PRACTICE EXAM

Problem 1. *Let* $f(x) = x^{11} - 42 \in \mathbf{Q}[x]$.

(a) *(1 point) Show that* f *is irreducible over* $\mathbf{Q}$.
(b) *(2 points) Give an explicit description of a splitting field* L *for* f *over* $\mathbf{Q}$.
(c) *(1 point) Compute* $[L : \mathbf{Q}]$. *Justify your answer.*
(d) *(1 point) Show that* $L/\mathbf{Q}$ *is Galois.*

Problem 2. *Let* f *and* L *be as in Problem 1.*

(a) *(2 points) Give generators and relations for* $\mathrm{Gal}(L/\mathbf{Q})$.
(b) *(2 points) Show that* $\mathrm{Gal}(L/\mathbf{Q})$ *is solvable.*
(c) *(2 points) Show that there is a unique extension* $K/\mathbf{Q}$ *of degree* 10 *which is contained in* L.
(d) *(2 poins) Show that there is a unique quadratic extension* $F/\mathbf{Q}$ *contained in* L *and describe* F *as* $\mathbf{Q}(\sqrt{D})$ *for some integer* D.

Problem 3. *Let* $\Phi_{143}(x) \in \mathbf{Z}[x]$ *be the cyclotomic polynomial of primitive* $143 = 11 \cdot 13$*th roots of unity. Let* ζ *be a root of* $\Phi_{143}(x)$ *in some finite extension of* $\mathbf{Q}$.

(a) *(2 points) Show that for every prime* p, *the reduction of* $\Phi_{143}(x)$ *modulo* p *is reducible in* $\mathbf{F}_p[x]$.
(b) *(1 point) Is the regular* 143*-gon constructible by straightedge and compass? Justify your answer.*
(c) *(2 point) Show that there are precisely three quadratic extensions of* $\mathbf{Q}$ *contained in* $\mathbf{Q}(\zeta)$.
(d) *(2 points) Describe the three distinct quadratic extensions of* $\mathbf{Q}$ *contained in* $\mathbf{Q}(\zeta)$ *in the form* $\mathbf{Q}(\sqrt{D})$, *where* $D \in \mathbf{Z}$ *is an integer.*

Problem 4.

(a) *(2 points) Let* p *be a prime and* $n \geq 1$ *an integer. Show that* $x^{p^n} - x$ *factors over* $\mathbf{F}_p$ *as the product of all the irreducible polynomials* $f \in \mathbf{F}_p[x]$ *of degree dividing* n, *each appearing with multiplicity one.*
(b) *(2 points) Let* G *be a subgroup of* S_7 *which contains a* 7*-cycle and a permutation of type* $(2, 2)$. *Show that* G *contains at least* 8 *Sylow* 7*-subgroups and that the order of* G *is at least* 56.
(c) *(2 points) Assume* k *is an integer which is divisible by* 5 *and not divisible by* 7. *Show that the Galois group of* $h(x) = x^7 - x + k \in \mathbf{Q}[x]$ *contains at least* 8 *Sylow* 7*-subgroups and that the order of* G *is at least* 56 *(one can probably show much more but it might be too hard) .*

Problem 5.

(a) *(1 point) Show that* $x^4 + x^3 + 1$ *divides* $x^{16} - x$ *in* $\mathbf{F}_2[x]$.
(b) *(1 point) Show that* $x^4 + 3x^3 - 1$ *divides* $x^9 - x$ *in* $\mathbf{F}_3[x]$.
(c) *(1 point) Show that the Galois group of* $x^4 + 3x^3 - 1 \in \mathbf{Q}[x]$ *is* S_4.
(d) *(1 point) Assume* $t(x) \in \mathbf{Q}[x]$ *is irreducible of degree* 4 *with Galois group* A_4 *and assume* $t(x)$ *has a real root* α. *Show that* α *is not constructible by straightedge and compass.*

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