FINAL EXAM

Instructions: Justify your answers. You may use results from the homework sets and from class, but make sure to carefully state such results. No calculators and no notes allowed.

Grading: This exam is worth 30 points. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

Problem 1. *Let $f(x) = x^5 - 3 \in \mathbf{Q}[x]$.*

(a) *(1 point) Show that f is irreducible over $\mathbf{Q}$.*
(b) *(2 points) Give an explicit description of a splitting field L for f.*
(c) *(1 point) Compute $[L : \mathbf{Q}]$.*
(d) *(1 point) Show that $L/\mathbf{Q}$ is Galois.*

Problem 2. *Let $f(x) = x^5 - 3 \in \mathbf{Q}[x]$ and L be as in* Problem 1.

(a) *(3 points) Give generators and relations for $\mathrm{Gal}(L/\mathbf{Q})$.*
(b) *(2 points) Show that $\mathrm{Gal}(L/\mathbf{Q})$ is solvable.*
(c) *(1 point) Show that f is solvable by radicals.*
(d) *(1 point) Let α be a root of f in L. Is α constructible by straightedge and compass? Explain.*

Problem 3. *Let ζ_7 be a primitive 7th root of unity in a field of characteristic zero.*

(a) *(1 point) Show that $\mathbf{Q}(\zeta_7)/\mathbf{Q}$ is Galois.*
(b) *(2 points) Give an explicit description of $\mathrm{Gal}(\mathbf{Q}(\zeta_7)/\mathbf{Q})$*
(c) *(2 points) Let $\alpha = \zeta_7 + \zeta_7^2 + \zeta_7^4$. Find $m_{\alpha,\mathbf{Q}}(x)$.*
(d) *(2 points) Let $\gamma = \zeta_7 + \zeta_7^{-1}$. Find $m_{\gamma,\mathbf{Q}}(x)$.*
(e) *(1 point) Find $m_{\zeta_7,\mathbf{Q}(\gamma)}(x)$.*

Problem 4.

(a) *(2 points) Construct a Galois extension of $\mathbf{Q}$ with Galois group $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.*
(b) *(1 point) Let $g(x) = x^3 - 2x + 4 \in \mathbf{Q}[x]$. What subgroup of S_3 is isomorphic to $\mathrm{Gal}(g)$? Explain.*
(c) *(2 points) Now view $g(x)$ as a polynomial in $\mathbf{Q}(i)[x]$, where i is a square root of -1. What subgroup of S_3 is isomorphic to $\mathrm{Gal}(g)$ in this case?*

Problem 5. *Let $h(x) = x^{12} + x^{11} + \cdots + x + 1 \in \mathbf{Z}[x]$.*

(a) *(1 point) Suppose p is a prime, $p \equiv 1 \pmod{13}$. Show that $h(x)$ splits completely in $\mathbf{F}_p[x]$.*
(b) *(2 points) Suppose p is a prime, $p \equiv 2 \pmod{13}$. Show that $h(x)$ is irreducible in $\mathbf{F}_p[x]$.*
(c) *(2 points) Show that $x^3 - x + 2$ divides $x^{125} - x$ in $\mathbf{F}_{125}[x]$.* Note: *Long division is highly discouraged in this problem.*

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