PRACTICE EXAM

Instructions: Justify your answers. You may use results from the homework sets and from class, but make sure to carefully state such results. No calculators and no notes allowed.

Grading: This exam is worth 30 points. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: [26.5, 30], B: [23, 26.5), C: [19.5, 23), D: [16, 19.5), E: [12.5, 16), F: [0, 12.5).

Problem 1. *Let* $f(x) = x^7 - 2 \in \mathbf{Q}[x]$.

(a) *(1 point) Show that* f *is irreducible over* $\mathbf{Q}$.
(b) *(2 points) Give an explicit description of a splitting field* L *for* f.
(c) *(1 point) Compute* $[L : \mathbf{Q}]$.
(d) *(1 point) Show that* $L/\mathbf{Q}$ *is Galois.*

Problem 2. *Let* $f(x) = x^7 - 2 \in \mathbf{Q}[x]$ *and* L *be as in* Problem 1.

(a) *(3 points) Give generators and relations for* $\mathrm{Gal}(L/\mathbf{Q})$.
(b) *(2 points) Show that* $\mathrm{Gal}(L/\mathbf{Q})$ *is solvable.*
(c) *(1 point) Show that* f *is solvable by radicals.*
(d) *(1 point) Let* α *be a root of* f *in* L. *Is* α *constructible by straightedge and compass? Explain.*

Problem 3. *Let* ζ_{11} *be a primitive* 11*th root of unity in a field of characteristic zero.*

(a) *(1 point) Show that* $\mathbf{Q}(\zeta_{11})/\mathbf{Q}$ *is Galois.*
(b) *(2 points) Give an explicit description of* $\mathrm{Gal}(\mathbf{Q}(\zeta_{11})/\mathbf{Q})$
(c) *(2 points) Let* $\alpha = \zeta_{11} + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9$. *Find* $m_{\alpha,\mathbf{Q}}(x)$.
(d) *(2 points) Let* $\gamma = \zeta_{11} + \zeta_{11}^{-1}$. *Find* $m_{\gamma,\mathbf{Q}}(x)$.
(e) *(1 point) Find* $m_{\zeta_{11},\mathbf{Q}(\gamma)}(x)$.

Problem 4.

(a) *(2 points) Construct a Galois extension of* $\mathbf{Q}$ *with Galois group* $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$.
(b) *(1 point) Let* $g(x) = x^3 + x^2 - 2x - 1 \in \mathbf{Q}[x]$. *What subgroup of* S_3 *is isomorphic to* $\mathrm{Gal}(g)$*? Explain.*
(c) *(2 points) Let* p *be a prime. Suppose* $x^3 + ax + b \in \mathbf{F}_p[x]$ *is irreducible. Show that* $-4a^3 - 27b^2$ *is a square in* $\mathbf{F}_p$.

Problem 5. *Let* $h(x) = x^{16} + x^{15} + \cdots + x + 1 \in \mathbf{Z}[x]$.

(a) *(1 point) Suppose* p *is a prime,* $p \equiv 1 \pmod{17}$. *Show that* $h(x)$ *splits completely in* $\mathbf{F}_p[x]$.
(b) *(2 points) Suppose* p *is a prime,* $p \equiv 2 \pmod{17}$. *Show that* $h(x)$ *factors in* $\mathbf{F}_p[x]$ *as a product of two irreducible polynomials, both of degree* 8.
(c) *(2 points) Show that* $x^3 + x + 1$ *divides* $x^{343} - x$ *in* $\mathbf{F}_7[x]$. Note: *Long division is highly discouraged in this problem.*

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