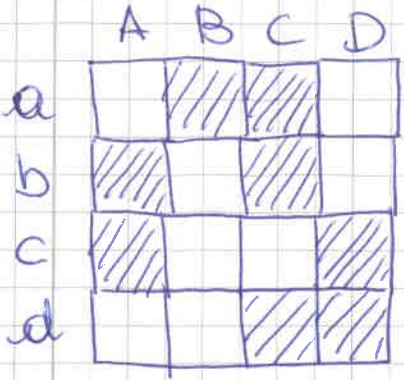
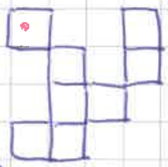
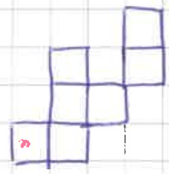
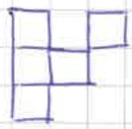
Solution to MM5023 exam 15/02/24

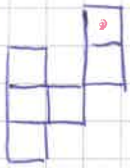
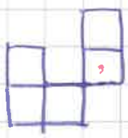
Exercise 1

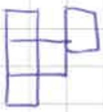
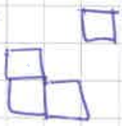
(a)



(b)

$r\Big($  $\Big) = r\Big($  $\Big) + x\,r\Big($  $\Big)$

$= r\Big($  $\Big) + x\,r\Big($  $\Big) + x\left(1+5x+5x^2+x^3\right)$

$= r\Big($  $\Big) + x\,r\Big($  $\Big) + x\,r\Big($  $\Big) + x^2 r\Big($  $\Big)$

$+ x + 5x^2 + 5x^3 + x^4$

$= (1+5x+5x^2+x^3) + x(1+4x+2x^2) + x(1+x)(1+3x+x^2)$

$+ x^2(1+2x) + x + 5x^2 + 5x^3 + x^4$

$= 1 + 5x + 5x^2 + x^3 + x + 4x^2 + 2x^3 + (x+x^2)(1+3x+x^2)$

$+ x^2 + 2x^3 + x + 5x^2 + 5x^3 + x^4$

$= 1 + 7x + 15x^2 + 10x^3 + x^4 + x + 3x^2 + x^3 + 3x^3 + x^4 + x^2$

$= 1 + 8x + 19x^2 + 14x^3 + 2x^4$

(c) Since the polynomial is of deg 4 it is possible to assign 4 jobs to 4 people (in two ways

Exercise 2

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} - 3\sum_{n=0}^{\infty} a_{n+1} x^{n+1} + 2\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=0}^{\infty} 3^{n+2} x^{n+2}$$

$$\Leftrightarrow \quad f(x) - a_0 - a_1 x - 3x\big(f(x) - a_0\big) + 2x^2 f(x) = \frac{1}{1-3x} - 1 - 3x$$

$$\Leftrightarrow \quad f(x) \cancel{-1} \cancel{-6x} - 3x f(x) \cancel{+3x} + 2x^2 f(x) = \frac{1}{1-3x} \cancel{-1} \cancel{-3x}$$

$$(1 - 3x + 2x^2)\, f(x) = \frac{1}{1-3x}$$

$$f(x) = \frac{1}{(1-3x+2x^2)} \cdot \frac{1}{1-3x} = \frac{1}{(1-x)(1-2x)(1-3x)}$$

$$= \frac{1}{2}\,\frac{1}{1-x} - 4\,\frac{1}{1-2x} + \frac{9}{2}\,\frac{1}{1-3x}$$

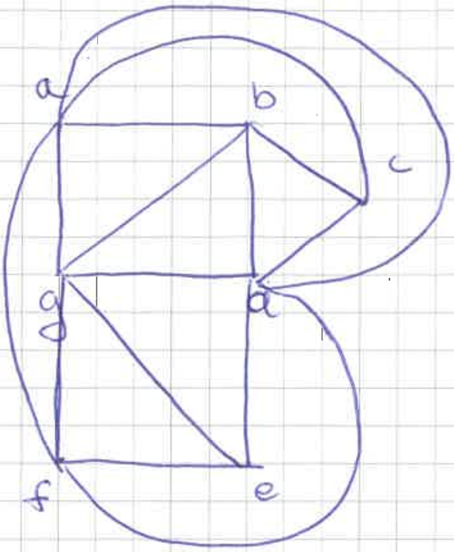
$$= \frac{1}{2}\sum_{n=0}^{\infty} x^n - 4\sum_{n=0}^{\infty} 2^n x^n + \frac{9}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$= \left(\sum_{n=0}^{\infty} \left(\frac{1}{2} - 4\cdot 2^n + \frac{9}{2}\, 3^n\right) x^n\right)$$

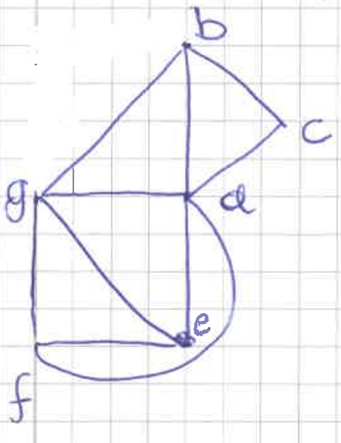
$$a_n = \frac{1}{2}\left(1 - 2^{n+3} + 3^{n+2}\right)$$

Exercise 3

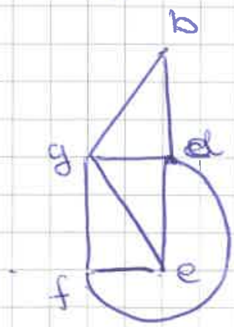
(1) The graph is planar. One way to see it is to argue that has no subgraph homeomorphic to K_5 or $K_{3,3}$. Giving a precise argument for this can be difficult though. Here is a planar rep



(2) collapse {a,d}



and then {d,c}



$= H$

(3) Chromatic poly: $G-e$

$G_e = K_4$

way 1

$$\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G_e}(\lambda) = \cancel{\lambda}\big(\lambda(\lambda-1)(\lambda-2)(\lambda-3)\big)$$

$$- \lambda(\lambda-1)(\lambda-2)(\lambda-3) - \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$= \big(\lambda(\lambda-1)(\lambda-2)(\lambda-3)\big)(\cancel{3}\lambda - 2)$$

$$= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)$$

Other way

$G = K_3 \cup K_4$ which intersect in a K_2

$$\chi_G = \frac{\cancel{\lambda(\lambda-1)}(\lambda-2) \cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3)}{\cancel{\lambda(\lambda-1)}} = \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)$$

Exercise 4

	S	1	2	3	4	5	6
Initialization	(- 0)	(-, +∞)	(-, +∞)	(-, +∞)	(-, +∞)	(-, +∞)	(-, +∞)
1st	(- 0)	$\boxed{(S\,3)}$	(S 4)	(- +∞)	(- +∞)	(- +∞)	(-, +∞)
2nd	(- 0)	(S 3)	$\boxed{(S\,4)}$	(- +∞)	(1 6)	(1 5)	(- +∞)
3rd	(- 0)	(S 3)	(S 4)	(2 7)	(1 6)	$\boxed{(1\,5)}$	(- ∞)
4th	(- 0)	(S 3)	(S 4)	(2 7)	$\boxed{(1\,6)}$	(1 5)	(5 8)
5th	(- 0)	(S 3)	(S 4)	$\boxed{(2\,7)}$	(1 6)	(1 5)	(4 7)

Now if (3 4) ~~is sent~~ has weight ~~to~~ changed to -2
All the iterations are the same till the 5th
when vertex 3 is added to S. ~~Now~~ Since the label
of 4 was then fixed it cannot be changed again
and the algorithm gives the shortest path to
4 as $S \to 1 \to 4$
but $S \to 2 \to 3 \to 4$ is shorter

~~Ex~~ Adding +2 to all weights will not work
as path of different length will have different
changes

Exercise 5

(a) a is invertible mod p

$$L(i,j) = L(i,k)$$

$$\Leftrightarrow ia + j \equiv ia + k \Rightarrow j \equiv k$$ no rep in row

$$L(i,j) = (k,j) \Leftrightarrow ia + j \equiv ka + j$$

$$\Leftrightarrow ia = ka \Leftrightarrow i = k$$

(no repetitions in the column.

(b) The two diagonal are ~~the~~ given by

$L(i,i)$ and $L(i, p-i-1$

$ai + i \equiv aj + j \mod p \qquad i \not\equiv j \mod p$

$$a(i-j) \equiv (j-i) \mod p$$

$$\Rightarrow a \equiv -1 \equiv p-1 \mod p$$

We have repetition on the main diagonal

$$\Rightarrow a = p-1$$

Suppose now that there is a repetition in the other diagonal

$$ai + p - i - 1 \equiv aj + p - j - 1 \mod p \quad i \neq j$$

$$\Leftrightarrow ai - i \equiv aj - j \mod p$$

$$a(i-j) \equiv (i-j) \mod p$$

$$\Rightarrow a \equiv 1 \mod p$$

Thus if $a \neq 1, p-1$ we have that there is no repetition in the diagonal

(b) the two standard latin squares of order 3 are

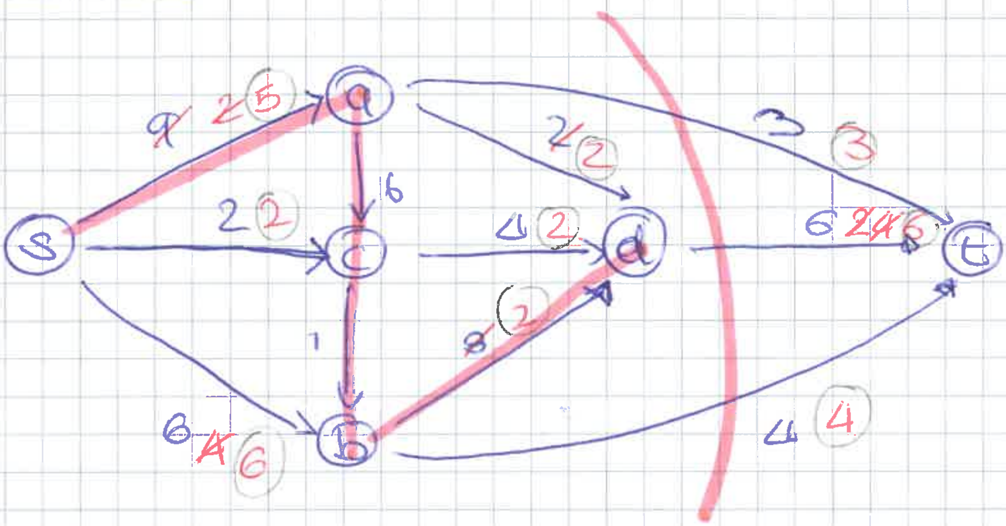
$$\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}$$ not diagonal

$n=5$ $a=2$

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{matrix}$$

this is diagonal

Exercise 6



this is a max flow

The flow value is 13. The associated cut is

$P = \{s, a, b, c, d\}$ $P^c = \{t\}$

$c(P, P^c) = 3 + 6 + 4 = 13 = val(f)$.