Introduction to Real Analysis

Lecture 7: Uniform convergence and continuity

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Questions?

Lecture Plan

Rudin Chapter 6

The problem

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$\sum \frac{1}{2^n} x^n$

ce

Suppose that we have a sequenc of functions $\{f_n : E \to \mathbb{R}\}$ such that for every $x \in E$ the sequence $f_n(x)$ or the series $\sum_{n=0}^{\infty} f_n(x)$ is convergent. Then we can define two functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ in the following was

$$f(x) = \lim_{n\to\infty} f_n(x)$$

if they exist

$$g(x) = \sum_{n=0}^{\infty} f_n(x)$$

We say that the sequence (series) of functions $\{f_n\}$ ($\sum_{n=0}^{k} f_n$) converges pointwise to f (g).

Secret desire

To compute limits, derivatives, integral of f and g using f_n.

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Secret desire

To compute limits, derivatives, integral of f and g using f_n.What we need is

$$\lim_{x\to t} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to t} f_n(x)$$

Sad example

$$f_n(x) = \frac{x^2}{(1+x^2)^n}$$

$\nearrow 0$

$x \to 0$

$\swarrow 1$

What is $g(x)$?

$$\sum_{k=0}^{\infty} \frac{x^2}{(1+x^2)^k} = x^2 \sum_{k=0}^{\infty} \boxed{\frac{1}{(1+x^2)^k}}$$

the series converges for every x

$$x^2 \frac{1}{1 - \frac{1}{1+x^2}} = \cancel{x^2} \frac{}{\frac{\cancel{1}+\cancel{x^2}-\cancel{1}}{1+x^2}} = 1+x^2$$

$$\lim_{n\to+\infty}\lim_{x\to 0} f_n(x) = \lim_{n\to+\infty} 0 = 0$$

$$\neq$$

$$\lim_{x\to 0}\lim_{n\to\infty} f_n(x) = 1+x^2 = 1$$

When can we have our dream

Uniform convergence

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Definition

We say that a sequence of functions $\{f_n : E \to \mathbb{R}\}$ converges uniformly to a function $f : E \to \mathbb{R}$ if for every $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that, for every $n > N(\varepsilon)$ we have that

$\hookrightarrow$ it does not dep & x

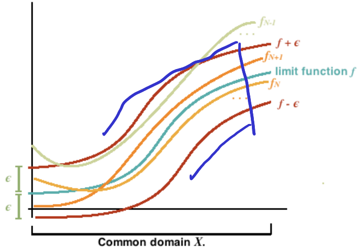
$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. In this case we write $f_n \to f$ unif

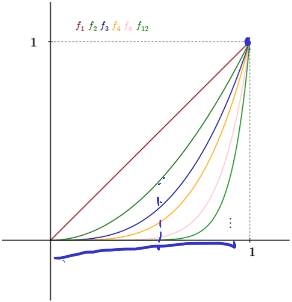
$$f_n \rightrightarrows f$$

In pictures

Yes



No



Criterions

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Cauchy Criterion

A sequence of functions $\{f_n : E \to \mathbb{R}\}$ converges uniformly to a function $f : E \to \mathbb{R}$ iff, for every $\varepsilon > 0$ there is $N(\varepsilon) > 0$ such that

$$|f_n(x) - f_m(x)| < \varepsilon$$

for alle $n, m > N(\varepsilon)$ and for every $x \in E$.

Theorem

A sequence of functions $\{f_n : E \to \mathbb{R}\}$ converging pointwise to $f : E \to \mathbb{R}$ converges uniformly (to f) iff,

$$M_n := \sup_E |f_n(x) - f(x)| \to 0$$

for all $n, m > N(\varepsilon)$ and for every $x \in E$.

Proof Cauchy $\quad f_n \rightrightarrows f$

fix $\varepsilon > 0 \quad \rightsquigarrow \quad N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for all}$$

$x \in E$ and $n \geq N(\varepsilon)$

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon$$

for all $x \in E$, $n, m \geq N(\varepsilon)$

$\Rightarrow$ Cauchy criterion.

Conversely suppose that the Cauchy criterion holds

Fix ε $\qquad f_n(x)$ is Cauchy $\forall x$

$$f_n(x) \longrightarrow f(x)$$

Need unif convergenc

Choos $N(\varepsilon)$ $\qquad |f_n(x) - f_m(x)| < \varepsilon$. does not dep of m

$$\Big\downarrow m \to +\infty$$

$$|f_n(x) - f(x)| < \varepsilon$$

for $x \in E$ $\quad n > N(\varepsilon)$

Weierstrass M-test

Theorem

Let $\{f_n : E \to \mathbb{R}\}$ a sequence of functions such that there is a sequence $\{M_n\}$ in $\mathbb{R}$ satisfying

$$|f_n(x)| < M_n$$

for all $x \in E$. If the series $\sum M_n$ converges, then the series of functions $\sum f_n(x)$ converges uniformly.

Uniform convergence and continuity

Theorem

Let $\{f_n : E \to \mathbb{R}\}$ a sequence of functions converging uniformly to a function $f : E \to \mathbb{R}$. Let $x \in E'$ and suppose that for every n $\lim_{t\to x} f_n(x)$ exists and s denoted by A_n. Then the sequence $\{A_n\}$ converges and

$$\lim_{n\to+\infty} \lim_{t\to x} f_n(x) = \lim_{t\to x} \lim_{n\to+\infty} f_n(x)$$

$$\lim_{n\to\infty} A_n = \lim_{t\to x} f(x).$$

Corollary

The uniform limit of a sequence of continuous function is continuous.

$p \in E$ Fix ε $N(\varepsilon)$

$$|f(s)-f(p)| \le \underbrace{|f(s)-f_n(s)|}_{<\varepsilon/3,\ n>N(\varepsilon)} + \underbrace{|f_n(s)-f_n(p)|}_{<\varepsilon/3,\ |s-p|<\delta(\varepsilon)} + |f_n(p)-f(p)| < \varepsilon/3 \quad n > N(\varepsilon)$$

Apply the theorem to

$$g_n(x) = \sum_{k=1}^{n} f_n(x)$$

$$g_n \rightrightarrows g$$

$$\lim_{t \to x} \sum_{k=1}^{\infty} g_n(x) = \sum_{k=1}^{\infty} \lim_{t \to x} g_n(x)$$

Derivatives are limit (under hp)

$$\frac{d}{dx} \lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \frac{d}{dx} f_n(x)$$

Proof (Exchange of limits) $x \in E'$

$f_n \rightrightarrows f$ $\quad f_n, f : E \to \mathbb{R}$

Cauchy: Fix ε $\exists N$ for $t \in E$ $n, m > N$

$$|f_n(t) - f_m(t)| < \varepsilon/2$$

$t \to x$ $\qquad A_n := \lim_{t \to x} f_n(x)$

($\exists$ for every n)

$$|A_n - A_m| < \varepsilon/2$$

$\iff$ $\{A_n\} \in \mathbb{R}$ is Cauchy

$A_n \longrightarrow A$ ($\mathbb{R}$ complete)

$f_n \rightrightarrows f \implies f_n \to f \quad \forall t \quad n >> 0$

because of $f_n \rightrightarrows f$

$$|f(t) - A| \underset{\forall n}{\overset{\Delta}{\leq}} \underbrace{|f(t) - f_n(t)|}_{< \varepsilon/3} + \underbrace{|f_n(t) - A_n|}_{< \frac{\varepsilon}{3}} + \underbrace{|A_n - A|}_{< \varepsilon/3}$$

$< \frac{\varepsilon}{3}$ for t close to x because $A_n = \lim_{t \to x} f_n(t)$

$n >> 0$ $A_n \to A$

if we choose $n > N$

$$|f(t) - A| < \frac{2\varepsilon}{3} + |f_n(t) - A_n|$$

$< \varepsilon$ for t st $|t - x| < \delta$

$$\lim_{t \to x} f(t) = A = \lim_{n \to +\infty} A_n = \lim_{n \to +\infty} \lim_{t \to x} f_n(t)$$

$$\|$$

$$\lim_{t \to x} \lim_{n \to +\infty} f_n(t)$$

Exchanged the limit signs!

$$\|$$

Uniform convergences on compact sets

Theorem

Let $K \subseteq X$ be a compact. Let $f_n : K \to \mathbb{R}$ a sequence of functions satisfying

1. f_n are continous on K
2. the sequence f_n converges *pointwise* to a function f which is continous
3. for all $x \in K$ the sequence $\{f_n(x)\} \subseteq \mathbb{R}$ is monotone.

Then the sequence f_n converges uniformly to f.

<u>Proof</u>: <u>Step 1</u> We can assume that

$f \equiv 0$

Suppose that this is true

$f_n \longrightarrow f$ satisfying the hp

$g_n := f_n - f$ satisfies 1, 2, 3

$g_n \longrightarrow 0$

$g_n \rightrightarrows 0$ $\quad f_n - f \rightrightarrows 0$

$f_n \rightrightarrows f$

$g_n \to 0$ $\quad g_n$ cont

$\forall x \quad g_n(x)$ monotone (decreasing)

$\to 0$ $\qquad g_n \nearrow 0$

Fix $\varepsilon > 0$

$$K_n = \{ x \in K \mid g_n(x) \geq \varepsilon \} \qquad -\varepsilon \leq g_n(x)$$

closed

$$= \overbrace{g_n^{-1}([\varepsilon, +\infty))}^{\text{closed}} \cap K \quad \subseteq K \text{ cpt}$$

cont $\qquad$ closed $\qquad g_n^{-1}((-\infty, -\varepsilon])$

$\Rightarrow \quad K_n$ is cpt

$K_n \supseteq K_{n+1}$ $\{f_n(x) \searrow\}$

$f_n(x) \geq f_{n+1}(x)$ for all $x \in K$

Sequence of nested cpt

if $K_n \neq \phi$ $\forall n$ $\Rightarrow$ $\bigcap K_n \neq \phi$

We are going to show that $\bigcap K_n = \phi$

$\Rightarrow$ for $n >> 0$ $K_n = \phi$

Let us see how $K_n = \phi$ for $n > N$

gives us $f_n \rightrightarrows 0$

for $n > N$ $K_n = \phi$

$0 < g_n(x) < \varepsilon$ for all $x \in K$.

$g_n \to 0$ $g_n(x) \to 0^+$

$\exists N > 0$ such that

$|g_n(x)| < \varepsilon$ for all $n > N$
on all $x \in K$

$g_n \rightrightarrows 0$.

Want $\bigcap K_n = \emptyset$ $\qquad g_n \to 0$ $\quad \varepsilon$ fixed

$x \in K \quad \exists \quad N(x, \varepsilon)$ such that

$|g_n(x)| < \varepsilon$ for all $n > g_n(x\, \varepsilon)$

$\exists \quad N(x, \varepsilon)$ such that

$x \notin K_{N(x, \varepsilon)}$

$x \notin \bigcap_{n \in \mathbb{N}} K_n = \emptyset$

A metric space of functions

$$\mathscr{C}(X) := \{f : X \to \mathbb{C} \text{ continuous and bounded }\}$$

$f_n \longrightarrow f$ in $\mathscr{C}(X)$

let

$$\|f\| = \sup_X |f(x)|$$

$f_n \rightrightarrows f$

and

$$d(f, g) = \|f - g\|$$

Theorem

We have that $(\mathscr{C}(X), d)$ is a complete metric space.

Banach space

Thank you for your attention!