Mathematics III : combinatorics:
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Ex: 1.2.10:

1^{st} method: Choose k books $(1 \le k \le 14)$ for the first bookshelf: $\frac{15!}{(15-k)!}$

Choose an ordering for the $15-k$ books left for the 2^{nd} shelf.

We get: $$\sum_{k=1}^{14} \frac{15!}{(15-k)!} \times (15-k)!$$

$$= \sum_{k=1}^{14} 15! = 14 \cdot 15!.$$

2^{nd} method: Choose an ordering of 15 books and then divide in two parts.

We get: $15! \times 14$.

Ex 1.2.16: 1) We can make 40^{25} messages.

2) Now we can make $30^2 \cdot 40^{23}$ messages.

Ex 1.3.10: 1) We get $\binom{12}{5} = \frac{12!}{5!\,(12-5)!}$ choices for the team.

2) We get $\binom{10}{3} = \frac{10!}{3!\,(10-3)!}$ choices for the team.

Ex 1.3.16: a) $\sum_{i=1}^{6} (i^2+1) = 1+1+2^2+1+3^2+1+4^2+1+5^2+1+6^2+1$
$= 2+5+10+17+26+37 = 97.$

b) $\sum_{j=-2}^{2} (j^3-1) = ((-2)^3-1) + ((-1)^3-1) + ((0)^3-1) + (1^3-1) + (2^3-1) = -9-2-1+7$
$= -5.$

c) $\sum_{i=1}^{10} [1+(-1)^i] = 12.$

d) $\sum_{k=1}^{2n} (-1)^k = 0$

e) $\sum_{i=1}^{6} i(-1)^i = -1 + 2 - 3 + 4 - 5 + 6$
$= 3.$

Ex 3.1.2: $A = \{1, \{1\}, \{2\}\}$

a) $1 \in A$ ✓ b) $\{1\} \in A$ ✓

c) $\{1\} \subseteq A$ ✓ d) $\{\{1\}\} \subseteq A$ ✓

e) $\{2\} \in A$ ✓ f) $\{2\} \subseteq A$ ✗

g) $\{\{2\}\} \subseteq A$ ✓ h) $\{\{2\}\} \subseteq A$ ✓

Ex 3.1.9: a) cardinality of $\mathcal{P}(A)$ is 2^n where A is of cardinality n.

Here we have $|\mathcal{P}(A)| = 64$ so that $|A| = 6$.

b) c) If B has cardinality n we can make $\binom{n}{k}$ subsets of cardinality k.

So we have the number of subsets of odd cardinality is:

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} = 2^{n-1}.$$

This equality from the facts that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k}(-1)^n = 0.$$

For B we have 64 subsets of odd cardinality:

$$2^{n-1} = 64$$

So $n-1 = 6$, $|B| = 7$.

Ex 5.3.10: a) Counting surjective maps from a set with 7 elements to a set with 4:

$$\underbrace{4^7}_{\text{Total number of functions}} - \binom{4}{1}\underbrace{3^7}_{\text{functions to a subset of 3}} + \binom{4}{2}2^7 - \binom{4}{1}1^7 = 8400.$$

b) Count surjective maps from a set with 6 elements to 3 elements and add maps with possibly more than one ball in the 2^{n-1} container which

are surjections from a set with 6 elements to a set with 4 elements.
We get 2100 possible choices.