Introduction to Real Analysis

Lecture 5: Continuous functions

Sofia Tirabassi
tirabassi@math.su.se

Stockholm University

Questions?

Stockholm University

Lecture Plan

Rudin Chapter 4

- Limits of functions
- Series (3.22-3.51)

Limits

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. Consieder $E \subseteq X$, $f : E \to Y$ and $p \in E'$ we say that

$$\lim_{x \to p} f(x) = q$$

if, for every $\varepsilon > 0$ there is a $\delta(\varepsilon, p) > 0$ such that $d_Y(f(x), q) < \varepsilon$ whenever $d_X(x, p) < \delta$

↳ depends of ε and p.

We say that a function $f : X \to Y$ is continuous at $p \in X$ if if, for every $\varepsilon > 0$ there is a $\delta(\varepsilon, p) > 0$ such that $d_Y(f(x), f(p)) < \delta$ whenever $d_X(x, p) < \epsilon$. The function f is said continuous if it is continuous at every $p \in X$.

↳ p ∈ Domain of X.

Exist.

$$\lim_{x \to p} f(x) = f(p).$$

Limits

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. Consieder $E \subseteq X$, $f : E \to Y$ and $p \in E'$ we say that

$$\lim_{x \to p} f(x) = q$$

if, for every $\varepsilon > 0$ there is a $\delta(\varepsilon, p) > 0$ such that $d_Y(f(x), q) < \delta$ whenever $d_X(x, p) < \epsilon$

We say that a function $f : X \to Y$ is continuous at $p \in X$ if if, for every $\varepsilon > 0$ there is a $\delta(\varepsilon, p) > 0$ such that $d_Y(f(x), f(p)) < \delta$ whenever $d_X(x, p) < \epsilon$. The function f is said continuous if it is continuous at every $p \in X$.

$\frac{1}{x}$ is continuous in its domain

$\mathbb{R} \setminus \{0\}$.

let $p \neq 0$ $\quad p \in \mathbb{R} \setminus \{0\}$

$$\lim_{x \to p} \frac{1}{x} = \frac{1}{p}$$

$-\varepsilon \quad : \quad : \quad \delta$

Fix ε

$$\left| \frac{1}{x} - \frac{1}{p} \right| \overset{?}{<} \varepsilon$$

$$\left| \frac{p - x}{xp} \right|$$

$|p-x| < \varepsilon |px| < \varepsilon |p| (p+2\varepsilon)$

$\delta = \delta(p, \varepsilon)$

Continuity

Remark

If p is not an isolated point of X, then f is continuous at p iff $\lim_{x \to p} f(x)$ exists in Y and it is equal to $f(p)$.
Every function is continuous at isolated points

Thus we have that

A function $f : X \to Y$ is continous iff for all $p_n \to p$ we have that $f(p_n) \to f(p)$.

Topological continuity

in ~~X~~

Theorem

A function $f : X \to Y$ is continuos iff $f^{-1}(V)$ is open for every $V \subseteq Y$ open set.

open in Y.

Proof $\Rightarrow$ f continuos

V open in Y

want $f^{-1}(V)$ open in X

$\boxed{p \in f^{-1}(V)}$ want $\exists\, r>0$ such that $N_r(p) \subseteq f^{-1}(V)$

$f(p) \in V$ open $\exists$ $\varepsilon > 0$ such that

$N_{\underline{\varepsilon}}(f(p)) \subseteq V$

$\exists$ $\delta(p, \varepsilon)$ such that | f contin.

$d_X(x, p) < \underline{\delta}(p, \varepsilon)$

$\Rightarrow$ $d_Y(f(x), f(p)) < \varepsilon$

$x \in N_{\delta(p,\varepsilon)}(p) \Rightarrow f(x) \in N_\varepsilon(f(p))$

$\cap$

V

$$N_\varepsilon(f(p)) \subseteq V$$

$$\boxed{N^X_{\underline{\delta(p,\varepsilon)}}(p) \subseteq f^{-1}(N_\varepsilon(f(p)))} \subseteq f^{-1}(V)$$

$\hookrightarrow$ OPEN.

take $r = \delta(p,\varepsilon)$

$\Leftarrow$ $f^{-1}(V)$ open when V open

Fix $\varepsilon > 0$ $\quad N^Y_{\varepsilon/2}(q)$ open in Y

$\Rightarrow$ $f^{-1}(N^Y_{\varepsilon/2}(q))$ open. in X

if $p \in f^{-1}(N_{\varepsilon}^{Y}(q))$ then $\exists$

$\boxed{\delta}$ such that $N_\delta(p) \subseteq f^{-1}(N_{\varepsilon/2}(q))$

$\Rightarrow$ $f(N_\delta(p)) \subseteq f(f^{-1}(N_{\varepsilon/2}(q)))$

$x \in \quad p \in$

$\subseteq N_{\varepsilon/2}(q)$

if $d(x,p) < \delta$

$\Rightarrow$ $d(f(x), q) < \varepsilon/2$

$d(f(x)\ f(p)) \overset{\Delta}{\leq} d(f(x)\ \ q) < \frac{\varepsilon}{2}$

$+ d(\underline{f(p)\ \ q}) < \varepsilon/2$ $\quad < \varepsilon$ $\quad \parallel$

Topological continuity

Stockholm University

Theorem

A function $f : X \to Y$ is continuos iff $f^{-1}(V)$ is open for every $V \subseteq Y$ open set.

Corollary

A function $f : X \to Y$ is continuos iff $f^{-1}(C)$ is open for every $C \subseteq Y$ closed set.

Continuity and Compactness

Stockholm University

Theorem

Let $f : X \to Y$ a continuous function. If $K \subseteq X$ is compact, then $f(K)$ is compact. In particular if $Y \simeq \mathbb{R}^n$ we have that $f(K)$ is closed and bounded

Corollary

If $f : X \to \mathbb{R}$ is continuous and $K \subseteq X$ is compact, then f has a max and a min value on K.

Theorem

If $f : K \to Y$ is continous and bijective, then the inverse funciton $f^{-1} : Y \to K$ is continuos (we say that f is an omeomorphism).

COR: $f : [a,b] \longrightarrow \mathbb{R}$ cont f has max & min

$f(K)$ is cpt in $\mathbb{R}$

$\hookrightarrow$ closed & bounded

$\downarrow$ closed: inf & sup belong to K (they are min & max)

$\downarrow$ bounded: inf sup of $f(K)$ $\exists$ in $\mathbb{R}$

Proof of thm 1:

$f(K)$ we take $\{U_\alpha\}_{\alpha \in A}$ open cover.

$$f(K) \subseteq \bigcup_\alpha U_\alpha$$

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha \underbrace{f^{-1}(U_\alpha)}_{\text{OPEN}}$$

$\exists\ \alpha_1 \ \dots \ \alpha_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$$

$$f(K) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})\right)$$

set theory: $\subseteq \boxed{\bigcup_{i=1}^{n} U_{\alpha_i}}$ finite subcov for $f(K)$

$x = f(b)$ $\quad b \in \bigcup f^{-1}(U_\alpha)$

$b \in f^{-1}(U_{\bar\alpha})$ for some $\bar\alpha$

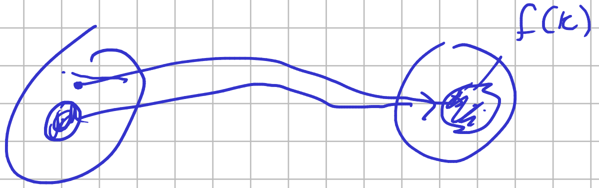
$\Rightarrow \quad f(b) \in U_{\bar{\alpha}} \subseteq \bigcup U_\alpha$

$$K \subseteq f^{-1}(f(K))$$

$x \qquad f(x) \in f(K)$

$x = k \qquad f(x) = f(k) \qquad k \in K$

$f(k)$



Proof of theorem 2

$f : K \longrightarrow Y$ cont and bijection

$g : Y \longrightarrow K$ inverse function

cont iff $g^{-1}(C)$ is closed

for every $C \subseteq K$ closed

$C \subseteq K$ closed is cpt since K is

$g^{-1}[C] = f(C)$ cpt $\Rightarrow$ closed

(think about it) $\Rightarrow$ g is continuous !! ☺

Uniform Continuity

Definition

We say that a function $f : X \to Y$ is uniformly continuous on X if for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ for all x and y in X such that $d_X(x, y) < \delta$.

Classical non example

$f : \mathbb{R}^+ \longrightarrow \mathbb{R}$

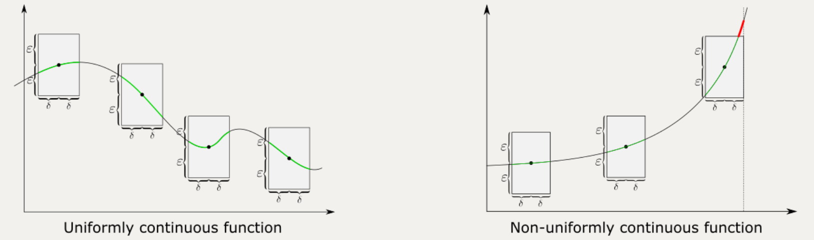
$f(x) = \frac{1}{x}$ $\longrightarrow$ the closer you get to 0 the smaller need to be the δ given ε.

Uniform Continuity

the δ is uniform!

Definition

We say that a function $f : X \to Y$ is uniformly continuous on X if for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ for all x and y in X such that $d_X(x, y) < \delta$.



Uniform continuity and compactness

Theorem

Let K be a compact metric space. If $f : K \to Y$ is continuous then it is uniformly continuous.

Proof : Fix $\varepsilon > 0$ f is continuos

for all $p \in K$ $\exists$ $\delta(\varepsilon, p) =: \delta(p)$

such that $f(N_{\delta(p)}(p)) \subseteq N_{\varepsilon/2}(f(p))$

$\{N_{\frac{1}{2}\delta(p)}(p)\}_{p \in K}$ open cover for K

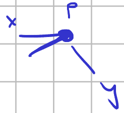
$\exists \quad p_1 \quad \dots \quad p_n$

$$K \subseteq \bigcup_{i=1}^{n} N_{\frac{1}{2}\delta(p_i)}(p_i)$$

K cpt

$d(x,y)$

$$\delta(\varepsilon) := \min_j \left\{ \frac{1}{2} \delta(p_j) \right\}$$



$x, y \in K \qquad d(x,y) <' \delta(\varepsilon)$

$$d(f(x)\ f(y)) \leq \underbrace{d(f(x)\ f(p_i))}_{< \varepsilon/2} + \underbrace{d(f(p_i),\ f(y))}_{< \varepsilon/2}$$

$$\Longrightarrow \quad d(f(x)\ f(y)) < \varepsilon$$

Continuity and connectedness

Theorem

Let $f : X \to Y$ be a continuous function. If $E \subseteq X$ is connected then $f(E)$ is connected.

Connected

Not the union of two separated set

↳ the of ten

There are no two open sets U_1 and U_2 $E \subseteq U_1 \cup U_2$

$E \cap U_1 \neq \phi$ $E \cap U_2 \neq \phi$ $E \cap U_1 \cap U_2 = \phi$

Proof $f(E) \subseteq U_1 \cup U_2$

U_i open with $U_1 \cap U_2 \cap f(E) = \phi$

$$\underline{E} \subseteq f^{-1}(f(E)) \subseteq f^{-1}(U_1 \cup U_2) = \underbrace{f^{-1}(U_1) \cup f^{-1}(U_2)}_{\text{open}}.$$

$$f^{-1}(U_1) \cap f^{-1}(U_2) \cap E$$

$$\cap\!|$$

$$f^{-1}(U_1) \cap f^{-1}(U_2) \cap f^{-1}(E)$$

$$= f^{-1}(U_1 \cap U_2 \cap E)$$

$$= f^{-1}(\phi) = \phi$$

$$f^{-1}(U_1) \cap f^{-1}(U_2) \cap E = \phi$$

Since E is connected

$$f^{-1}(U_1) \cap E = \phi$$

$$\Rightarrow U_1 \cap f(E) = \phi$$

or

$$f^{-1}(U_2) \cap E = \phi$$

$$\Rightarrow U_2 \cap f(E) = \phi$$

$\Rightarrow f(E)$ is conected

Example

$GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is not connected

$\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$

↳ cannot be connected.

$(-\infty, 0) \cup (0, +\infty)$

Section 1
Discontinuities

What can go wrong?

Stockholm
University

What can go wrong?

Stockholm University

Let $f : (a, b) \to Y$ a function and let p be a point such that f is not continuous at p. We set (if they exist)

$$f(p+) := \lim_{x \to p^+} f(x) \qquad f(p-) := \lim_{x \to p^-} f(x)$$

Definition

We say that f has a discontinuity of the first kind at p if $f(p+)$ and $f(p-)$ exist. Otherwise we say that it has a discontinuity of the second kind.

Discontinuities for monotone functions

What is a monotone function $f : (a, b) \to \mathbb{R}$?

Increasing $\quad x < y \Rightarrow f(x) \leq f(y)$

decreasing $\quad x < y \Rightarrow f(x) \geq f(y)$

Discontinuities for monotone functions

f monotone $f: (a,b) \longrightarrow \mathbb{R}$

~~What is a monotone function $f : (a, b) \to \mathbb{R}$?~~

Theorem

If $f : (a, b) \to \mathbb{R}$, then it has no discontinuities of the second kind.

Theorem

If $f : (a, b) \to \mathbb{R}$, then it has at most countably many discontinuity points.

Proof 1 : (Sketch) $\to$ f increasing

$x \in (a,b)$

$t < x \Rightarrow f(t) \leq f(x)$

$A := \{f(t) \mid t \in (a,x)\}$ bounded by $f(x)$ above

$\exists \quad \alpha := \sup A$

$B = \{ f(t) \mid t\ (x, b) \}$ bounded below

$\beta = \inf B$

Using the def of limit

$$\alpha = \lim_{t \to x^-} f(t)$$

$$\beta = \lim_{t \to x^+} f(t)$$

Fix ε $\quad \alpha - \varepsilon$ not an upper bound

$\exists \quad t \in (a, x) \quad \alpha > f(t) > \alpha - \varepsilon$

f increasing for all $t' > t$

$\alpha > f(t') > f(t) > \alpha - \varepsilon$

$\delta = |x - t|$

if $|x - t'| < \delta$ $\quad t' < x$

$\Rightarrow t' > t$

$\Rightarrow |\alpha - f(t)| < \varepsilon$

Proof 2 (Sketch) $\quad x \in (a, b)$

disc. $\quad \exists \; f(x^+) \quad$ choose $\in \mathbb{Q}$ $\quad f(x^-)$

$f(x^-) < r(x) < f(x^+)$

at most countable.

$r : \left\{ \text{Discontinuities of } f \right\} \xrightarrow{\text{injective}} \mathbb{Q}$

$\hookrightarrow$ At most cardinal $|\mathbb{Q}|$

injective $\quad x' > x \quad f$ incr

$r(x) < \underbrace{f(x+) \leq f(x'-)} < r(x') < f(x'+)$

for different x they belong to the interior of different interval.

Thank you for your attention!