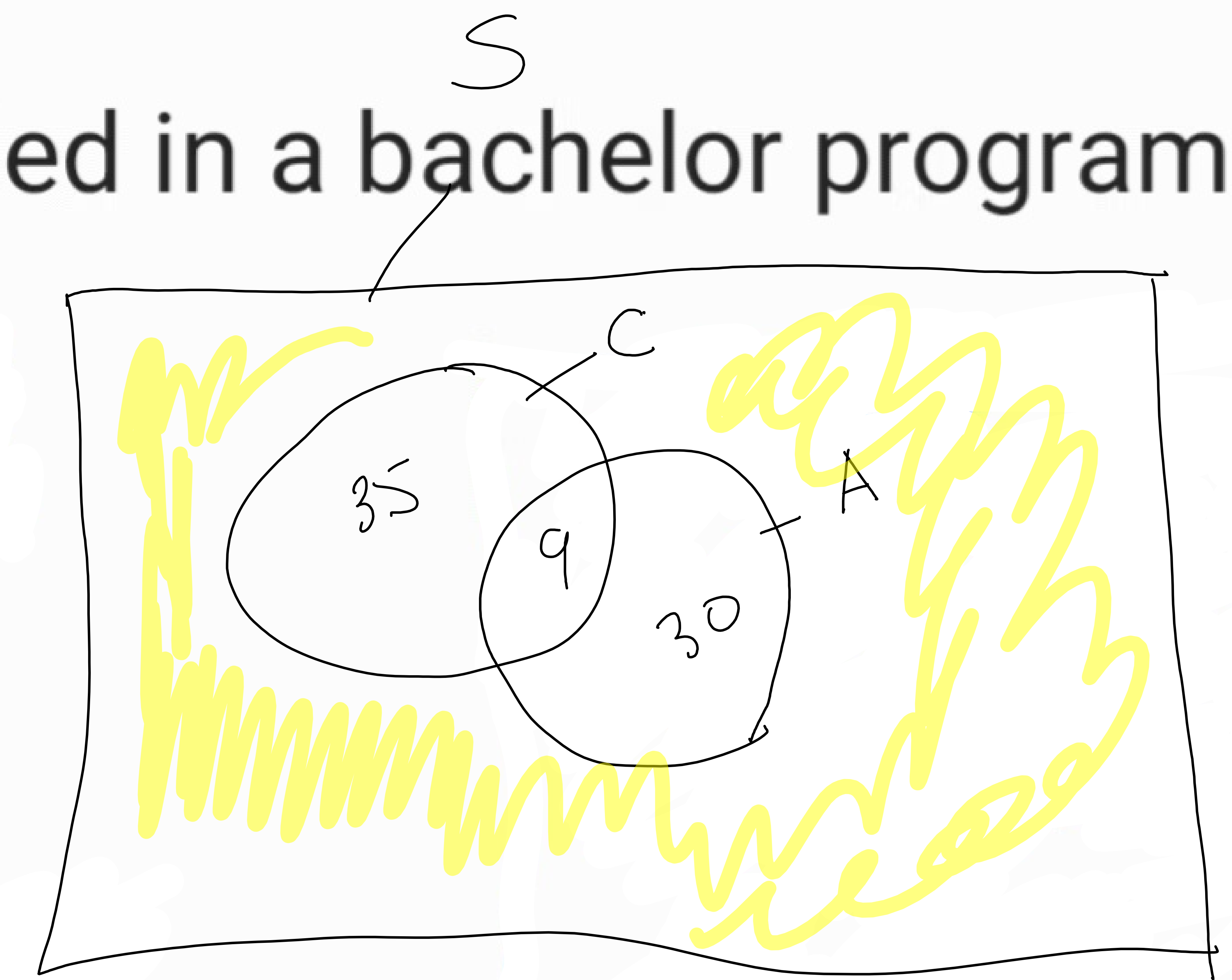
MM5023 - Lecture 2
Inclusion exclusion principle

Example

There are 100 students enrolled in a bachelor program

- 35 take combinatorics
- 30 take abstract algebra
- 9 take both

How many take neither?

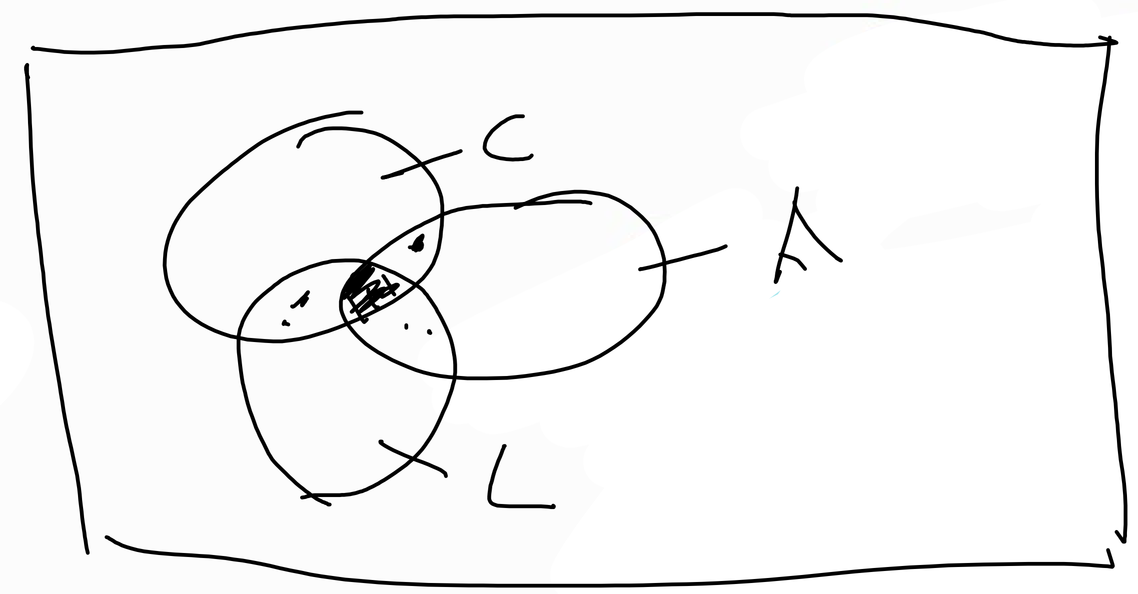


$$100 - |C \cup A| = 100 - 56 = 44$$

$$|C \cup A| = |C| + |A| - |C \cap A| = 35 + 30 - 9 = 56$$

Example There are 100 students enrolled in a bachelor program

- 35 take combinatorics
- 30 take abstract algebra
- 30 take Logic
- 9 both combinatoric and logic
- 10 take both combinatorics and algebra
- 11 take algebra and logic.
- 5 take all 3



How many take neither? $100 - |C \cup A \cup L| = 100 - 70 = 30$

$$|C \cup A \cup L| = |C| + |A| + |L| - |C \cap A| - |C \cap L| - |A \cap L| + |C \cap L \cap A|$$

$$= 35 + 30 + 30 - 9 - 10 - 11 + 5 = 70$$

$\underline{n} = \{1, 2 \dots n\}$

The Principle

Let S be a finite set and let $A_1 \dots A_n$ be subsets of S. Denote by

$$\alpha_j := \sum_{\substack{I \subseteq \underline{n} \\ |I| = j}} \left| \bigcap_{i \in I} A_i \right|$$

$$\Rightarrow \left| \bigcup_{j=1}^{n} A_j \right| = \sum_{j=1}^{n} (-1)^{j+1} \alpha_j$$

$\underline{n=2}$

$$\alpha_1 = |A_1| + |A_2| \qquad \Rightarrow |A_1 \cup A_2| = |\alpha_1 - \alpha_2$$

$$\alpha_2 = |A_1 \cap A_2| \qquad = |A_1| + |A_2| - |A_1 \cap A_2|$$

$\underline{n=3}$

$$\alpha_1 = |A_1| + |A_2| + |A_3|$$

$$\alpha_2 = |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|$$

$$\alpha_3 = |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$
$$+ |A_1 \cap A_2 \cap A_3| \quad \checkmark$$

Corollary: Same notation as before

$$\left|\bigcap A_i^c\right| = \sum_{j=0}^{n} (-1)^j \alpha_j$$

$$\alpha_0 = |S|$$

Proof

$$\left|\bigcap_i^n A_i^c\right| = \left|\left(\bigcup_{i=1}^n A_i\right)^c\right| = |S| - \left|\bigcup_{i=1}^n A_i\right|$$

$$= \alpha_0 - \sum_{j=1}^n (-1)^{j+1} \alpha_j = (-1)^0 \alpha_0 + \sum_{j=1}^n (-1)^j \alpha_j = \sum_{j=0}^n (-1)^j \alpha_j$$

$n = 2$

$$\alpha_1 = \sum_{\substack{I \subseteq \{1,2\} \\ |I| = 1}} \left|\bigcap_{i \in I} A_i\right| = \left|\bigcap_{i \in \{1\}} A_i\right| + \left|\bigcap_{i \in \{2\}} A_2\right|$$

$$= |A_1| + |A_2|$$

$I = \{1\}$. $I = \{2\}$

The Characteristic function of a set

S a set (not nec finite)

$A \subseteq S$ → characteristic / index function on A

$$\mathbb{1}_A : S \longrightarrow \mathbb{R}$$

$$\mathbb{1}_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

If A is finite

$$|A| = \sum_{s \in S} \mathbb{1}_A(s)$$

Proof of the theorem (S is finite)

$$f : S \longrightarrow \mathbb{R}$$

$$f(s) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{I \subseteq \underline{n} \\ |I| = j}} \mathbb{1}_{\bigcap_{i \in I} A_i}(s)$$

Want to show $\boxed{f(s) = \mathbb{1}_{\bigcup_{i=1}^{n} A_i}(s)}$

$$\sum_{s\in S} f(s) = \sum_{s\in S}\sum_{j=1}^{n}(-1)^{j+1}\sum_{\substack{I\subseteq n\\ |I|=j}} \mathbb{1}_{\bigcap_{i\in I} A_i}(s)$$

$$\left|\bigcup_{i=1}^{n} A_i\right| \overset{?}{=} \sum_{j=1}^{n}(-1)^{j+1}\underbrace{\sum_{\substack{I\subseteq n\\ |I|=j}} \underbrace{\sum_{s\in S} \mathbb{1}_{\bigcap_{i\in I} A_i}(s)}_{\left|\bigcap_{i\in I} A_i\right|}}_{\alpha_j}$$

$$= \sum_{j=1}^{n}(-1)^{j+1}\alpha_j$$

So if $f = \mathbb{1}_{\bigcup_{i=1}^{n} A_i}$ we would be done

$$\mathbb{1}_{\bigcup_{i=1}^{n} A_i}(s) \overset{?}{=} \boxed{f(s) = \sum_{j=1}^{n}(-1)^{j+1}\sum_{\substack{I\subseteq n\\ |I|=j}} \mathbb{1}_{\bigcap_{i\in I} A_i}(s)}$$

WANT $\quad f(s) = \begin{cases} 1 & \text{if } s \in \bigcup_{i=1}^{n} A_i \\ 0 & \text{if } s \notin \bigcup_{i=1}^{n} A_i \end{cases}$

$s \notin \bigcup_{i=1}^{n} A_i$ it meanst that $s \notin A_i$ for some i
$\Rightarrow$ it is not in any of the intersections $\bigcap_{i \in I} A_i$ $I \subseteq n$

$\subseteq \bigcup_{i=1}^{n} A_i$

$$f(s) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{I \subseteq n \\ |I|=j}} \mathbb{1}_{\bigcap_{i \in I} A_i}(s) = 0$$

$= 0$

$s \in \bigcup_{i=1}^{n} A_i$ $\quad f(s) \overset{?}{=} 1$

let $r = r(s)$ be the number of A_i ~~to whom~~ which s belongs to

$$f(s) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{I \subseteq n \\ |I|=j}} \mathbb{1}_{\bigcap_{i \in I} A_i}(s)$$

if $|I| > r$ $\quad \mathbb{1}_{\bigcap_{i \in I} A_i}(s) = 0$

$$= \sum_{j=1}^{r} (-1)^{j+1} \boxed{\sum_{|I|=j} \mathbb{1}_{\bigcap_{i \in I} A_i}(s)}$$

$\mathbb{1}_{\bigcap_{i \in I} A_i}(s) \neq 0$ iff s belongs to all the A_i with $i \in I$

$$= \sum_{j=1}^{n} (-1)^{j+1} \binom{r}{j}$$

$$= -\sum_{j=1}^{n} (-1)^j \binom{r}{j} + (-1)^0 \binom{r}{0} - (-1)^0 \binom{r}{0}$$

$$= -\left(\sum_{j=0}^{n} (-1)^j \binom{r}{j}\right) + \underbrace{(-1)^0 \binom{r}{0}}_{1}$$

$$= -(1-1)^r + 1 = 1$$

The book: Start with a finite set S

c_i condition on S

Example: $S = \{\text{bachelor student}\}$

c_i = "students enrolled in course i"

$\bar{c_i}$ negation of c_i

$N(c_1 \dots c_n)$ = # of element of S satisfying the c_i

IEP

$$N(\bar{c}_1 \dots \bar{c}_n) = \sum_{j=0}^{n} (-1)^j \sum_{|I|=j} \underbrace{N(c_i \mid i \in I)}_{= \alpha_j}$$

Connection with the book notation

Notation: S a set $c_1, c_2, c_3 \ldots c_k$ conditions on S

$$N(c_1 \ldots c_k) = \left| \{x \in S \mid x \text{ satisfies } c_1 \ldots c_k\} \right|$$

$\overline{c_i}$ = not c_i

$A_i = \{x$ ~~st~~ satisfy $c_i)$

$$|A_1 \cap \cap A_k|$$

$$N(\overline{c}_1 \ldots \overline{c}_k) = |A_1^c \cap \cap A_k^c|$$

Exclusion Inclusion principle:

Let S be a finite set of size N with condition $c_1 \dots c_t$

$$N(\overline{c_1} \dots \overline{c_t}) = \sum_{i=0}^{t} (-1)^i \left[\sum_{\substack{I \subseteq \{1 \dots t\} \\ |I| = i}} N(c_k \mid k \in I) \right]$$

Generalization of inclusion exclusion

S finite set

$A_1 \dots A_n$ subset $\qquad t \in \mathbb{N}$

$E_t := \{ s \in S \mid s \text{ belongs to exactly } t \text{ of the } A_i\text{'s} \}$

Thm (8.1)

$$|E_t| = \sum_{j=t}^{n} (-1)^{j-t} \binom{j}{j-t} \alpha_j$$

Rmk $t=0$ $\qquad E_0 = \{ s \in S \mid s \notin A_i \}$

$$= \bigcap_{i=1}^{n} A_i^c$$

$$|E_0| = \sum_{j=0}^{n} (-1)^{j-0} \binom{j}{j} \alpha_j = \sum_{j=0}^{n} (-1)^j \alpha_j \rightarrow$$ size of the intersection of comp / comp of the uni.

Proof: $f : S \longrightarrow \mathbb{R}$

$$|E_t| = \sum_{s \in S} f(s) = \sum_{s \in S} \sum_{j=t}^{n} (-1)^{j-t} \binom{j}{j-t} \sum_{|I|=j} \mathbb{1}_{\bigcap_{i \in I} A_i}(s) = \sum_{j=t}^{n} (-1)^{j-t} \binom{j}{j-t} \alpha_j$$

As before we want $\boxed{f = \mathbb{1}_{E_t}}$

$$f(s) = \sum_{j=t}^{n} (-1)^{j-t} \binom{j}{j-t} \sum_{|I|=j} \mathbb{1}_{\bigcap_{i\in I} A_i}(s)$$

Suppose $s \in S$ $s \notin E_t$ there are two cases

① s belongs to less than t of the A_i

But then, if $|I| \geq t$ $s \notin \bigcap_{i\in I} A_i$

$$f(s) = \sum_{j=t}^{n} (-1)^{j-t} \binom{j}{j-t} \sum_{|I|=j} \mathbb{1}_{\bigcap_{i\in I} A_i}(s) = 0$$

$= 0$

② s belongs to more than t of the A_i's

let $k > t$ the number of A_i's such that $s \in A_i$

$j > k$ $|I| = j$ $\mathbb{1}_{\bigcap_{i\in I} A_i}(s) = 0$

$$f(s) = \sum_{j=t}^{k} (-1)^{j-t} \binom{j}{j-t} \sum_{|I|=j} \mathbb{1}_{\bigcap_{i\in I} A_i}(s) \quad \binom{k}{j}$$

$$= \sum_{j=t}^{k} (-1)^{j-t} \binom{j}{j-t} \binom{k}{j} \qquad m = j-t$$

$$= \sum_{j=t}^{k} (-1)^{j-t} \frac{\cancel{j!}}{(j-t)!\, t!} \cdot \frac{k!}{\cancel{j!}\,(k-j)!} \qquad j = m+t$$

$$= \sum_{m=0}^{k-t} (-1)^{m} \boxed{\frac{k!}{m!\, t!} \, \frac{(k-t)!}{(k-t)!} \, \frac{1}{(k-m-t)!}}$$

$$\|$$

$$\binom{k}{t} \binom{k-t}{m}$$

$$= \frac{k!}{t!\, \cancel{(k-t)!}} \, \frac{\cancel{(k-t)!}}{(k-t-m)!\, m!}$$

$$= \sum_{m=0}^{k-t} (-1)^{m} \binom{k}{t} \binom{k-t}{m}$$

$$= \binom{k}{t} \sum_{m=0}^{k-t} (-1)^{m} \binom{k-t}{m} = \binom{k}{t} \left(1 + (-1)\right)^{k-t} = 0$$

We showed $f(s) := 0$ if $s \notin E_t$

Now it remains to show $f(s) = 1$ is $s \in E_t$

$$f(s) = \sum_{j=t}^{n} (-1)^{j-t} \binom{j}{j-t} \sum_{|I|=j} \mathbb{1}_{\bigcap_{i\in I} A_i}(s)$$

s cannot belong to any intersection of more than t elements

$$= \underbrace{\sum_{j=t}^{t} (-1)^{j-t} \binom{j}{j-t} \sum_{|I|=j} \boxed{\mathbb{1}_{\bigcap_{i\in I} A_i}(s)}}_{1} \neq 0$$

only if it is the intersection of the t sets where s lies

$$= (-1)^0 \binom{j}{0} \cdot 1 = 1$$

$L_t := \{ s \in S$ that belongs to at least t condition$\}$

$$|L_t| = \sum_{k=t}^{n} |E_k| = \sum_{j=t}^{n} (-1)^{j-t} \binom{j}{t-1} \alpha_j$$

Examples & application

Find the number of positive integers ≤ 100 not divisible by 2 or 5.

$$A_1 = \{ n \in \mathbb{Z} \mid 100 \geq n \geq 1 \quad 2 \mid n \}$$

$$A_2 = \{ n \in \mathbb{Z} \mid 100 \geq n \geq 1 \quad 5 \mid n \}$$

$$|A_1| = \frac{100}{2} = 50$$

$$|A_2| = \frac{100}{5} = 20 \qquad A_1 \cap A_2 = \{ 100 \geq n \geq 1 \quad 10 \mid n \}$$

$$|A_1 \cap A_2| = 10$$

$$|A_1^c \cap A_2^c| = 100 - 50 - 20 + 10 = 40$$

$n \in \mathbb{Z} \qquad n \geq 1$

$$\phi(n) = \{ n \geq k \geq 1 \quad \gcd(k, n) = 1 \}$$

$$= n \prod_{\substack{p \mid n \\ \text{prime}}} \left(1 - \frac{1}{p}\right)$$

$n = p_1^{n_1} \cdots p_t^{n_t}$ prime factorization p_i distinct

$$A_i = \{ 1 \leq k \leq n \mid p_i \mid k \}$$

$$\phi(n) = \left| \bigcap_{i=1}^{t} A_i^c \right|$$

$$|A_i| = \frac{n}{p_i}$$

$$|A_i \cap A_j| = \frac{n}{p_i p_j}$$

$$|A_{i_1} \cap \dots \cap A_{i_s}| = \frac{n}{p_{i_1} \cdots p_{i_s}}$$

$$\Phi(n) = n - n\left(\frac{1}{p_1} + \quad + \frac{1}{p_t}\right) + n\left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \dots + \frac{1}{p_{n-1}p_n}\right)$$

$$- n\left(\frac{1}{p_1p_2p_3} + \dots\right) + \dots + (-1)^t n\left(\frac{1}{p_1\cdots p_t}\right)$$

$$= n \underbrace{\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_n}\right)}$$

Derangements:

A derangement is a permutation $\sigma \in S_n$ such that $\sigma(i) \neq i$ (no fixed point)

$$d_n := \# \text{ derangements } \approx n!\, e^{-1}$$

$$|n!\, e^{-1} - d_n| \leq \frac{1}{n+2}$$

$|S_n| = n!$ $\qquad i = 1 \dots n$

$A_i = \{\sigma \in S_n \mid \sigma(i) = i\}$

$$d_n = \left| \bigcap_{i=1}^{n} A_i^c \right|$$

$$\overset{IEP}{=} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! =$$

$$|A_{i_1} \cap \dots \cap A_{i_k}| = |\{\sigma \in S_n \text{ leave fixed } i_1 \dots i_k\}|$$
$$= (n-k)!$$

How many intersections are there: $\binom{n}{k}$

$$= \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$$

$$|n!\, e^{-1} - d_n| = \left| n! \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} - n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right|$$
$$\left| n! \cdot \sum_{i=n+1}^{\infty} (-1)^i / i! \right| \leq \frac{1}{n+2}$$