Introduction to Real Analysis

Lecture 6: The Riemann–Stieljes integral

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Questions?

Lecture Plan

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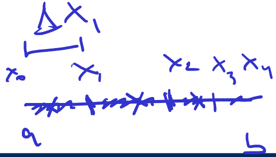
Rudin Chapter 6

Partitions



Definition

Given $[a, b]$ a closed bounded interval, a partition of its is a finite set $\{x_0, \ldots, x_n\} \subset [a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_n = b,$$

Given a partition $P = \{x_0, \ldots x_n\}$ we set

$$\Delta x_i := x_i - x_{i-1} \qquad i = 1 \ldots n$$

$\Delta x = x_1 - x_0$

Definition

A refinement of a partition P is another partition P' such that $P' \subseteq P$.

Partitions

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Definition

A refinement of a partition P is another partition P' such that $P' \subseteq P$.

Observe that given two partitions P_1 and P_2, the partition $P^* = P_1 \cup P_2$ gives a common refinement.

Upper and lower sums (not assuming cont)

[ ] → bounded

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\alpha : [a, b] \to \mathbb{R}$ a monotonically increasing function.

Remark

Observe that, since both $\alpha(a)$and $\alpha(b)$ are real numbers, we have that α is bounded.

Given P a partition of $[a, b]$ we set $\Delta\alpha_1 = \alpha(x_1) - \alpha(x_0) \geq 0$

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$$

α increas

and

$$\mathbb{R} \ni \quad M_i := \sup_{[x_{i-1}, x_i]} f \qquad m_i := \inf_{[x_{i-1}, x_i]} f$$

Theen we have

$$U(P, f, \alpha) := \sum_{i=1}^{n} M_i \cdot \Delta\alpha_i \quad L(P, f, \alpha) := \sum_{i=1}^{n} m_i \cdot \Delta\alpha_i$$

Upper & lower sum associated to P, f, α

Example $[a,b] = [0, 2]$

$$P = \{0, 1, 2\}$$

$$f : [0,2] \longrightarrow \mathbb{R}$$
$$x \longmapsto x$$

$$\alpha : [0, 2] \longrightarrow \mathbb{R} = f$$

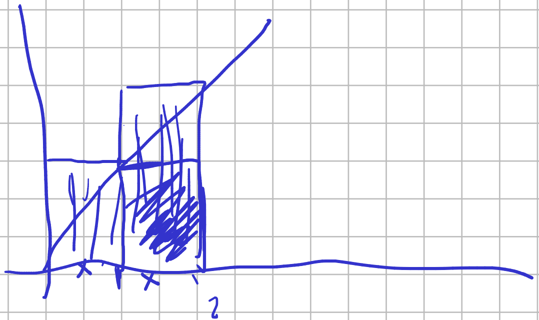
$M_1 = \sup_{[0,1]} f = 1$ $\quad m_1 = 0$

$M_2 = \sup_{[1,2]} f = 2$ $\quad m_2 = 1$

$\Delta\alpha_1 = 1$ $\qquad \Delta\alpha_2 = 1$

$U(P, f, \alpha) = 1 \cdot 1 + 2 \cdot 1 = 3$

$L(P f \alpha) = 0 \cdot 1 + 1 \cdot 1 = 1$



Upper and Lower integrals

upper integral of f w r α

We set

$$\overline{\int_a^b} f\, d\alpha := \inf\{\underline{U(P, f, \alpha) \mid P \text{ is a partition}}\}$$

and

the lower integral

$$\underline{\int_a^b} f\, d\alpha := \sup\{L(P, f, \alpha) \mid P \text{ is a partition}\}$$

If they both exists and are equal, we denote the common value by

$$\int_a^b f\, d\alpha$$

The Stieltjes integral w.r.t. α

and we say that f is Stieljes integrable on $[a, b]$ with respect to α - we write

$$\boxed{f \in \mathscr{R}(\alpha)}$$

$\alpha(x) = x$

$\int_a^b f\, dx$ usual (Riemann)

Upper and Lower integrals

We set

$$\overline{\int_a^b} f\, d\alpha := \inf\{U(P, f, \alpha) \mid P \text{ is a partition}\}$$

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and we say that f is Stieljes integrable on $[a, b]$ with respect to α - we write

$$f \in \mathscr{R}(\alpha)$$

Note: if $\alpha(x) = x$, then this is the "usual" integral you have seen in previous analysis courses.

Questions

for which f?

- When does it exist?
- How we compute it?

Refinements

Theorem

If P^* is a refinements of P we have that

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

and

$$U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

Sketch of the proof (for U)

By induction on $|P^* \setminus P| < \infty$

Base case $P^* = P \cup \{x^*\}$

$x^* \in [x_{i-1}, x_i]$ $\quad P = \{x_0 \ldots x_n\}$

$$w_1 = \sup_{[x_{i-1} x^*]} f \leq M_i$$

$$w_2 = \sup_{[x^* x_i]} f \leq M_i$$

$< 0 \quad M_j \Delta\alpha_i$

$$\underline{U(P^*, f, \alpha) - U(P f \alpha)} \quad w_1$$

$$= \sum_{j=1}^{i-1} M_j \Delta\alpha_j + w_1 \left(\alpha(x^*) - \alpha(x_{i-1})\right)$$

$$+ \; w_2 \left(\underline{\alpha(x_i) - \alpha(x^*)} \right)$$

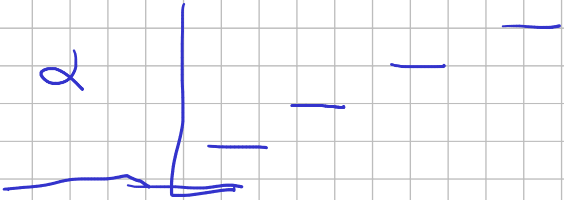
$$+ \sum_{j=i+1}^{n} \Pi_j \, \Delta\alpha_j \; - \sum_{j=1}^{n} \Pi_j \, \Delta\alpha_j$$

$$\begin{aligned}
= \; & \underline{w}_1 \left(\alpha(x^*) - \alpha(x_{i-1}) \right) \\
& + \underline{w}_2 \left(\alpha(x_i) - \alpha(x^*) \right) \\
& \left. \begin{aligned} & - \underline{\Pi}_i \left(\alpha(x_i) - \alpha(x^*) \right) \\ & - \Pi_i \left(\alpha(x^*) - \alpha(x_{i-1}) \right) \end{aligned} \right\} - \Pi_i \Delta\alpha_i
\end{aligned}$$

$$\overset{<0}{(w_1 - M_i)}\,(\Delta\alpha^*)^{>}$$

$$+\ \overset{<0}{(w_2 - M_i)}\,(\Delta\alpha_2^*)^{>0} \quad < 0$$

sum of two negative



Refinements

Theorem

If P^* is a refinements of P we have that

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

and

$$U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

Corollary

$$\underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha$$

$$\overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha = 0$$

Equivalent condition for the existence

Now on $f : [a,b] \longrightarrow \mathbb{R}$ bounded

Theorem

Let $f :\in \mathscr{R}(\alpha)$ iff, for every $\varepsilon > 0$ there is a partition P such that

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \tag{1}$$

$\mathscr{R}_{[a,b]}(\alpha)$

$\exists \int_a^b f dt.$

Consequences

Theorem

1. If (1) holds for some ε and some partition P then it holds for every refinement of P.
2. If (1) holds, $P = \{x_0, \dots, x_n\}$, and s_i and t_i denote elements of $[x_{i-1}, x_i]$ then
$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon,$$
3. If $f \in \mathscr{R}(\alpha)$, $P = \{x_0, \dots, x_n\}$, $t_i \in [x_{i-1}, x_i]$ then
$$\left| \sum_{i=1}^{n} |f(t_i)| \Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon,$$

1) P (1) holds for P

P^* refinement

$$0 < U(P^* f \alpha) - L(P^* f, \alpha) <$$

$$\begin{matrix} \wedge & & \vee \\ U(P f \alpha) & & L(P f, \alpha) \end{matrix}$$

$$U(P f \alpha) - L(P f \alpha) < \varepsilon$$

$$t_i \; s_i \in [x_{i-1}, x_i]$$

(b)

$$\boxed{\sum_i |f(s_i) - f(t_i)| \Delta\alpha_i} < \varepsilon$$

$$|f(s_i) - f(t_i)| \leq \underbrace{\sup_{[x_{i-1} x_i]} f}_{=M_i} - \underbrace{\inf_{[x_{i-1} x_i]} f}_{=m_i}$$

$$\leq \sum_i (M_i - m_i) \Delta\alpha_i$$

$$= \sum_i M_i \Delta\alpha_i - \sum_i m_i \Delta\alpha_i$$

$$= U(P, f, \alpha) - L(P, f, \alpha) \overset{(1)}{<} \varepsilon$$

Integrability

A

Theorem

If f is continuous on $[a, b]$ then $f \in \mathscr{R}(\alpha)$.

B

Theorem

If f is monotonic on $[a, b]$ and α is continuous then $f \in \mathscr{R}(\alpha)$.

C

Theorem

If f is bounded on $[a, b]$, has finitely many discontinuity points and α is continuous at those points $f \in \mathscr{R}(\alpha)$.

D

Theorem

If $f \in \mathscr{R}(\alpha)$, $m \leq f(x) \leq M$ for all $x \in [a, b]$, and ϕ is continuous on $[m, M]$ then the composition $\phi \circ f \in \mathscr{R}(\alpha)$

Proof A

f continuos on $[a,b]$ cmpct

$\Rightarrow$ f is unif. cont (δ does not depend of x)

$\boxed{\text{Fix } \varepsilon}$

$$\frac{\alpha(b) - \alpha(a)}{n} < \varepsilon \text{ for some } n >> 0$$

f unif $\Rightarrow$ $\exists\, \delta\left(\frac{1}{n}\right)$ such that

$|f(s) - f(t)| < \frac{1}{n}$ whenever $|s-t| < \delta\left(\frac{1}{n}\right)$

Let $\boxed{P \text{ a partition}}$ such that

$$\Delta x_i = x_i - x_{i-1} < \delta(1/n)$$

Eg $\quad x_i \leftarrow a + i\,\dfrac{\delta}{2}$ $\quad s^* \quad t_*$

$$\underbrace{\sup_{[x_{i-1},x_i]} f}_{M_i} \overset{= f(s^*)}{} - \underbrace{\inf_{[x_{i-1},x_i]} f}_{m_i} \overset{= f(t_*)}{} < \frac{1}{n} \qquad \in [x_{i-1}, x_i]$$

$$U(P\, f\, \alpha) - L(P\, f\, \alpha)$$

$$= \sum_i (M_i - m_i)\,\Delta\alpha_i$$

$$\leq \sum_i \frac{1}{n}\,\Delta\alpha_i = \sum_i \frac{1}{n}\big(\alpha(x_i) - \alpha(x_{i-1})\big)$$

$$= \frac{1}{n}\left(\alpha(b) - \alpha(a)\right) < \varepsilon$$

$$\Rightarrow \quad f \in R(\alpha)$$

Proof of C

$$M := \sup_{[a,b]} |f(x)|$$

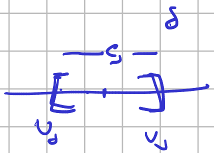
$$E = \{ x \in [a,b] \mid f \text{ not cont at } x \}$$
$$= \{ e_1 \ \dots \ e_N \}$$

Fix ε (Goal get a Partition P such that $U - L < \varepsilon$)

$e_j \in E$ $\quad \exists$ δ_j such that

$$|s - e_j| < \delta_j/2 \implies |\alpha(s) - \alpha(e_j)| < \frac{\varepsilon}{2}$$

α cont at e_j



Take a partition

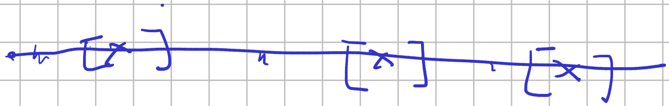
$< \frac{\delta_j}{2}$

$$e_j \in [u_j, v_j] \implies$$

$$|\alpha(v_j) - \alpha(u_j)| < \varepsilon$$

$$\underbrace{\alpha(v_j) - \alpha(e_j)}_{\substack{\wedge \\ \varepsilon/2}} + \alpha(e_j) - \alpha(u_j) < \varepsilon$$

$\wedge / \varepsilon/2$



I can control $\Delta\alpha$

$$\underbrace{\underset{\displaystyle K}{\underset{\shortparallel}{[a,b]}} \setminus \left(\bigcup_{j=1}^{N} (u_j - v_j) \right)}_{K'}$$

closed in $[a,b]$ $\Rightarrow$ compact

f is cont on K' $\Rightarrow$ f uni's cont

With the same ε has before

$\exists\ \delta(\varepsilon)$

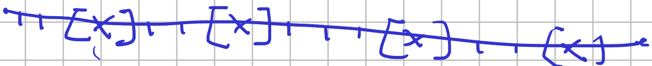
$|f(s) - f(t)| < \varepsilon$ for all

$|s_i - t| < \delta(\varepsilon)$

Construct a partition P such

that $x_i - x_{i-1} < \delta(\varepsilon)$

for $x_i \neq y_j$ $x_{i-1} \neq y_i$



$$M_i - m_i \le \underset{\underset{\sup|f|}{\shortparallel}}{M} + M = 2M$$

$$U(P\,f\,\alpha) - L(P\,f\,\alpha)$$

$$= \sum_i (M_i - m_i)\,\Delta\alpha_i$$

$$= \sum_{x_j = v_j} (M_i - m_i)\,\Delta\alpha_i^{\,<} + \sum_{x_i = v} (M_i - m_i)\,\Delta\alpha_i$$

$$< \sum 2M\,\Delta\alpha_i^{\,<\varepsilon} + (\alpha(b) - \alpha(a))\,\varepsilon$$

$$< 2M\varepsilon + (\alpha(b) - \alpha(a))\varepsilon$$

$$\leq C \cdot \varepsilon$$

$U(P f \alpha) - L(P f \alpha)$ can make it as small as poss

$\wedge$

$C \cdot \varepsilon \to$ as small as possi

Properties

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1. If f_1 and f_2 are in $\mathscr{R}(\alpha)$ then so is $f_1 + f_2$ and

$$\int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

2. Let $c \in \mathbb{R}$ and $f \in \mathscr{R}(\alpha)$. Then $cf \in \mathscr{R}(\alpha)$ and

$$\int_a^b cf \, d\alpha = c \int_a^b f \, d\alpha$$

linearity in $\int$ of the int

3. If $f_1 \leq f_2$ and they are integrable then

$$\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$$

monotonicity of the integral

4.

$$\left| \int_a^b f \, d\alpha \right| \leq \sup_{[a,b]} |f| \cdot (\alpha(b) - \alpha(a))$$

Properties II

1. $$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$
2. Let $c \in \mathbb{R}$
$$\int_a^b f\, dc \cdot \alpha = c \int_a^b f\, d\alpha$$

Theorem

1. If f and g are in $\mathscr{R}(\alpha)$ so is $f \cdot g$.
2. if $|f|$ is in $\mathscr{R}(\alpha)$ so is f and
$$\left| \int_a^b f\, d\alpha \right| \leq \int_a^b |f|\, d\alpha$$

$f \in R(\alpha_1)$ $\qquad$ $f \in R(\alpha_2)$

for $\varepsilon > 0$ $\qquad$ for $\varepsilon > 0$

$\exists\ P_1$ $\qquad$ $\exists\ P_2$

$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \varepsilon/2$ $\qquad$ $U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \varepsilon/2$

P^* common refinement of P_1 and P_2

$U(P^*, f, \alpha_1 + \alpha_2) - L(P^*, f, \alpha_1 + \alpha_2)$

$= U(P^*, f, \alpha_1) + U(P^*, f, \alpha_2)$

$- L(P^*, f, \alpha_1) - L(P^*, f, \alpha_2)$

$\varepsilon/2 >$ $\qquad$ $< \varepsilon/2$

$< \varepsilon$

Calculations

The unit step function is the function $I : \mathbb{R} \to [0, 1]$ defined by

$$I(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Theorem

Let $s \in [a, b]$ and set $\alpha(x) = I(x - s)$. If f is continuous we have that

$$\int_a^b f \, d\alpha = f(s)$$

Corollary

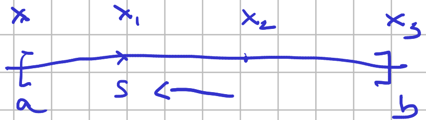
Let $\{s_k\} \in (a, b)$ and set $\alpha(x) = \sum_{k=1}^{\infty} c_k I(x - s_k)$ for some sequence $\{c_k\}$ in $\mathbb{R}$ such that $\sum_{k=1}^{\infty} c_k$ is convergent. Then

$$\int_a^b f \, d\alpha = \sum_{k=1}^{n} c_k f(s_k)$$

0 when $x - s \le 0$ $\quad x \le s$

$I(x-s)$

Proof



$$P = \{a, s, x_2, b\}$$

$$\alpha(a) = \alpha(s) = 0$$

$$\alpha(x_2) = \alpha(b) = 1$$

$$\Delta\alpha_i \begin{cases} 0 & i = 1, 3 \\ 1 & i = 2 \end{cases}$$

$M_2 \quad \sup_{[s, x_2]} f \qquad m_2 = \sup_{[s, x_2]} f$

$U(P\ f\ \alpha) - L(P\ f\ \alpha)$

$M_1 \cdot 0 + M_2 \cdot 1 + M_3 \cdot 0$

$- m_1 \cdot 0 - m_2 \cdot 1 - m_3 \cdot 0$

$= \underset{f(s)}{\underset{\downarrow}{M_2}} - \underset{f(s)}{\underset{\downarrow}{m_1}} \longrightarrow 0 \quad \Rightarrow f \in R(\alpha)$

$U(P\ f\ \alpha) \xrightarrow{x_2 \to s} \int_a^b f\, d\alpha = \inf \{U(P, f\alpha)\}$

$$\int_a^b f \, d\alpha = \lim_{x_2 \to s} U(P, f, \alpha) = f(s) \cdot 1$$

$$= f(s)$$

Further Computations

Theorem

Assume that α is strictly increasing and that α' is in $\mathscr{R}(\mathrm{id})$. Suppose that f is bounded. Then $f \in \mathscr{R}(\alpha)$ iff $f\alpha' \in \mathscr{R}(\mathrm{id})$ and we have

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)dx$$

Rieman integral

$\alpha = id$ $\quad\quad$ $\alpha(x) = x$

α differentiable $\quad\quad$ $f\alpha' \in R(id)$

$\alpha'(x)dx$ the differential of α $\quad$ $\underline{d\alpha}$

$$\int_0^1 e^{x^2} \cdot \underline{x \; dx}$$

$$\int_0^1 \frac{1}{2} e^{\alpha} d\alpha$$

$$\alpha = x^2$$

$$d\alpha = 2x \, dx$$

$$= \frac{1}{2} \int_0^1 e^{\alpha} d\alpha = \frac{1}{2} \left[e^{\alpha} \right]_0^1$$

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Thank you for your attention!