MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Tentamensomskrivning i
Combinatorics
7.5 hp
February 4th, 2022

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.

GOOD LUCK!

1. **Counting:** (3 points)
Decide how many integer solutions has the equation $x_1 + x_2 + x_3 + x_4 = 10$ when x_1 is odd and non-negative, $x_2 \geq 2$, $1 \leq x_3 \leq 5$ and x_4 is non-negative.
Solution: The generating function of compositions (ordered partitions) (x_1, x_2, x_3, x_4) such that x_1 is odd and non-negative, $x_2 \geq 2$, $1 \leq x_3 \leq 5$ and x_4 is non-negative is

$$f(x) = \frac{x^2}{1-x}(x + x^2 + x^3 + x^4)\frac{1}{1-x}\frac{x}{1-x^2} = \frac{x^4 + x^5 + x^6 + x^7 + x^8}{(1-x)^2(1-x^2)}.$$

We need to compute the coefficient of degree 10. To do that we need to compute the first 6 coefficient of

$$\frac{1}{(1-x)(1-x^2)}.$$

We know that

$$\frac{1}{(1-x^2)} = (1 + 0x + x^2 + 0x^3 + x^4 + 0x^5 + x^6 + \cdots)$$

Using that the generating function of $\frac{1}{1-x}$ is the summation operator, we compute that

$$\begin{aligned}
\frac{1}{(1-x)(1-x^2)} &= 1 + (1+0)x + (1+0+1)x^2 + (1+0+1+0)x^3 + (1+0+1+0+1)x^4 \\
&= \qquad +(1+0+1+0+1+0)x^5 + (1+0+1+0+1+0+1)x^6 + \cdots) \\
&= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + \cdots
\end{aligned}$$

We reiterate and find that

$$\begin{aligned}
\frac{1}{(1-x)^2(1-x)^2} &= 1 + (1+1)x + (1+1+2)x^2 + (1+1+2+2)x^3 + (1+1+2+2+3)x^4 \\
&= \qquad +(1+1+2+2+3+3)x^5 + (1+1+2+2+3+3+4)x^6 + \cdots) \\
&= 1 + 2x + 4x^2 + 6x^3 + 9x^4 + 12x^5 + 16x^6 + \cdots
\end{aligned}$$

Thus

$$f(x) = (x^4 + x^5 + x^6 + x^7 + x^8)(1 + 2x + 4x^2 + 6x^3 + 9x^4 + 12x^5 + 16x^6 + \cdots)$$

the coefficient of degree 10 is:

$$\boxed{16 + 12 + 9 + 6 + 4 = 47}$$

2. **Rook polynomials:**

   (a) (3 points) Let $C_{n,m}$ be complete (without forbidden cells) $n \times m$ chessboard. Show that

   $$r(C, x) = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} k! x^k$$

   (b) (3 points) Compute the rook polynomial of a 6×13 chessboard with a forbidden cell in one of the corners.

Solution: (a) We compute $r_k(C)$. We need to place k rooks on a $n \times m$ chessboard without forbidden places. In order to do that we first choose in which rows we place the rooks, there are $\binom{n}{k}$ ways to do that. Now, for every row we have chosen before we have to chose a column where to place the rook for that row. As swapping rows will give us different rooks configuration, the order in which the column are chosen matters! Thus we have $\binom{P(m,k)=k!m}{k}$ ways to chose the column for each rook. We get that

$$r_k(C) = \binom{n}{k} \cdot \binom{m}{k} \cdot k!.$$

The statement then follows from the definition of rook polynomial.
(b) Let C a 6×13 chessboard with a forbidden place in one of the corner. Let C_1 a complete 6×13 chessboard and C_2 a complete 5×12 chessboard. Then we know that

$$r(C_1,x) = r(C,x) + xr(C_2,x).$$

Thus

$$\begin{aligned} r(C,x) &= r(C_1,x) - r(C_2,x) \\ &= \boxed{\sum_{k=0}^{6}\binom{6}{k}\binom{13}{k}k!x^k - \sum_{k=0}^{5}\binom{5}{k}\binom{12}{k}k!x^{k+1}} \end{aligned}$$

3. **Recursion:** Consider the following recursion relation

$$a_{n+2} - 6a_{n+1} + 5a_n = 300$$

With boundary conditions $a_0 = 0$ and $a_1 = 1$.

(a) (3 points) Solve the relation finding a closed formula for a_n.
(b) (2 points) Express the generating function of the sequence $\{a_n\}_{n\in\mathbb{N}}$ as a quotient of polynomials.

Solution:(a) The characteristic polynomial of the recursion relation is r^2-6r+5 which factors as $(r-5)(r-1)$ We deduce that the general solution of the homogeneous relation is

$$a_n^{(h)} = A5^n + B1^n = A5^n + B,$$

for A and B two constants to be determined. To get a particular solution to the inhomogenous problem we have to be careful: the guess $a_n^{(p)} = C$ will not work in this case as this will be a solution of the homogeneous problem. Thus, we set $a_n^{(p)} = Cn$ and get

$$C(n+2) - 6C(n+1) + 5Cn = 300.$$

We expand the products and we get

$$-4C = 300,$$

so $C = -75$. Thus the general solution of the recursion relation is

$$a_n = A5^n + B - 75n.$$

Now we have to use the boundary conditions to find the values of A and B. We get the following linear system

$$\begin{cases} A+B &= 0 \\ 5A+B-75 &= 1 \end{cases}$$

with solution $A = 19$ and $B = -19$. Thus the solution to the recursion problem is

$$\boxed{a_n = 19\cdot 5^n - 75n + 19}$$

(b) We use the methods of generating functions. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$. Then we have that

$$f(x) - a_1x + a_0 - 6x(f(x) - a_0) + 5x^2 f(x) = 300\frac{x^2}{1-x}.$$

We use the boundary conditions and we get

$$f(x)(1-6x+5x^2) - x = 300\frac{x^2}{1-x}.$$

Thus

$$\boxed{f(x) = \frac{300x^2 - x(1-x)}{(1-x)(1-6x+5x^2)}}.$$

4. **Graphs:** Let $S = \{1, 2, \ldots, k\}$ be a set with k elements. Construct the graph G_k in the following manner:

   - the vertices of G_k are subsets of S with 3 elements;
   - there is an edge connecting two vertices A and B if and only if $A \cap B = \emptyset$

   (a) (3 points) Compute $|V(G_k)|$ and the degree of each vertex (**Hint:** If there is a edge connecting A with B, then B is a subset of the complementary of A. How many such B are there?).

   (b) (1 point) For any simple undirected graph $G = (V, E)$, give a formula relating $|E|$ with the degrees of the vertices of G.

   (c) (1 point) Use the formula in the above point to find $|E(G_k)|$.

Solution:(a) The size of the set of vertices of G_k is the number of subsets with 3 elements of S:

$$\boxed{|V(G_k)| = \binom{k}{3}}.$$

Using the Hint, let A be a vertex of G_k. Then A is a subset with 3 elements of S. The vertex/set A is adjacent to a vertex/set B if, and only if, B is a subset with three elements of $S \backslash A$. The set $S \backslash A$ has $k - 3$ elements and has exactly $\binom{k-3}{3}$ subsets. Thus foe each vertex A we have that

$$\boxed{\deg(A) = \binom{k-3}{3}}$$

(b) For a graph $G = (V, E)$ we have that

$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

(c) We use the above formula and what we found out in (a):

$$2|E(G_k)| = \binom{k}{3}\binom{k-3}{3} = \frac{k!}{(3!)^2(k-6)!}.$$

Thus

$$\boxed{|E(G_k)| = \frac{1}{2} \cdot \frac{k!}{(3!)^2(k-6)!}}$$

5. **Minimal spanning trees:** Consider the weighted graph in Figure 1:

   (a) (2 points) Detrmine the possible values for the weight w knowing that someone is constructing a minimal spanning tree using **Kruskal's algorithm** and so far has selected the dashed edges.

   (b) (2 points) Set $w = 3$ and use **Prim's algorithm** to construct a minimal spanning tree for the graph.

   (c) (1 point) Set again $w = 3$ and find the total weight of a minimal spanning tree.

Solution: (a) The last edge selected with Kruskal's algorithm has weight 5. Thus w must have weight greater or equal 5.
(b) Attention the solution might change if we change the choice of the starting point. Here we set $v_1 = G$ A minimal spanning tree is given by the edges (selected in the given order)

$$\{GD, GC, GF, CB, GA, FE\}$$

(c) 28

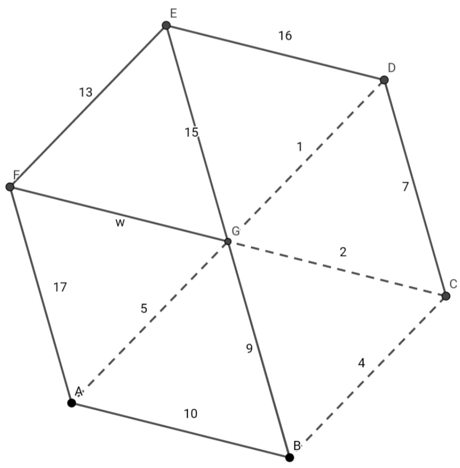

Figure 1: Weighed Graph

6. **Transport Networks:** Consider the transport network in Figure 2.

   (a) (3 points) Find a flow with the maximum value for the network.

   (b) (3 points) Give a cut with the minimum capacity for the network. Determine the capacity of such cut.

   Solution: (a) A maximum flow can be found in Figure 3. Its value is $9 + 15 = 24$. (b) A minimum cut is $P = \{A, B, C\}$ and its capacity is 24.

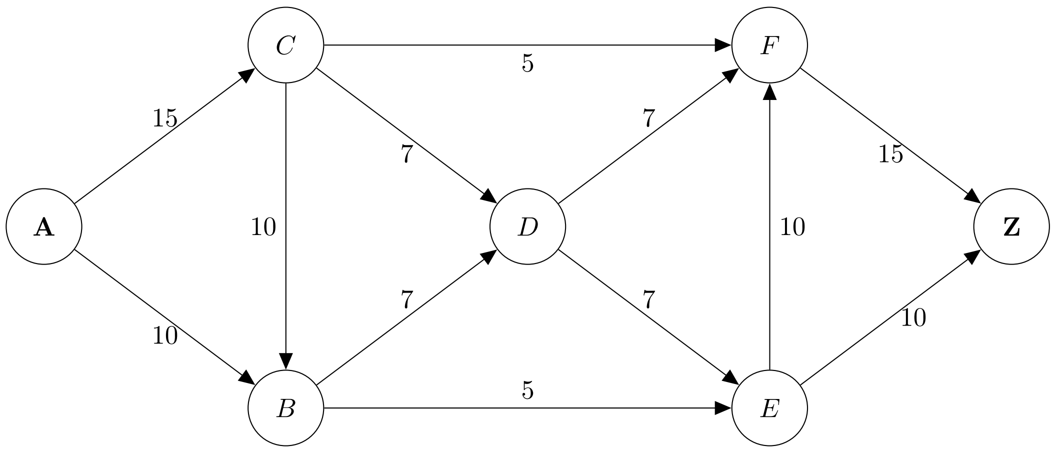

Figure 2: Network

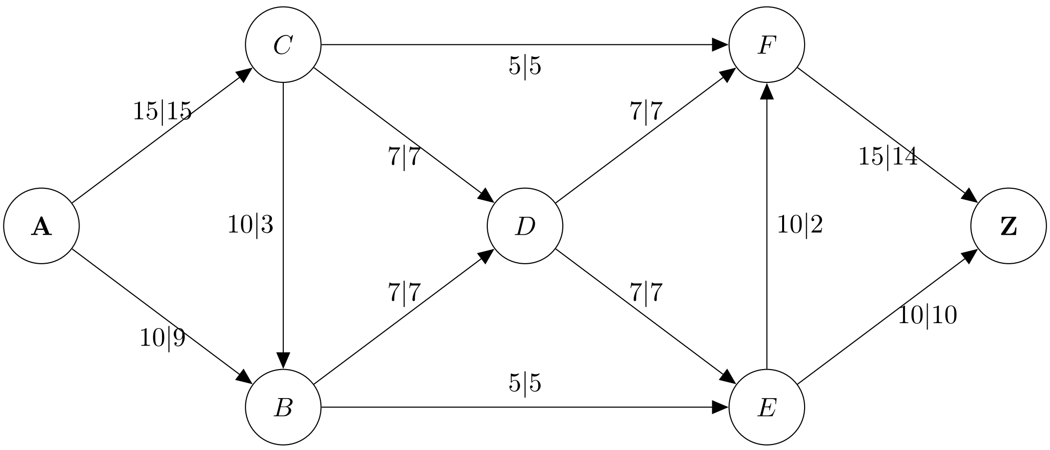

Figure 3: Maximum flow