MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Facit solutions for the written exam in
Ordinary differential equations,
May 23, 2024

OBS: This is a facit with comments!
As a solution at the exam it would not give full points as parts are missing!!!

1. Determine the solution(s) of the differential equation

$$xx' = t(x^2 + 1)$$

satisfying the initial condition:

(a) $x(0) = 1$
(b) $x(0) = -1$
(c) $x(0) = 0$.

Comment on your result with respect to the *existence and uniqueness theorem*! (4p)

$$xx' = t(x^2+1) \iff \frac{2x \cdot x'}{x^2+1} = 2t \iff \ln|x^2+1| = t^2 + C$$

$$\iff x^2 = D \cdot e^{t^2} - 1 \iff \underline{x(t) = \pm\sqrt{D \cdot e^{t^2} - 1}} \qquad \underline{D \in \mathbb{R}}$$

a) $x(0) = 1 \Rightarrow \ldots \Rightarrow \underline{x(t) = \sqrt{2e^{t^2} - 1}}$

b) $x(0) = -1 \Rightarrow \ldots \Rightarrow \underline{x(t) = -\sqrt{2e^{t^2} - 1}}$

entydiga lösningar, in accordance with the existence and uniqueness theorem, as $f(t,x) = \frac{t(x^2+1)}{x}$ is C^1 as long as $x \neq 0$.

c) $x(0) = 0 \Rightarrow \ldots \Rightarrow \underline{x(t) = \pm\sqrt{e^{t^2} - 1}}$

Non-unique, no contradiction to the theorem as in $(t_0, x_0) = (0,0)$ the assumptions are not satisfied!

OBS: In principle we also should check that the solutions are differentiable!

$$\lim_{h \to 0^\pm} \frac{\sqrt{e^{h^2}-1} - 0}{h - 0} = \lim_{h \to 0^\pm} \frac{\sqrt{e^{h^2}-1}}{\pm\sqrt{h^2}} = \lim_{h \to 0^\pm} \pm\sqrt{\frac{e^{h^2}-1}{h^2}} = \pm 1$$

So the solutions are $x_+(t) := \begin{cases} \sqrt{e^{t^2}-1} & t \geq 0 \\ -\sqrt{e^{t^2}-1} & t \leq 0 \end{cases}$ and $x_-(t) := -x_+(t)$

2. Is there an *ordinary linear homogeneous second order differential equation with constant coefficients* for which the functions $x_1(t) = \cos 2t$ and $x_2(t) = t^2 e^{-7t}$ are solutions? If yes, give an example. If no, explain why! (2p)

No!

The first solution $x_1(t) = \cos 2t$ is related to the characteristic values $\pm 2i$, whereas the second solution $x_2(t) = t^2 e^{-7t}$ is related to the characteristic value -7 with multiplicity at least 3.
Hence a linear homogenous ODE with const. coefficients has order at least $2+3=5$.

Alternative: One might also try to find $a, b \in \mathbb{R}$ such that x_1 and x_2 satisfy

$$x'' + a x' + b x = 0,$$

and see that this is not possible.

3. Use Laplace transform to solve the initial value problem (4p)

$$x'' - 2x' = -2e^{2t}, \quad \text{with} \quad x(0) = 1, \quad x'(0) = -1.$$

With $X := \mathcal{L}(x(t))$ we obtain

$$s^2 X - s\,x(0) - x'(0) - 2(sX - x(0)) = -2 \underbrace{\mathcal{L}(e^{2t})}_{= \frac{1}{s-2}}$$

$$\Rightarrow \dots \Rightarrow X(s) = \frac{s^2 - 5s + 4}{s(s-2)^2} = \dots = \frac{1}{s} - \frac{1}{(s-2)^2}$$

$$\Rightarrow \underline{x(t) = 1 - t \cdot e^{2t}}$$

4. (a) Determine a fundamental matrix for the homogeneous system (2p)

$$\vec{x}\,' = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \vec{x}.$$

(b) Give the general solution of the inhomogeneous system (4p)

$$\vec{x}\,' = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}.$$

a) There are several possibilities to solve the problem. We give here two alternatives.

Variant 1: $A = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \Rightarrow \dots \Rightarrow \lambda_1 = -1 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1 \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$T := \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \Rightarrow A = T \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot T^{-1}$$

$$\Rightarrow e^{tA} = T \cdot \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \cdot T^{-1} = \dots = \underline{\begin{pmatrix} e^{-t} & \frac{3}{2}(e^{t} - e^{-t}) \\ 0 & e^{t} \end{pmatrix}}$$

Variant 2: two linearly independent solutions are

$e^{-t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e^{t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Hence

$\underline{F(t) := \begin{pmatrix} e^{-t} & 3e^{t} \\ 0 & 2e^{t} \end{pmatrix}}$ is a fundamental matrix

b) Variant 1: $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

Ansatz för $\vec{x}_p(t) = e^{2t} \cdot \vec{v} \Rightarrow \ldots \Rightarrow \vec{v} = (A - 2I)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \ldots = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad \vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Variant 2: Use formula $\vec{x}(t) = F(t) \left(\vec{c} + \int_0^t F(\tau)^{-1} \vec{b}(\tau) \, d\tau \right)$

5. Consider the following boundary value problem (BVP)

$$-x'' = \lambda x$$
$$x(0) = x'(0)$$
$$x(1) = x'(1).$$

(a) For which values of $\lambda \in \mathbb{R}$ does the BVP have a non-trivial solution? Determine also these solutions. (3p)

(b) Are there values of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ for which the BVP has a non-trivial solution? Motivate your answer! (1p)

a) We distinguish 3 cases:

1) $\lambda < 0 \Rightarrow \lambda = -\omega^2$ with $\omega > 0$.

$$x'' - \omega^2 x = 0 \Rightarrow x(t) = A e^{\omega t} + B \cdot e^{-\omega t}$$
$$x'(t) = \omega A e^{\omega t} - \omega B e^{-\omega t}$$

boundary conditions give:

$$\left.\begin{array}{l} A + B = \omega A - \omega B \\ A e^{\omega} + B e^{-\omega} = \omega A e^{\omega} - \omega B e^{-\omega} \end{array}\right\} \Leftrightarrow \begin{array}{l} (1-\omega) A + (1+\omega) B = 0 \\ (1-\omega) e^{\omega} A + (1+\omega) e^{-\omega} B = 0 \end{array}$$

there is a non-trivial solution (i.e. $(A,B) \neq (0,0)$) if and only if

$$\det \begin{pmatrix} 1-\omega & 1+\omega \\ (1-\omega) e^{\omega} & (1+\omega) e^{-\omega} \end{pmatrix} = 0.$$

this is $(1-\omega)\underbrace{(1+\omega)}_{>0}\underbrace{(e^{-\omega} - e^{\omega})}_{\neq 0 \text{ for } \omega \neq 0} = 0$

Hence $\omega = 1$. So $\lambda = -1$ and $x(t) = e^{-t}$.

2) $\lambda = 0 \Rightarrow x'' = 0 \Rightarrow x(t) = A + Bt$

$\Rightarrow A = B$ and $A + B = B \Rightarrow A = B = 0$

3) $\lambda > 0 \Rightarrow \lambda = \omega^2$ with $\omega > 0$

$x'' + \omega^2 x = 0 \Rightarrow x(t) = A\cos\omega t + B\sin\omega t$

$x'(t) = -\omega A\sin\omega t + \omega B\cos\omega t$

$\Rightarrow A = \omega B$ and $\cancel{A\cos\omega} + B\sin\omega t = -\omega A\sin\omega t + \cancel{\omega B\cos\omega t}$

$\Rightarrow \ldots \Rightarrow B\underbrace{(1+\omega^2)}_{>0}\sin\omega t = 0$

There exists a nontrivial solution if and only if $\sin\omega t = 0$. This is $\omega = k\pi$ $k = 1, 2, \ldots$

So $\lambda_k = (k\pi)^2$ and $x_k(t) = k\pi\cos k\pi t + \sin k\pi t$.

Answer: The real eigenvalues are $\lambda_0 = -1$ and $\lambda_k = (k\pi)^2$ for $k = 1, 2, \ldots$ with corresponding eigenfunction $x_0(t) = e^t$ and $x_k(t) = k\pi\cos k\pi t + \sin k\pi t$.

b) The boundary value problem is symmetric and hence the eigenvalues real.

6. Consider the system

$$\dot{x} = -y + x\cos(x^2+y^2)$$
$$\dot{y} = x + y\cos(x^2+y^2).$$

Find all equilibrium points (i.e. stationary points). Sketch the phase portrait and describe it also briefly by word. What can be said about the stability of the equilibrium point(s)?

It might be helpful to use polar coordinates! (4p)

equilibrium points:

$$\left.\begin{array}{ll} -y + x\cos(x^2+y^2) = 0 & \cdot(-y) \\ x + y\cos(x^2+y^2) = 0 & \cdot x \end{array}\right\} +$$

$\Rightarrow y^2 + x^2 = 0 \Rightarrow x = y = 0$ (0,0) is only equilibrium

Linearization:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

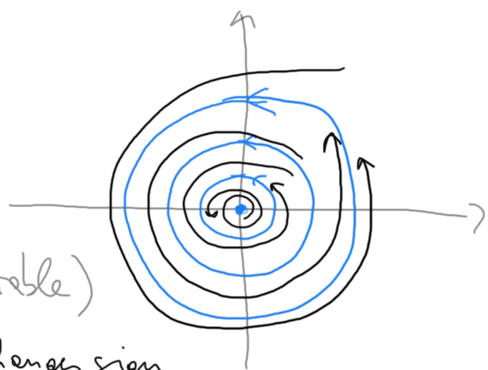
trace $A = 2 > 0 \Rightarrow$ (0,0) unstable

(alternative: $\lambda_{1,2} = 1 \pm i \Rightarrow \operatorname{Re}\lambda_{1,2} > 0$
$\Rightarrow$ (0,0) unstable)

polar coordinates: $x = r\cos\varphi$
$y = r\sin\varphi$

$$\Rightarrow \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi = -r\sin\varphi + r\cos\varphi\cdot\cos r^2$$
$$\dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi = r\cos\varphi + r\sin\varphi\cdot\cos r^2$$

$$\Rightarrow \dot{r} = r\cos r^2$$
$$\dot{\varphi} = 1$$



$\dot{r} = 0 \Leftrightarrow r = \sqrt{\frac{2k+1}{2}\pi}$ $k = 0,1,2,\ldots$

periodic orbits (every second stable)

inbetween spirals, positive orientated; $\dot{r}$ changes sign

Sketch by hand, only qualitative picture!!