STOCKHOLMS UNIVERSITET,
MATEMATISKA INSTITUTIONEN,
Avd. Matematisk statistik

Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2024-08-22

Problem 1

(A) The present value of the bond is

$$10 \cdot e^{-0.02} + 10 \cdot e^{-0.02*2} + (100 + 10) \cdot e^{-0.02*3} \approx 123.0040.$$

(B) The present value (equal to the price P_0) of the zero coupon bond is given by $P_0 = 1 \cdot e^{-rT}$. Thus,

$$r = -\frac{\ln(P_0)}{T} = 5.27\%.$$

Problem 2

See Capinski & Zastawniak e.g., ch. 6.2.2 and 6.3.

(A) The value of the European version of the derivative is

$$\begin{aligned} H_E(0) &= \frac{1}{(1+R)} \left[p_* f(S^u) + (1-p_*) f(S^d) \right] \\ &= \frac{1}{(1+R)} \left[\frac{R-D}{U-D} (S^u)^2 + \left(1 - \frac{R-D}{U-D} \right) (S^d)^2 \right] \\ &= \frac{1}{(1+0)} \left[0.5 \cdot 11^2 + 0.5 \cdot 9^2 \right] \\ &= 101. \end{aligned}$$

(B) The value of an American derivative is generally for the present one-period model

$$H_A(0) = max\{f(S(0)); H_E(0)\}.$$

In our case we hence obtain

$$H_A(0) = max\{10^2; 101\} = 101.$$

Problem 3

We want to find the portfolio that solves

$$\max_{\mathbf{w}} \frac{\mathbf{w}\mathbf{m}^T - R}{\sqrt{\mathbf{w}C\mathbf{w}^T}}$$

where $\mathbf{w} = (w_1, w_2)$ under the condition $w_1 = 1 - w_2$ (compare with Capinski & Zastawniak around p. 82). In our case we have

$$\mathbf{m} = (0.2, 0, 1)$$

and

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we set $w_1 = x$ then $\mathbf{m} = (x, 1 - x)$ and we find

$$\begin{aligned} \mathbf{w}\mathbf{m}^T - R &= 0.2x + 0.1(1 - x) = 0.1x + 0.1 \\ \mathbf{w}C\mathbf{w}^T &= x^2 + (1 - x)^2 \end{aligned}$$

Hence our problem boils down to

$$\max_x 0.1 \frac{x + 1}{\sqrt{x^2 + (1 - x)^2}}$$

Differentiating the expression to be maximized and setting that derivative to zero gives:

$$0.1 \frac{2 - 3x}{(x^2 + (1 - x)^2)^{3/2}} = 0.$$

Solving this equation gives $x = \frac{2}{3}$.[1] The market portfolio is therefore

$$\mathbf{w}_m = \left(\frac{2}{3}, \frac{1}{3} \right).$$

Problem 4

Using mainly sections 6.1-6.2 in Capinski & Zastawniak we find

$$\begin{aligned} p_* &= \frac{R - D}{U - D} = 0.75, \\ S^{uu} &= S(0)(1 + U)^2 = 121, \\ S^{ud} &= S(0)(1 + U)(1 + D) = 99 \\ S^{dd} &= S(0)(1 + D)^2 = 81. \end{aligned}$$

The risk-neutral valuation formula (p. 154) thus gives

$$\begin{aligned} P_E &= \frac{1}{(1 + R)^2} \left[p_*^2 (X - S^{uu})_+ + 2p_*(1 - p_*)(X - S^{ud})_+ + (1 - p_*)^2 (X - S^{dd})_+ \right] \\ &= \frac{1}{1.05^2} \left[0 + 0 + 0.25^2 \cdot 9 \right] \\ &\approx 0.5102. \end{aligned}$$

[1] The value $x = \frac{2}{3}$ can be seen to be the maximizer by e.g., observing that $x \to \infty$ leads to the value $0.1/\sqrt{2}$ and $x \to -\infty$ leads to the value $-0.1/\sqrt{2}$, while $x = \frac{2}{3}$ yields a value that is larger than the two limits.

Problem 5

The forward price is

$$F(0,T) = \frac{S(0)}{B(0,T)} = S(0)e^{rT}.$$

See Capinski & Zastawniak, around pages 93-94 (as well as page 44) for a derivation and an interpretation.