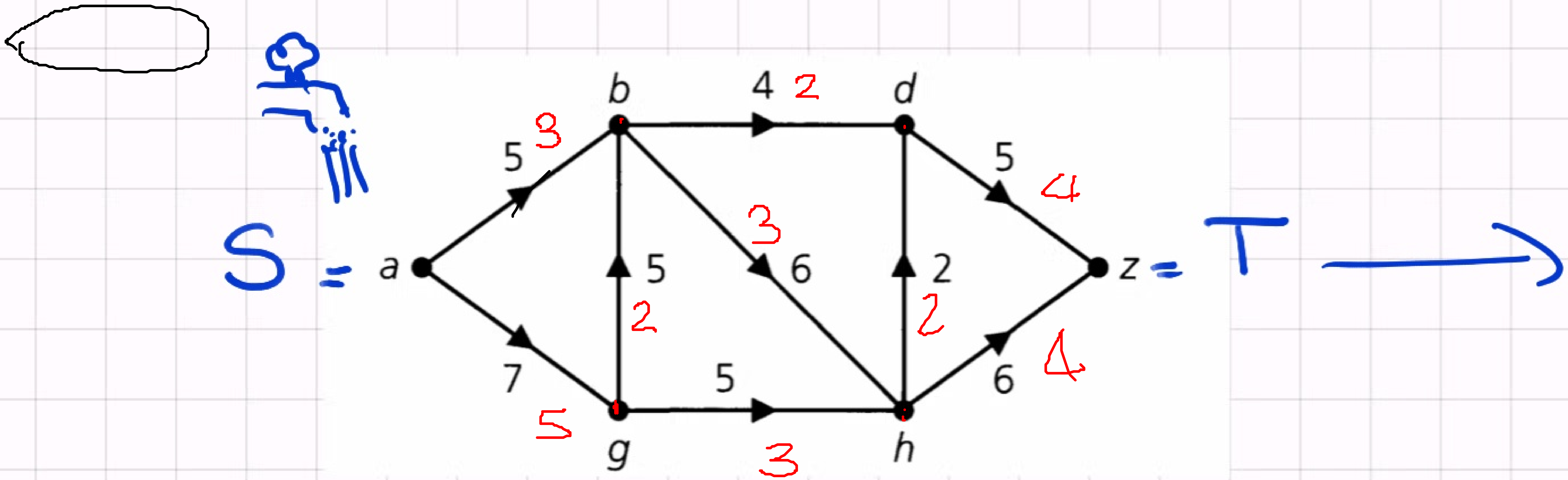
   $\text{outdeg } T = 0$ — T is called the sink

2) $c(v, w) = 0$ when $v \nsim w$

$c(a,h) = 0$

$c(a,b) = 5$

Example



Network = special weighted directed graph

① weight are in $\mathbb{N}^+$

② There are Source and sink

Flows

Let (G, c, S, T) be a transport network N
a FLOW on N is a function

$$f : E(G) \longrightarrow \mathbb{N}$$

such that

1) $f(v,w) \leq c(v,w)$ for all $(v,w) \in E(G)$

2) (balancing condition)
$\forall v \in V(G) \setminus \{S, T\}$

$$\sum_{w \in V} f(w,v) = \sum_{w \in V} f(v,w)$$

$$\left(f : V^2 \longrightarrow \mathbb{N} \quad f(v,w) = 0 \text{ if } (v,w) \notin E(G) \right)$$

"No new transported material is created other than in S
No material exist other than in T"

Given a flow f on a Network $N = (G, c, S, T)$
the value of the flow

is

$$\mathrm{Val}(f) := \sum_{v \in V} f(S, v)$$

We are going to see that

$$\mathrm{Val}(f) = \sum_{v \in V} f(v, T)$$

The max flow problem

Given a network $N = (V, E, c)$ provide a flow f such that $Val(f)$ is maximal.

s, t

Capacities limit the # of possible flow

This exist by the well ordering of $\mathbb{N}$

$\{ Val(f) \mid f \in \text{flow on } \mathbb{N} \} \subseteq \mathbb{Z}$

bounded above by $\sum_{v \in V} c(s, v)$ $\Rightarrow$ has a max

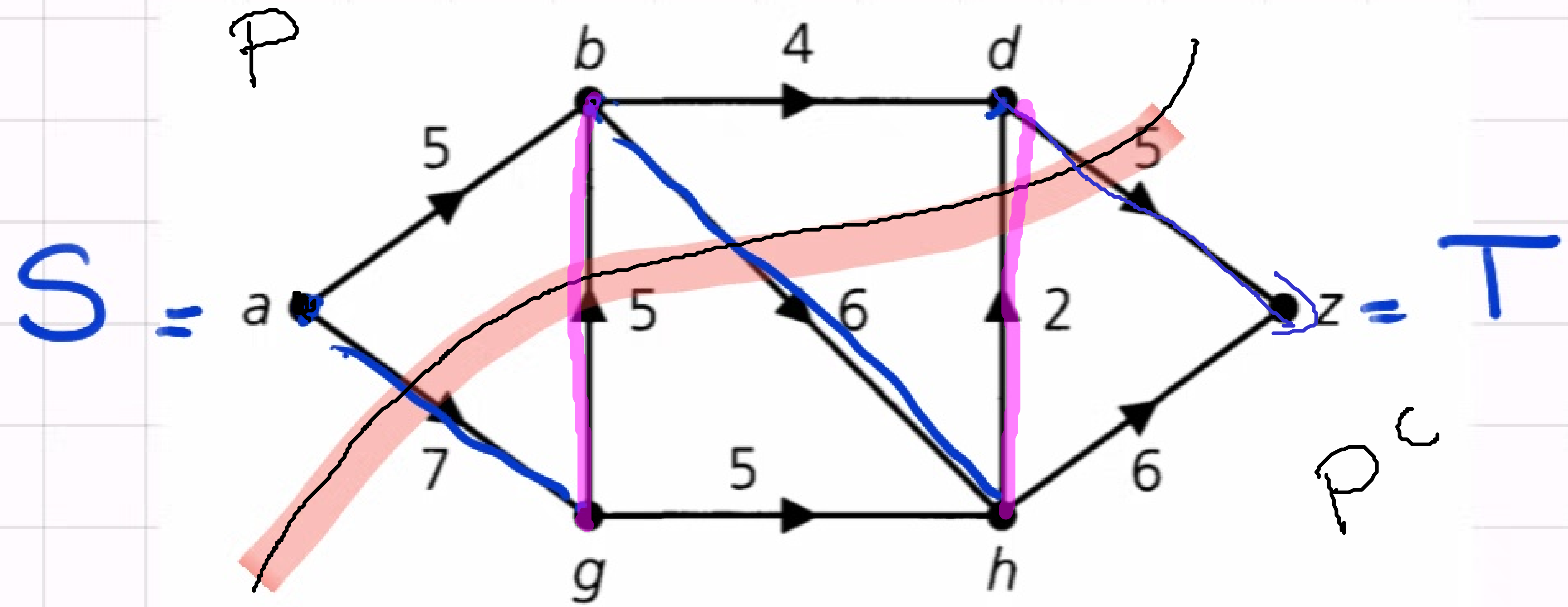
Cuts in a network

Def a CUT in a directed graph is a partition of its vertices in two sets P P^c

Given a network $N = (G, c, S, T)$ a CUT in N is a cut $P \cup P^c = V(G)$ of G such that

$\boxed{S \in P \text{ and } T \in P^c}$

Example



$P = \{a, b, d\}$

$P^c = \{g, h, z\}$

$P \cup P^c = V(G)$

$a \in P$

$z \in P^c$

Given a Network N and a cut P, P^c the capacity of the cut is

$$c(P, P^c) := \sum_{\substack{v \in P \\ w \in P^c}} c(v, w)$$

$7 + 6 + 5 = 18$

in the example

Problem

Given a Network N find a cut (P, P^c) such that $c(P, P^c)$ is minimal

N finite.

there are finitely many possible cuts.

$\Rightarrow$ there is one with the minimum value

The max flow / min cut theorem

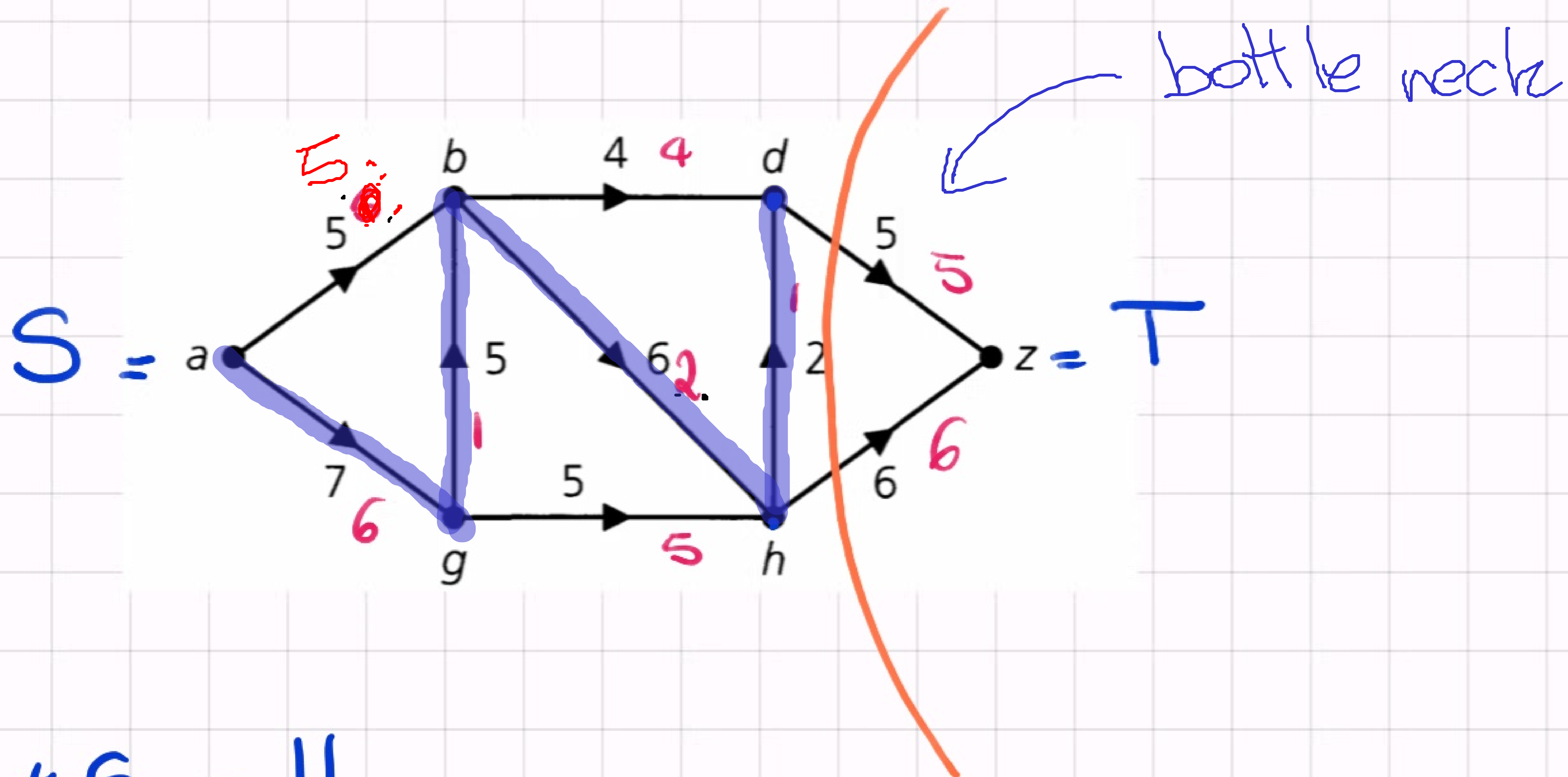
Given a network $N = (G, c, S, T)$ we have that a flow f has maximum value iff tha exist a cut (P, P^c) with $\boxed{c(P, P^c) = Val(f)}$

If this happens, we have that $c(P, P^c)$ is a minimum cut.

Characterization / Stop condition

Example

In the first example with this network we had val (f) = 8 was not maxim



$Val(f) = 5 + 6 = 11$
$c(P, P^c) = 5 + 6 = 11$)) $\Rightarrow$ the flow is maximal.

Lemma 1

Given a network $N = (G, c, ST)$, a cut (P, P^c) and a flow f we have that

$$Val(f) = \sum_{\substack{x \in P \\ y \in P^c}} f(x,y) - \sum_{\substack{x \in P \\ y \in P^c}} f(y,x)$$

what goes out from P - what come to P

$$\leq c(P, P^c)$$

Proof

$$\mathrm{Val}(f) := \sum_{v \in V(G)} f(s,v) = \sum_{v \in V(G)} \Big(f(s,v) - \underbrace{f(v,s)}_{= 0} \Big)$$

$$\mathrm{indeg}(s) = 0$$

$$= \sum_{v \in V(G)} \big(f(s,v) - f(v,s) \big)$$

$$+ \underbrace{\sum_{v \in V} \sum_{\substack{w \in P \\ w \neq s}} f(w,v) - f(v,w)}_{0}$$

$$w \in P \qquad \begin{matrix} w \neq T \\ w \neq S \end{matrix}$$

$$\sum_{\substack{w \in P \\ w \neq s}} \Big(\sum_{v \in V} f(w,v) - f(v,w) = 0 \Big)$$

$$= \sum_{v \in V} \underbrace{f(s,v) - f(v,s)} + \sum_{\substack{w \in P \\ w \neq s}} f(w,v) - f(v,w) \Big) \Big)$$

$$= \sum_{v \in V} \sum_{w \in P} f(w,v) - f(v,w)$$

$$= \sum_{v \in P} \sum_{w \in P} \Big(f(w,v) - f(v,w) \Big) + \sum_{v \notin P} \sum_{w \in P} f(w,v) - f(v,w)$$

$$= \sum_{v \in P} \sum_{w \in P} \underbrace{\Big(f(w,v) - f(v,w) \Big)}_{= 0} + \underbrace{\sum_{v \notin P} \sum_{w \in P} f(w,v) - f(v,w)}_{\text{SURVIVON}}$$

$$\neq \sum_{\substack{v \in P \\ w \in P}} f(w,v) - \sum_{\substack{v \in P \\ w \in P}} f(v,w)$$

$$= \sum_{\substack{w \in P \\ v \in P^c}} f(w, v) - f(v, w)$$

Corollary

If we take $P = V \setminus \{T\}$ $P^c = \{T\}$ we have

$$\mathrm{Val}(f) = \sum_{v \in V} f(vT)$$

Proof

$$\mathrm{Val}(f) = \sum_{\substack{v \in P \\ w \in P^c}} f(vw) - f(wv)$$

$$= \sum_{\substack{v \neq T \\ w = T}} f(v,w) - f(Tv)$$

$$= \sum_{v \neq T} f(vT) - \overbrace{f(Tv)}^{0}$$

outdeg $T = 0$

$(T, v) \notin E(G)$

Corollary 2 : for any cut (P, P^c) we have $\mathrm{val}(f) \leq c(P, P^c)$

Proof

$$\mathrm{Val}(f) = \sum_{\substack{v \in P \\ w \in P^c}} f(v,w) - \underbrace{f(w,v)}_{\geq 0}$$

$$\overset{\mathrm{I}}{\leq} \sum_{\substack{v \in P \\ w \in P^c}} f(v,w) \overset{\mathrm{II}}{\leq} \sum_{\substack{v \in P \\ w \in P^c}} c(v,w)$$

$$= C(P, P^c)$$

#

If f is a flow with $\mathrm{val}(f) = C(P, P^c)$ for some cut then $\mathrm{val}(f)$ is maximal.

<u>Rmk</u> When do we have $val(f) = c(P, P^c)$?

Iff both (I) and (II) above are $=$

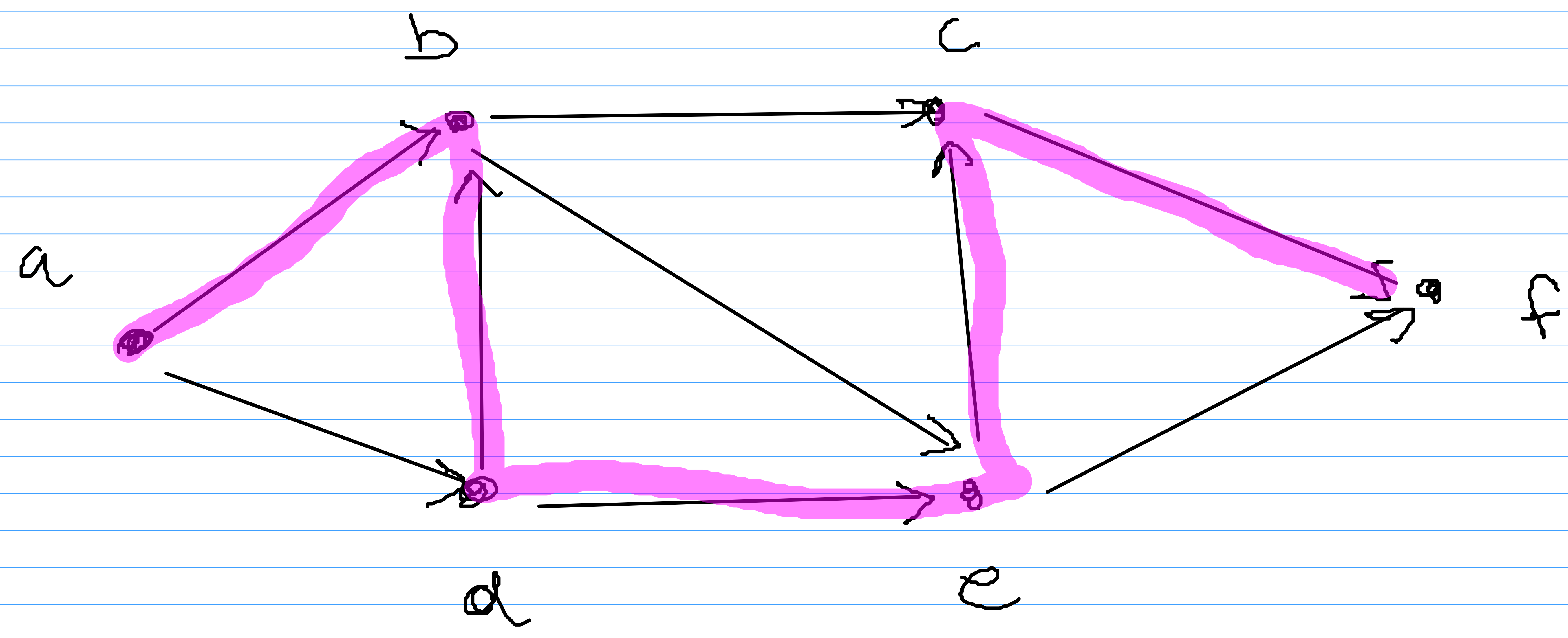
$\iff$ ① $\sum_{\substack{v \in P \\ w \in P^c}} f(w,v) = 0$ &

(II) $f(w,v) = c(w,v)$ for all $v \in P^c$, $w \in P$

$\rightsquigarrow$ Motivate the following definition.

A chain in a directed graph is a path in the corresponding undirected graph

Given $(v_1 \dots v_n)$ a chain in a directed graph (v_i, v_{i+1}) is a forward edge if $(v_i, v_{i+1}) \in E(G)$; (v_i, v_{i+1}) is a backward edge if $(v_{i+1}, v_i) \in E(G)$



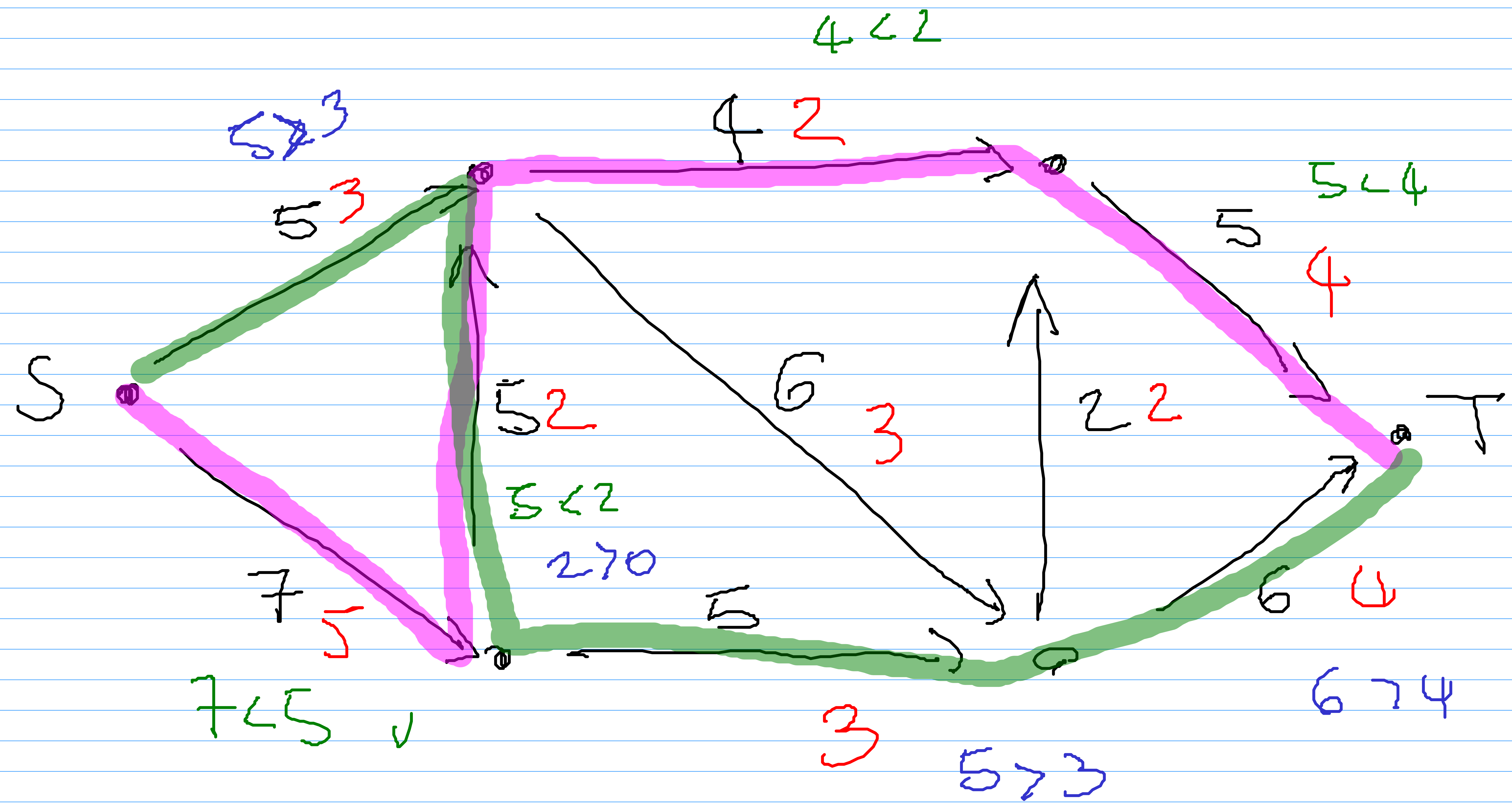
(a, b, d, e, c, f) is a chain

all the edges but (b d) are forward

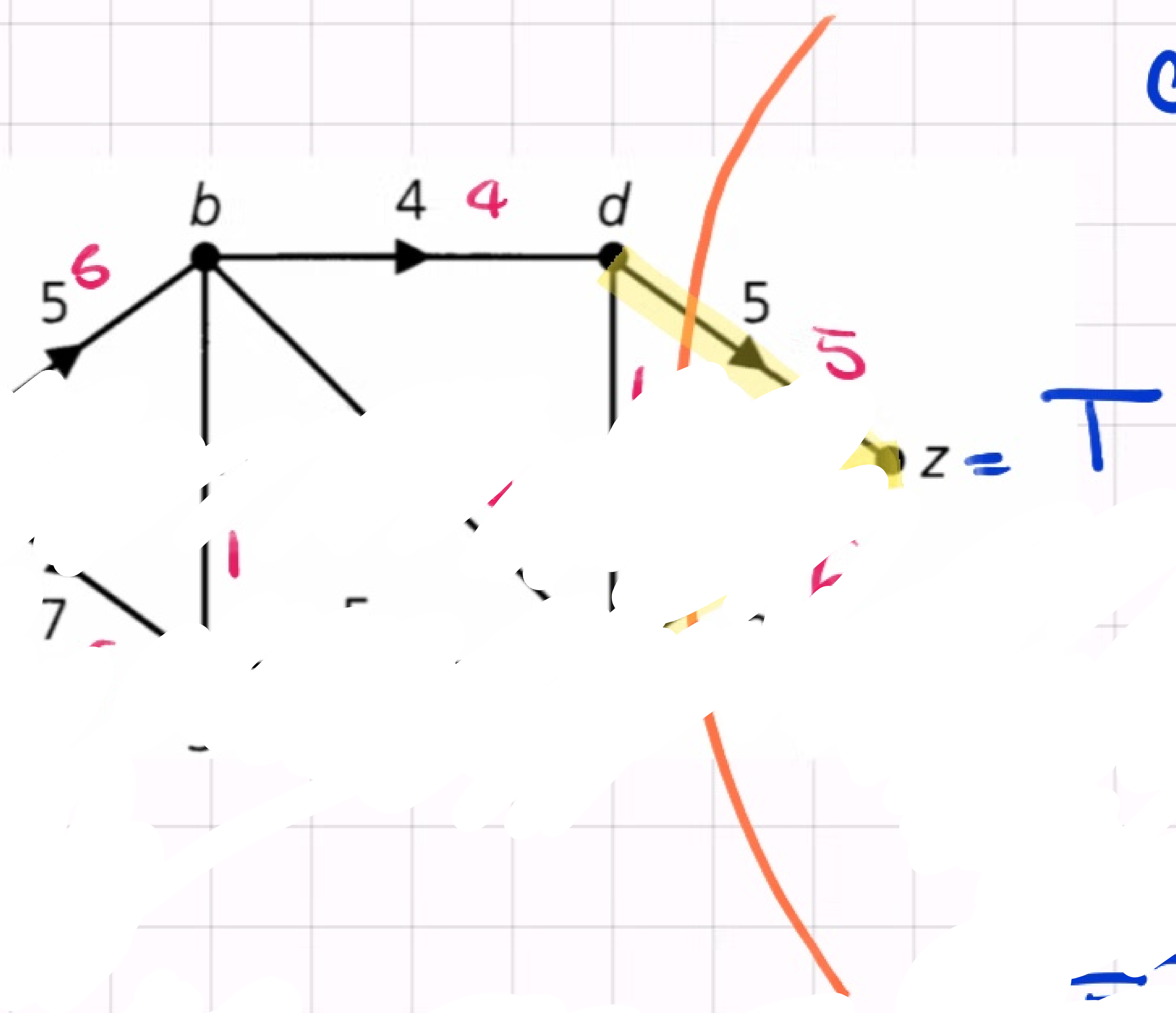
$(v_1, \ldots, v_n)$

A chain is <u>f-augmenting</u> if $v_1 = S$, $v_n = T$

- $f(v_i\, v_{i+1}) < C(v_i\, v_{i+1})$ for every forward edge
- $f(v_i\, v_{i+1}) > 0$ for every backward edge



There is no edge from P^c to P and the capacity of the edges from P



... is maxim...

... $V(f)$ ... $c(P, P')$

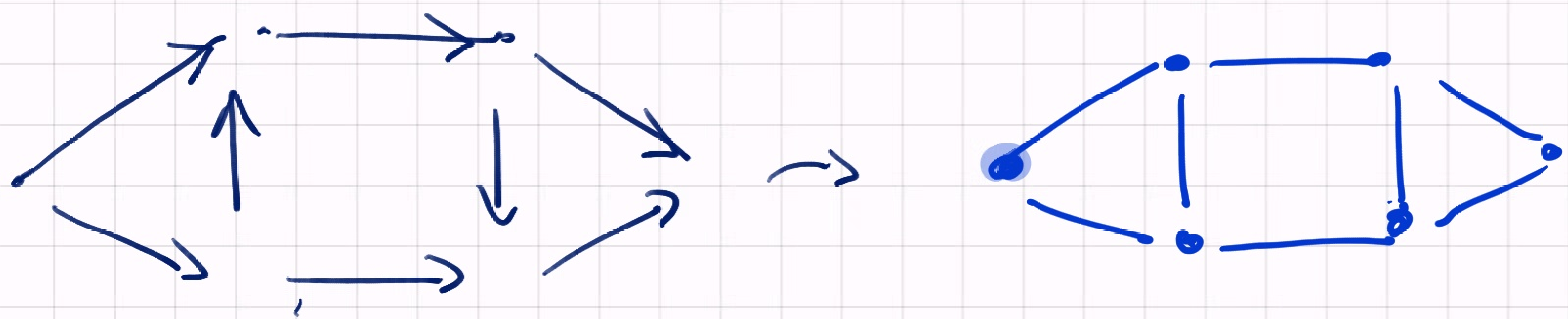
Some definitions

Given a <u>directed</u> graph $G=(V,E)$ we can associate to it a undirected graph $G'=(V',E')$

$V'=V$

$E'=\{\{s(e),r(e)\}\ /\ e\in E\}$

forget the directions of the arrow

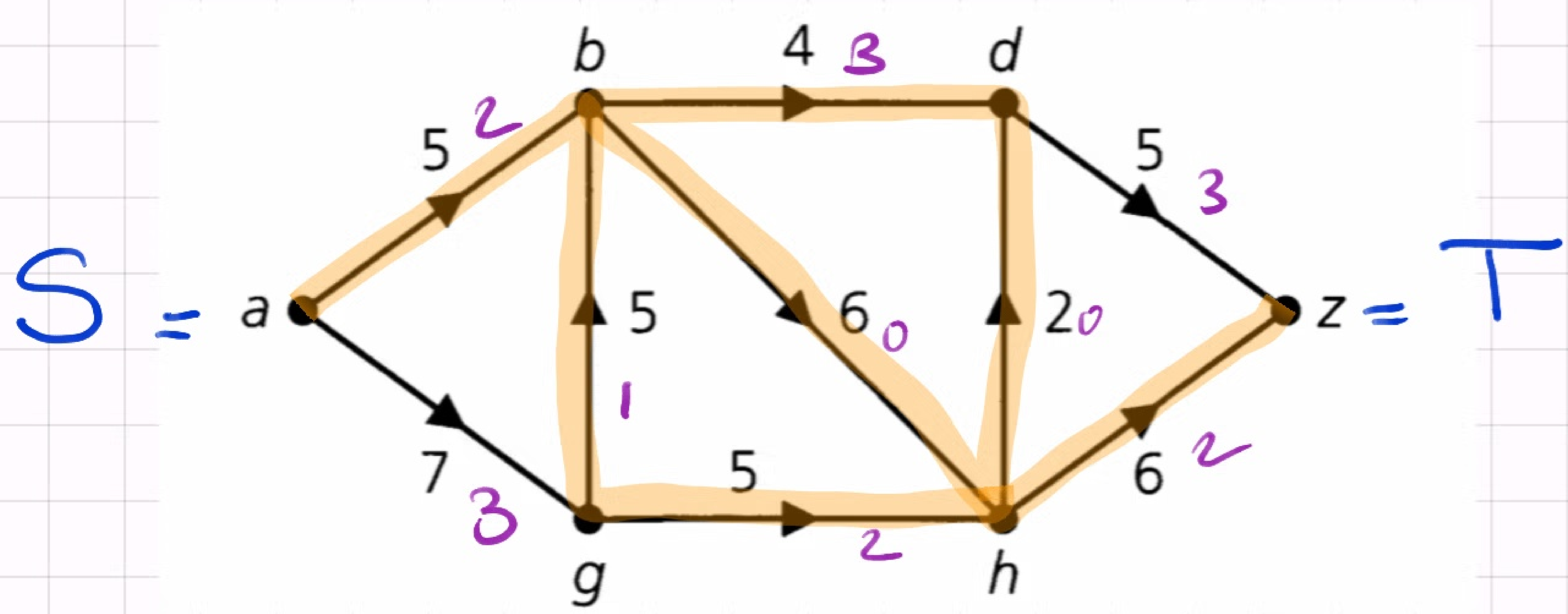


A chain in a directed graph is a walk in the associated undirected graph. That is is a sequence $(v_0 \ldots\ldots v_n)$ with $v_i \sim v_{i+1}$

$i = 0 \ldots n-1$

If $(v_i \; v_{i+1}) \in E(G)$ we say that this is a forward edge

If $(v_{i+1} v_i) \in E(G)$ then this is a backward edge.



(a b g h d b h z)

(a b g h z)

Lemma

If no f- augmenting path exist then there is a cut (P, P^c) with

$$c(P, P^c) = Val(f)$$

(the flow is maximal)

Proof

$$P = \left\{ \begin{array}{l} v \in V(G) \text{ such that there is a chain} \\ 1)\ (v_1 \ldots v) \text{ with } v_1 = S \\ 2)\ f(v_i\, v_{i+1}) < c(v_i\, v_{i+1}) \text{ for all forward} \\ 3)\ f(v_i\, v_{i+1}) > 0 \text{ for all } (v_i\, v_{i+1}) \text{ backwar} \end{array} \right\}$$

$S \cup$

$s \in P$ $\quad (v_1)$

$t \notin P$ or there is an f augmenting path.

forward edges in a pd

$$\mathrm{val}(f) = \sum_{\substack{v \in P \\ w \in P^c}} f(v,w) - \underbrace{f(w\,v)}_{0} = \sum_{\substack{v \in P \\ w \in P^c}} f(v,w)$$

$$\geq \sum c(v,w)$$

$$\geq c(P,P')$$

or we can go from P to P^c in the opposite direction with a path $f(w\,v) > 0$

to find a max flow we have to take away the f-augmenting path.

f flow and $(v_1 \ldots v_n)$ is an f-augment path
$v_1 = s$, $v_n = t$

$e = (v_i\ v_{i+1})$

$$\Delta_e = \begin{cases} c(v_i, v_{i+1}) - f(v_i, v_{i+1}) > 0 & \text{if } (v_i\ v_{i+1}) \text{ forward} \\ f(v_i\ v_{i+1}) > 0 & \text{if } (v_i\ v_{i+1}) \text{ backward} \end{cases}$$

$\Delta_p = \min \{ \Delta_e$ e is an edge of the path $p = (v_1 \text{ — } v_n \}$

Lem: $N = (G, c, S, T)$ f a flow.

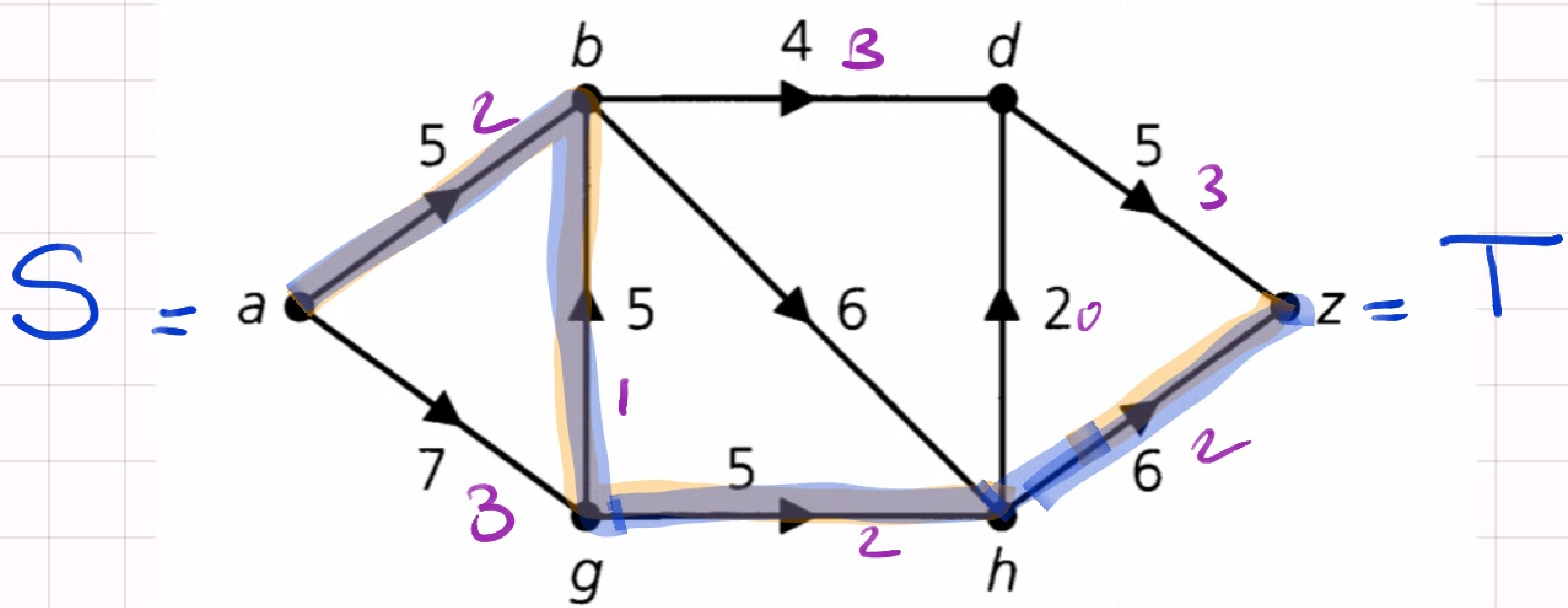
$P = (S\ v_2 \quad v_{n-1}\ T)$ an f-augmenting path

we defin

$$g(v,w) = \begin{cases} f(v,w) + \Delta p & \text{if } (v,w) \text{ is a forward edge} \\ f(v,w) - \Delta p & \text{if } (v\,w) \text{ is a backward edge} \\ f(v,w) & \text{if } (v\,w) \text{ is not ``walked'' by } p \end{cases}$$

We have that
$g(v,w)$ is a flow on N

$$Val(g) = Val(f) + \Delta p$$
$$> Val(f)$$



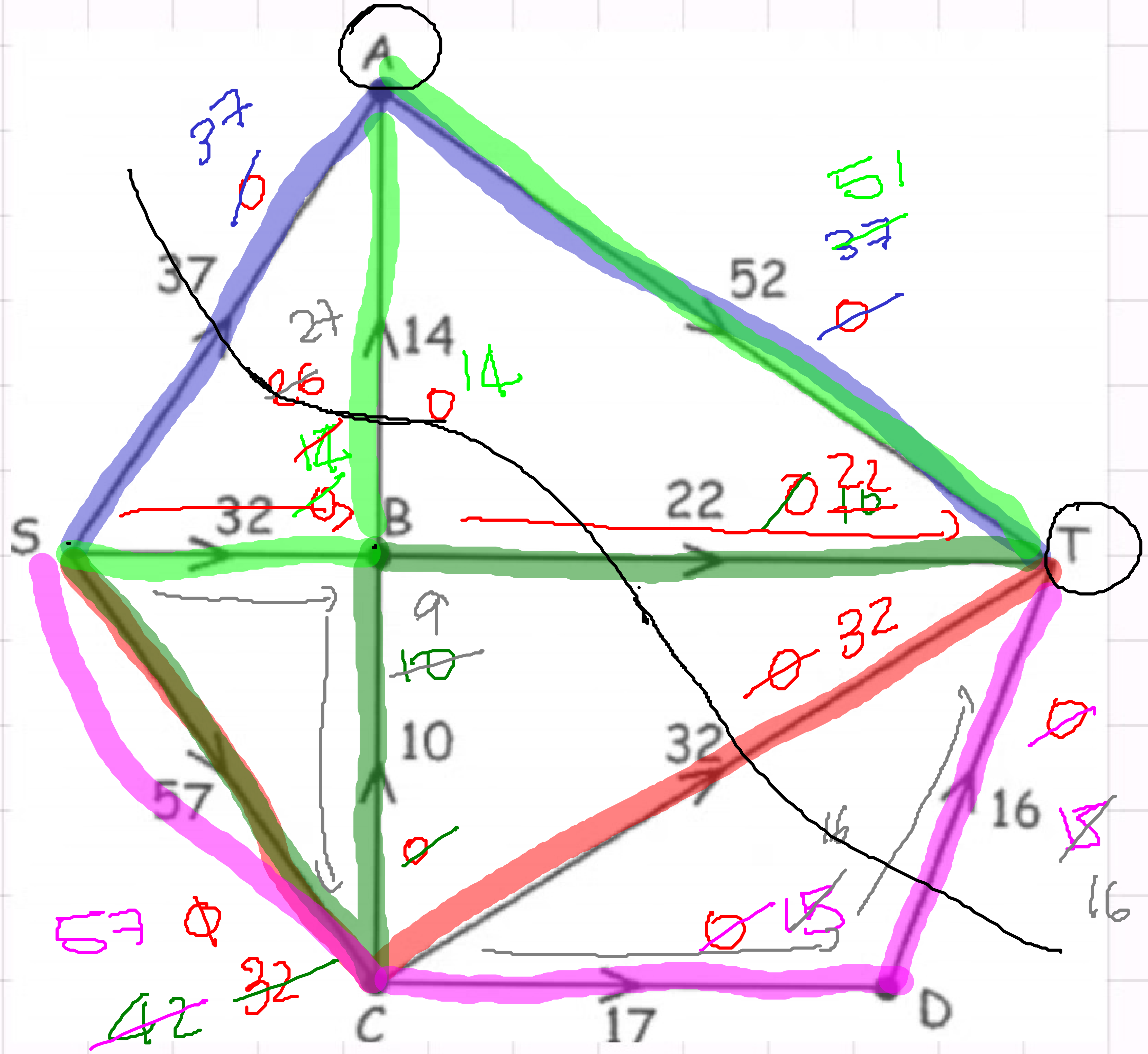
$(a\,b\,g\,h\,z) = c$

$\Delta c = \min\{5-2,\ 1,\ 5-2,\ 6-2\} = \{3, 1, 3, 4\} = 1$

Lemma

Given a Network N, a flow f and an f-augmenting path $p = (v_0 \dots v_n)$ then there is $\Delta_p > 0$ and a flow g with

$$val(f) + \Delta_p = val(g)$$



$$\Delta_p = \min\{37, 52\} = 37$$

$$\Delta_p = 32$$

$$\Delta_p = 10$$

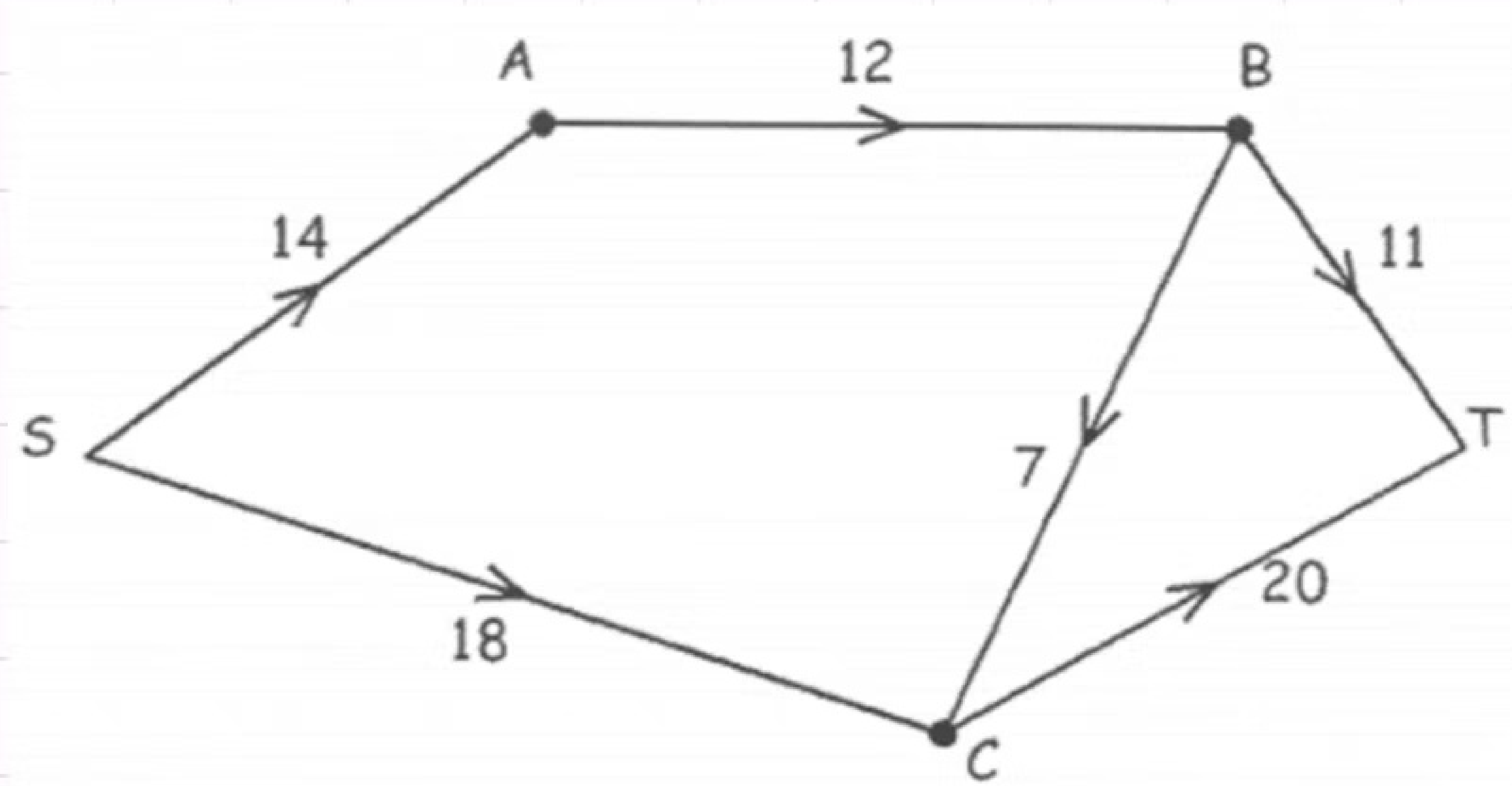
$$\Delta_p = 15 = \min\left\{\begin{matrix} 57-42 \\ 17 \\ 16 \end{matrix}\right\}$$

$$\Delta_p = 14$$

$$\Delta_p = 12$$

$$\mathrm{Val}(f_{max}) = 57 + \cancel{37} + 27 = 37 + 84$$

$$P = \{S, B, C, D\} \qquad c(P, P^c) = \cancel{37} + 14 + 22 + 32 + 16 = 37 + 84$$


A
12
B
14
11
S
T
7
20
18
C

Applications

Enthusiastic celebration of a sunny day at a prominent northeastern university has resulted in the arrival at the university's medial clinic of 169 students in need of emergency treatment. Each of the 169 students requires a transfusion of one unit of whole blood. The clinic has supplies of 170 units of whole blood. The number of units of blood available in each of the four major blood groups and the distribution of patients among the groups is summarized below.

Blood type	A	B	O	AB
Supply	46	34	45	45
Demand	39	38	42	50

Type A patients can only receive type A or O; type B patients can receive only type B or O; type O patients can receive only type O; and type AB patients can receive any of the four types.