Abstract Algebra

SU — 2024 Spring

Practice

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§ Field Extensions §

This homework is worth 10 points that is 1 bonus point. You are welcome to collaborate with your classmates to find a solution, but write and submit your own solution.

Problem 1: Minimal Polynomial

Find the minimal polynomial of $\sqrt{2+\sqrt{5}}$ over $\mathbb{Q}$. Let $\mathbb{Q}\left(\sqrt{2+\sqrt{5}}\right)$ the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $\sqrt{2+\sqrt{5}}$, compute $[\mathbb{Q}\left(\sqrt{2+\sqrt{5}}\right) : \mathbb{Q}]$

Problem 2: Basis

Let $\alpha := \sqrt{2}+\sqrt{3}$. Find a basis of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $\sqrt{2}+\sqrt{3}$ Show that they are isomorphic subgroup of S_4, but are not conjugate.

Problem 3: Degree of an extension

Let m and n two positive integers such that $\gcd(m, n) = 1$. Denote by $\mathbb{Q}(\sqrt[m]{7}, \sqrt[n]{7})$ the smallest subfield of $\mathbb{C}$ containing both $\sqrt[m]{7}$ and $\sqrt[n]{7}$. Show that $[\mathbb{Q}(\sqrt[m]{7}, \sqrt[n]{7}) : \mathbb{Q}] = mn$. Show with an example that this si not the case if m and n share a prime divisor.

Problem 4: Extensions of finite fields

Consider the polynomial $p(x) = x^3 + x + 1 \in \mathbb{Z}/2\mathbb{Z}[x]$.

(1) Show that $\mathbb{Z}/2\mathbb{Z}[x]/$ is a field, which we will denote by F.

(2) Let $\alpha = x + (p(x)) \in F$. Show that $p(x)$ is the minimal polynomial of α over $\mathbb{Z}/2\mathbb{Z}$. Find a basis for F over $\mathbb{Z}/2\mathbb{Z}$.

(3) Express α^6 as a linear combination of elements in the basisi you found in the previous point.

(4) Express the inverse of $\alpha + 1$ as a linear combination of the elements of your chosen basis.

(5) Express the inverse of $\alpha^2 + 1$ as a linear combination of the elements of your chosen basis.

Problem 5: Extesnions of the rational numbers

Given α a complex number we denote by $\mathbb{Q}(\alpha)$ the smallest subfield of $\mathbb{C}$ containing both $\mathbb{Q}$ and α. Similarly if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, we denote by $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and all the $\alpha_i's$.

(1) Compute the minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}(\sqrt[3]{3})$. Compute $[\mathbb{Q}(\sqrt[3]{3}, \sqrt{2}) : \mathbb{Q}]$

(2) Show that $\sqrt{2 + \sqrt[3]{3}}$ is algebraic over $\mathbb{Q}$.

(3) Let $\alpha = 2 + \sqrt[3]{3}$ and $\beta = \sqrt[3]{3}$. Explain why the minimal polynomials of α and β over $\mathbb{Q}$ have the same degree.

(4) Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{3}$ over $\mathbb{Q}$. Give a basis of $\mathbb{Q}(\sqrt{2} + \sqrt[3]{3})$ over $\mathbb{Q}$.

(5) Express the inverse of $(\sqrt{2} + \sqrt[3]{3})$ in the chosen basis.