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Solutions
Logic 7.5 hp / Logik 7,5 hp
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Basic part

1. Let C be a subset of the natural numbers $\mathbb{N}$ inductively defined by

- $0 \in C$;
- if $n \in C$ then $n + 2 \in C$.

(a) Explain in words what the set C consists of, that is to say, give a non-inductive definition of the set C.

(b) Give a proof by induction of your answer in (a).

(c) Give a non-recursive definition of the function $f : C \to \mathbb{N}$ defined recursively by

$$\begin{aligned} f(0) &= 0 \\ f(n+2) &= f(n) + 1. \end{aligned}$$

4 p

Solution.

(a) The set C consists of all non-negative even numbers.

(b) We prove by induction on $n \in C$ that every element of C is of form $2k$ for some k in $\mathbb{N}$, i.e., even. Clearly, $0 = 2 \cdot 0$, so the base case holds. If $n = 2k$ is an element of C then $n + 2 = 2(k + 1)$, so the induction step holds.

We show the converse, that every even number $2k$ lies in C, we proceed by induction on $k \in \mathbb{N}$. The case $k = 0$ is immediate, and if $2k \in C$ then $2(k + 1) = 2k + 2$ is also an element of C, so we are done.

(c) We have shown above that $C = \{2k \mid k \in \mathbb{N}\}$. The function f sends $2k$ to k.

2. Let P_1 and P_2 be propositional variables, and let $\mathcal{A}$ be an interpretation such that $P_1^{\mathcal{A}}$ and $P_2^{\mathcal{A}}$ are both true.

(a) Compute the truth-value of $((\neg P_1) \vee P_2) \to \neg(P_1 \wedge P_2)$; 2 p

(b) Based on your answer in (a), can you say whether the formula in (a) is valid or not? Why/why not? 1 p

Solution.

(a) Since in the interpretation $\mathcal{A}$ the propositional variables P_1 and P_2 are true, $P_1 \wedge P_2$ is true and $\neg(P_1 \wedge P_2)$ is false. Further, $(\neg P_1) \vee P_2$ is true which implies that $((\neg P_1) \vee P_2) \to \neg(P_1 \wedge P_2)$ is false.

(b) We were given an interpretation of the propositional variables under which the formula did not hold, hence it is not valid.

3. Give a natural deduction proof of $(\neg\psi \wedge (\phi \vee \psi)) \to \phi$.

2 p

Solution.

$$\dfrac{\dfrac{\dfrac{[\neg\psi \wedge (\phi \vee \psi)]^1}{\phi \vee \psi}\wedge E \qquad [\phi]^2 \qquad \dfrac{\dfrac{\dfrac{[\neg\psi \wedge (\phi \vee \psi)]^1}{\neg\psi}\wedge E \qquad [\psi]^2}{\bot}\to E}{\phi}\bot E}{\phi}\vee E_2}{(\neg\psi \wedge (\phi \vee \psi)) \to \phi}\to I_1$$

4. Let Γ be a set of propositional formulas and let ϕ be a propositional formula.

(a) What does $\Gamma \vdash \phi$ mean? What does $\Gamma \vDash \phi$ mean? 2 p

(b) State the soundness theorem and the completeness theorem for propositional logic. 1 p

Solution.

(a) $\Gamma \vdash \phi$ means that the formula ϕ can be deduced using the rules of natural deduction from finitely many formulas belonging to the set Γ. $\Gamma \vDash \phi$ means that in any model in which every formula from Γ is true then the formula ϕ is also true.

(b) The soundness theorem says that if a formula ϕ can be derived from a theory Γ, meaning that there exists a derivation in natural deduction ending in ϕ whose undischarged assumptions lie in Γ, then Γ entails ϕ, i.e., ϕ holds in all the models of Γ. In its symbolic form soundness is the implication $\Gamma \vdash \phi \implies \Gamma \vDash \phi$. The completeness theorem is the converse, $\Gamma \vDash \phi \implies \Gamma \vdash \phi$.

5. In the language with only equality ($\langle ; \rangle$) consider the sentence $\exists x \exists y \exists z \forall w ((w = x) \vee (w = y) \vee (w = z))$. Give a structure $\mathcal{A}$ where this sentence is true. Give a structure $\mathcal{B}$ where the sentence is false. 2 p

Solution. If we consider a model $\mathcal{A}$ in which the domain $|\mathcal{A}|$ contains at most three elements then the above sentence is true since we can assign x, y, z to the distinct elements of $|\mathcal{A}|$. If $|\mathcal{A}|$ has more than 3 elements the sentence is false.

6. We think of the language with only one binary relation symbol ($\langle E; \rangle$) as the language of directed graphs, with $E(x, y)$ meaning that there is an edge from node x to node y.

- A node a in a graph is said to have *out-degree one* if there is exactly one node to which there is an edge from a.

- A node a in a graph is said to have *in-degree one* if there is exactly one node from which there is an edge to a.

Write a sentence in the language stating that every node has out-degree one, and a sentence stating that every node has in-degree one.

2 p

Solution. The sentence displayed below expresses that every node has out-degree one.

$$\forall x \exists y E(x, y) \wedge \forall x \forall y \forall z((E(x, y) \wedge E(x, z)) \to y = z)$$

Simply swapping the arguments of E yields a formula expressing that every node has in-degree one, as shown below.

$$\forall x \exists y E(y, x) \wedge \forall x \forall y \forall z((E(y, x) \wedge E(z, x)) \to y = z)$$

7. In the formula $\forall x_0 \exists x_2 \forall x_5 (R_1(x_0, x_1) \to R_2(f(x_3, x_2), x_0))$ which variables are free and which are bound? Is the term $f_2(x_6, x_5)$ free for x_1 in this formula?

2 p

Solution. The set of free variables FV for the above formula consists of $\{x_1, x_3\}$. The variables $\{x_0, x_2, x_5\}$ are bound. The term $f_2(x_6, x_5)$ is not free for x_1 since the variable x_5 will be captured.

8. What does it mean that a sentence in predicate logic is a *tautology*? Prove carefully that the sentence $\exists x P_1(x)$ is not a tautology.

2 p

Solution. A sentence is a tautology if it is true in any interpretation. The sentence $\exists x P_1(x)$ is not a tautology since if we for example consider the interpretation with $|\mathcal{A}| = \mathbb{R}$ and $P_1(x)$ meaning $x^2 < 0$ then the sentence $\exists x P_1(x)$ will be false.

Problem part

9. Provide derivations in natural deduction without any undischarged assumptions of the following formulas:

(a) $((\phi \to \delta) \wedge (\psi \to \sigma)) \to ((\phi \vee \psi) \to (\delta \vee \sigma))$ 2 p

(b) $\exists y \forall x (P_1(x, y)) \to \forall x \exists y (P_1(x, y))$ 3 p

Solution.

(a)

$$\dfrac{\dfrac{\dfrac{[\phi \vee \psi]^2 \qquad \dfrac{\dfrac{\dfrac{[(\phi \to \delta) \wedge (\psi \to \sigma)]^1}{\phi \to \delta}\wedge E \qquad [\phi]^3}{\delta}\to E}{\delta \vee \sigma}\vee I \qquad \dfrac{\dfrac{\dfrac{[(\phi \to \delta) \wedge (\psi \to \sigma)]^1}{\psi \to \sigma}\wedge E \qquad [\psi]^3}{\sigma}\to E}{\delta \vee \sigma}\vee I}{\delta \vee \sigma}\vee E_3}{(\phi \vee \psi) \to (\delta \vee \sigma)}\to I_2}{((\phi \to \delta) \wedge (\psi \to \sigma)) \to ((\phi \vee \psi) \to (\delta \vee \sigma))}\to I_1$$

(b)

$$\dfrac{\dfrac{\dfrac{[\exists y\forall x(P_1(x,y))]^1 \qquad \dfrac{\dfrac{[\forall x(P_1(x,y))]^2}{P_1(x,y)}\,\forall E}{\exists y(P_1(x,y))}\,\exists I}{\exists y(P_1(x,y))}\,\exists E_2}{\forall x\exists y(P_1(x,y))}\,\forall I}{\exists y\forall x(P_1(x,y)) \to \forall x\exists y(P_1(x,y))}\,\to I_1$$

10. For each of the following formulas decide, using a method of your choice, whether it is derivable in natural deduction (and justify your answer):

(a) $\forall x\exists y(P_1(x,y)) \to \exists y\forall x(P_1(x,y))$ 2 p

(b) $\exists y\forall x(f(x) = y) \to \forall x(f(f(x)) = f(x))$ 2 p

Solution.

(a) Consider the interpretation $\mathcal{A}$ where $|\mathcal{A}|$ is $\mathbb{N}$ and $P_1(x,y)$ means $x < y$. The formula $\forall x\exists y(P_1(x,y))$ is true in $\mathcal{A}$ since one can simply let $y = x+1$, as indeed $x < x+1$. However, $\exists y\forall x(P_1(x,y))$ does not hold in $\mathcal{A}$ since $\mathbb{N}$ does not have a maximum element.

In conclusion, the implication $\forall x\exists y(P_1(x,y)) \to \exists y\forall x(P_1(x,y))$ does not hold in $\mathcal{A}$, which means that it is not derivable in natural deduction by (the contrapositive of) soundness.

(b) This is derivable, as shown below.

$$\dfrac{\dfrac{\dfrac{[\exists y\forall x(f(x)=y)]^1 \qquad \dfrac{\dfrac{[\forall x(f(x)=y)]^2}{f(f(x))=y}\,\forall I \qquad \dfrac{\dfrac{}{f(x)=f(x)}\,\text{refl} \qquad \dfrac{[\forall x(f(x)=y)]^2}{f(x)=y}\,\forall I}{y=f(x)}\,\text{repl}}{f(f(x))=f(x)}\,\text{repl}}{f(f(x))=f(x)}\,\exists E_2}{\forall x(f(f(x))=f(x))}\,\forall I}{\exists y\forall x(f(x)=y) \to \forall x(f(f(x))=f(x))}\,\to I_1$$

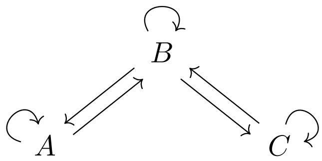
11. Let $(\langle E; \rangle)$ be the language of graphs, as in Problem 7 above. Recall that a set of sentences is said to be *independent* if no sentence is a logical consequence of the other sentences in the set. Show that the set of the following three sentences is independent (you may define simple structures by drawing diagrams with nodes and directed edges, as long as you are careful and do it clearly):

1. $\forall x\forall y\forall z(E(x,y) \wedge E(y,z) \to E(x,z))$
2. $\forall x E(x,x)$
3. $\forall x\forall y(E(x,y) \to E(y,x))$

4 p

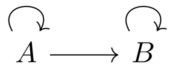
Solution. Denote by ϕ_1, ϕ_2, ϕ_3 the three formulas above. It suffices to show that $\phi_2, \phi_3 \nvdash \phi_1$ and $\phi_1, \phi_3 \nvdash \phi_2$ and $\phi_1, \phi_2 \nvdash \phi_3$. By (the contrapositive of) soundness, it is enough to provide models of $\Gamma_1 = \{\neg\phi_1, \phi_2, \phi_3\}$, $\Gamma_2 = \{\phi_1, \neg\phi_2, \phi_3\}$, and $\Gamma_3 = \{\phi_1, \phi_2, \neg\phi_3\}$.

(Γ_1) Any reflexive, symmetric, non-transitive graph is a model of Γ_1, such as the one displayed below.



(Γ_2) The graph with one vertex but no edges will vacuously satisfy ϕ_1 and ϕ_3 but not ϕ_2, hence it is a model of Γ_2.

(Γ_3) Any reflexive, transitive, non-symmetric graph is a model of Γ_3, such as the one displayed below.



12. Let Γ be a set of sentences (in some fixed language). Let $\mathrm{Thry}(\Gamma)$ be the set of sentences ϕ such that $\Gamma \vdash \phi$. Show that for all sentences ψ, it is the case that $\Gamma \vdash \psi$ if and only if $\mathrm{Thry}(\Gamma) \vdash \psi$. 3 p

Solution. Since $\Gamma \subseteq \mathrm{Thry}(\Gamma)$, it is clear that the models of $\mathrm{Thry}(\Gamma)$ are also models of Γ. Moreover, each sentence ϕ such that $\Gamma \vdash \phi$, soundness tells us, has to be true in every model of Γ. As a result, models of Γ are already models of $\mathrm{Thry}(\Gamma)$.

These two theories having exactly the same models means that $\Gamma \vDash \psi$ if and only if $\mathrm{Thry}(\Gamma) \vDash \psi$. The conclusion then follows of the equivalence $\vdash \iff \vDash$ provided the combination of soundness and completeness.

13. The purpose of this exercise is to prove that the notion of connected graph is not first-order definable (the proof here is different from the one in class). Let $(\langle E; \rangle)$ be the language of directed graphs, as in Problem 7 and Problem 12 above. Suppose that Γ is a set of sentences in this language such that for any structure $\mathcal{G}$ it is the case that $\mathcal{G} \vDash \Gamma$ if and only if $\mathcal{G}$ is a connected graph, in the sense that for all $a, b \in |\mathcal{G}|$ there exists a path from a to b, i.e. a sequence of elements $c_1, \ldots, c_n \in |\mathcal{G}|$ such that $\langle a, c_1 \rangle \in E^{\mathcal{G}}$, $\langle c_i, c_{i+1} \rangle \in E^{\mathcal{G}}$ for all $1 \leq i \leq n-1$, and $\langle c_n, b \rangle \in E^{\mathcal{G}}$.

(a) Let ϕ be the sentence from Problem 7 stating that every node has out-degree one, and ψ be the sentence stating that every node has in-degree one (you can take both sentences as given, whether you wrote them out in 7 or not). Show that $\Gamma \bigcup \{\phi, \psi\}$ has models of arbitrary finite size n. (Hint: a connected graph where every node has in-degree and out-degree one is a cycle.)

(b) Show that for every natural number n there exists a sentence σ_n such that $\mathcal{G} \vDash \sigma_n$ if and only if $|\mathcal{G}|$ has at least n elements (you may write e.g. σ_3 explicitly and then just indicate how to do it for arbitrary n).

(c) Use the compactness theorem and (a) and (b) to show that $\Gamma \bigcup \{\phi, \psi\}$ must have an infinite model.

As there are no infinite cycles, this concludes the proof. 4 p