MM5023

Lecture 3

Rook polynomials

(the first example of generating functions)

Lecture plan

- Definitions and easy examples
- Finer computational methods
- Examples (old exams)

Definition: Rook numbers

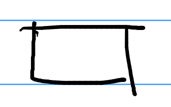
Let C be a chessboard - that is a grid made up of cells, some chells might be shaded, and they are to be considered forbidden. For every integer k , the k-th rook number of C

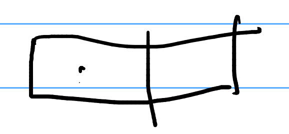
$$r_k(C)$$

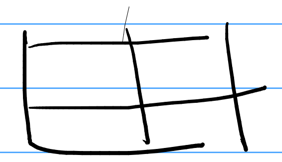
is the number of placing k rooks (place holders) on C in such away that every rows and column of C contains at most 1 rook.

By definition we set $\boxed{r_0(C) = 1}$

Examples

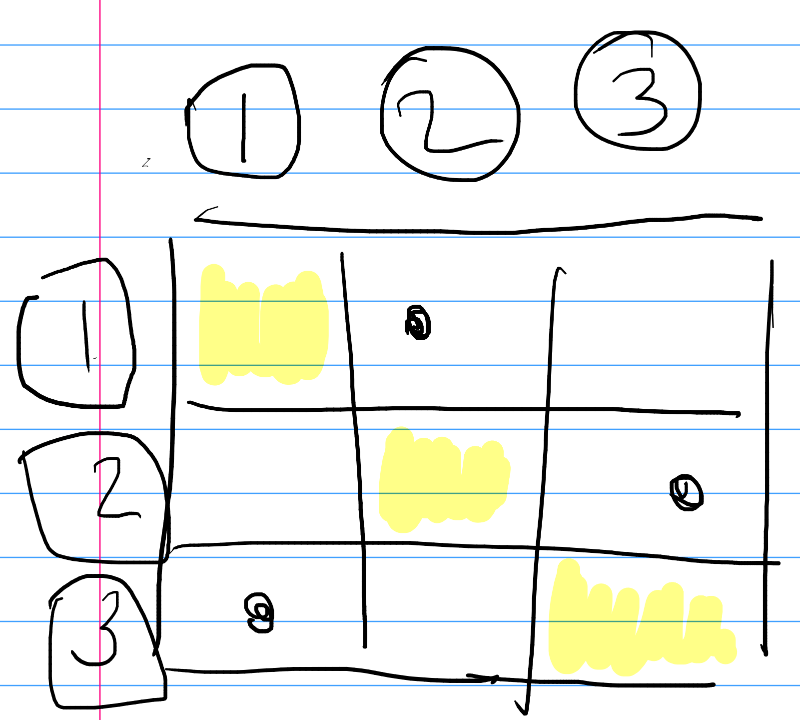
 $r_0(C) = 1$ $r_1(C) = 1$

 $r_0(C) = 1$ $r_1(C) = 2$ $r_i(C) = 0 \ \forall i > 1$

 $r_0(C) = 1$ $r_1(C) = 4$ $r_2(C) = 2$

always the # of cells

$r_i(C) = 0 \quad \forall i \geq 3$



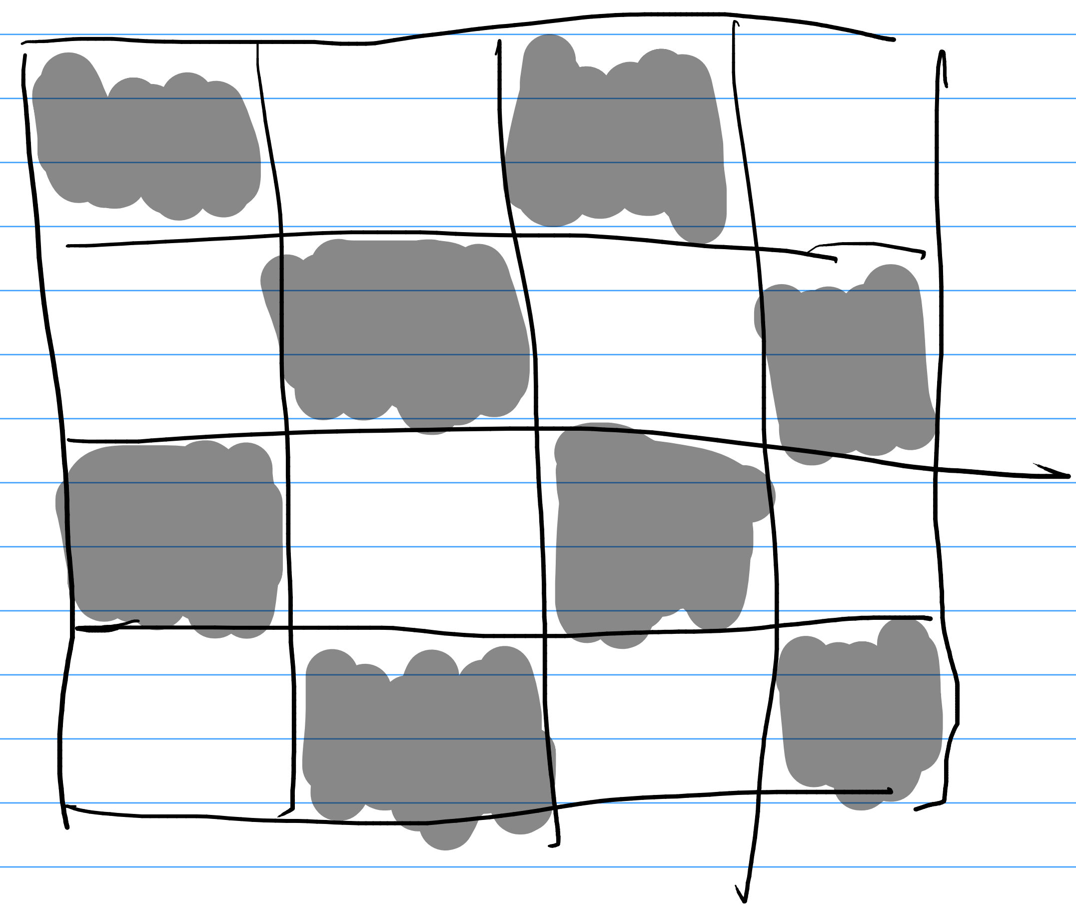
$r_0(C) = 1$

$r_1(C) = 6$ $r_2(C)$ we will see

$r_3(C) = d_3$

rook placement = permutation that avoid the "diagon

Shaded cell are forbidden



Gray cells are forbidden

$r_1(c) = 8$

Rook Polynomial

The rook polynomial of a chessboard C

$$r(C,x) = \sum_{k=0}^{\infty} r_k(C)\, x^k$$

Why is it a polynomial ?

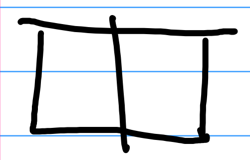
with at least 1 non forbidden

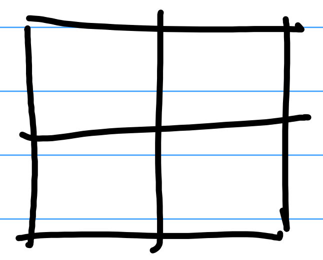
C finite chessboard m row n colum

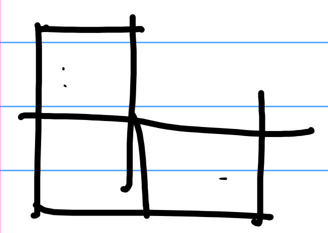
$$k > \min\{m,n\} \implies r_k(C) = 0$$

$$\deg r(C,x) \le \min\{m,n\}$$

Examples

 $1+x$

 $1+4x+2x^2$

 $1+3x+x^2$

$m \times n$ chessboard (no forbidden places)

want to place k rooks

Assume that they are different

1st	2st	3	k
$m \cdot n$	$(m-1)(n-1)$	$(m-2)(n-2)$	$(m-k+1)(n-k+1)$

I have to divide by $k!$

$$r_k(C) = \frac{1}{k!} \frac{m!\, n!}{(m-k)!\,(n-k)!} = \binom{m}{k}\binom{n}{k} k!$$

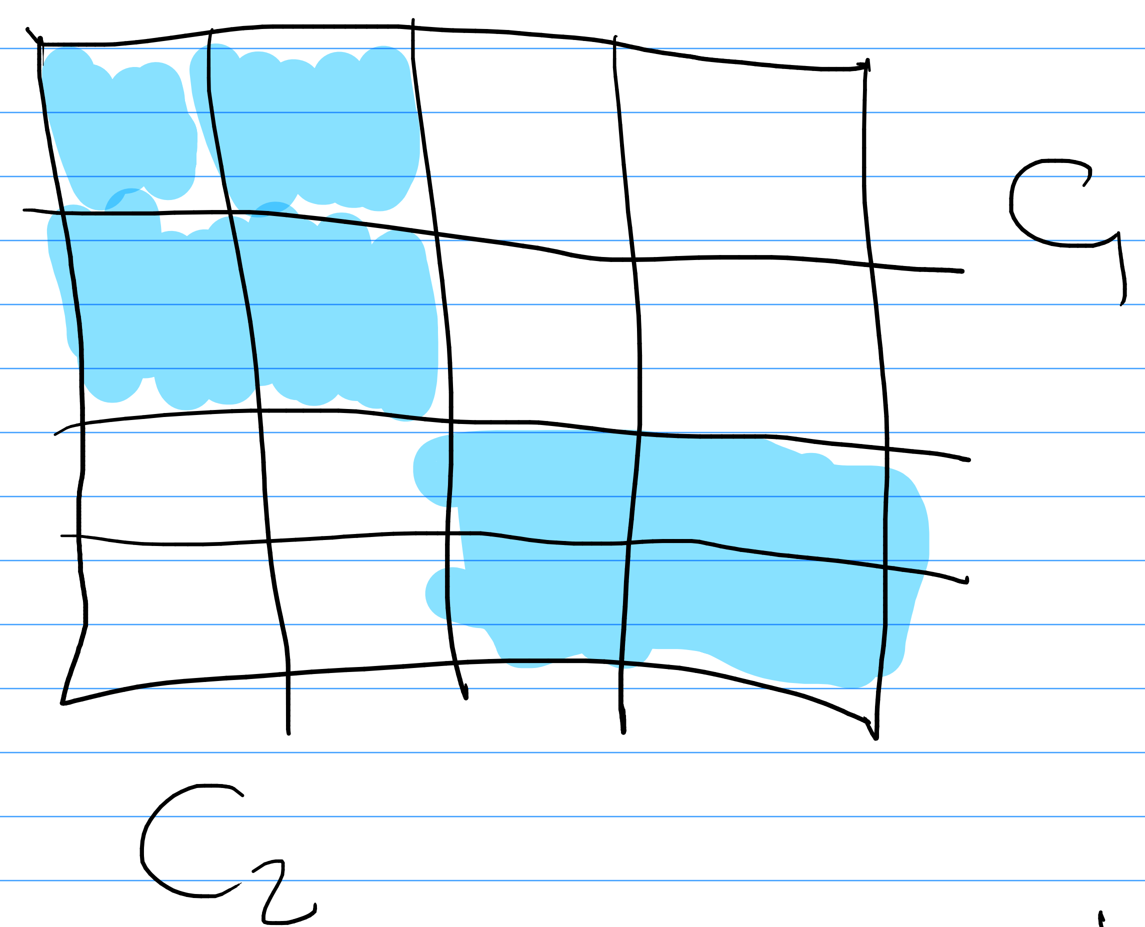
$$r(C, x) = \sum_{k=0}^{\infty} \binom{m}{k}\binom{n}{k} k!\, x^k$$

Disjoint Chessboard and Rook Polynomials

We say that a chessboard C is composed of two <u>disjoint</u> chessboards C_1 and C_2

if $C = C_1 \cup C_2$

and C_1 and C_2 have no common row or column.



<u>Prop</u> If C is composed by two disjoint chessboards C_1 and C_2 then

$$r(C, x) = r(C_1, x)\, r(C_2, x)$$

<u>Proof</u>: We want to show that they have the same coefficients

$$r_k(C) = \sum_{j=0}^{k} r_j(C_1)\, r_{k-j}(C_2)$$

coeff of deg k of $r(C_1 x)\, r(C_2 x)$

To place k rooks on C we place j rooks on C_1 and the remaining on C_2

~~as~~ C_1 & C_2 disjoint makes the two placement independent PRODUCT

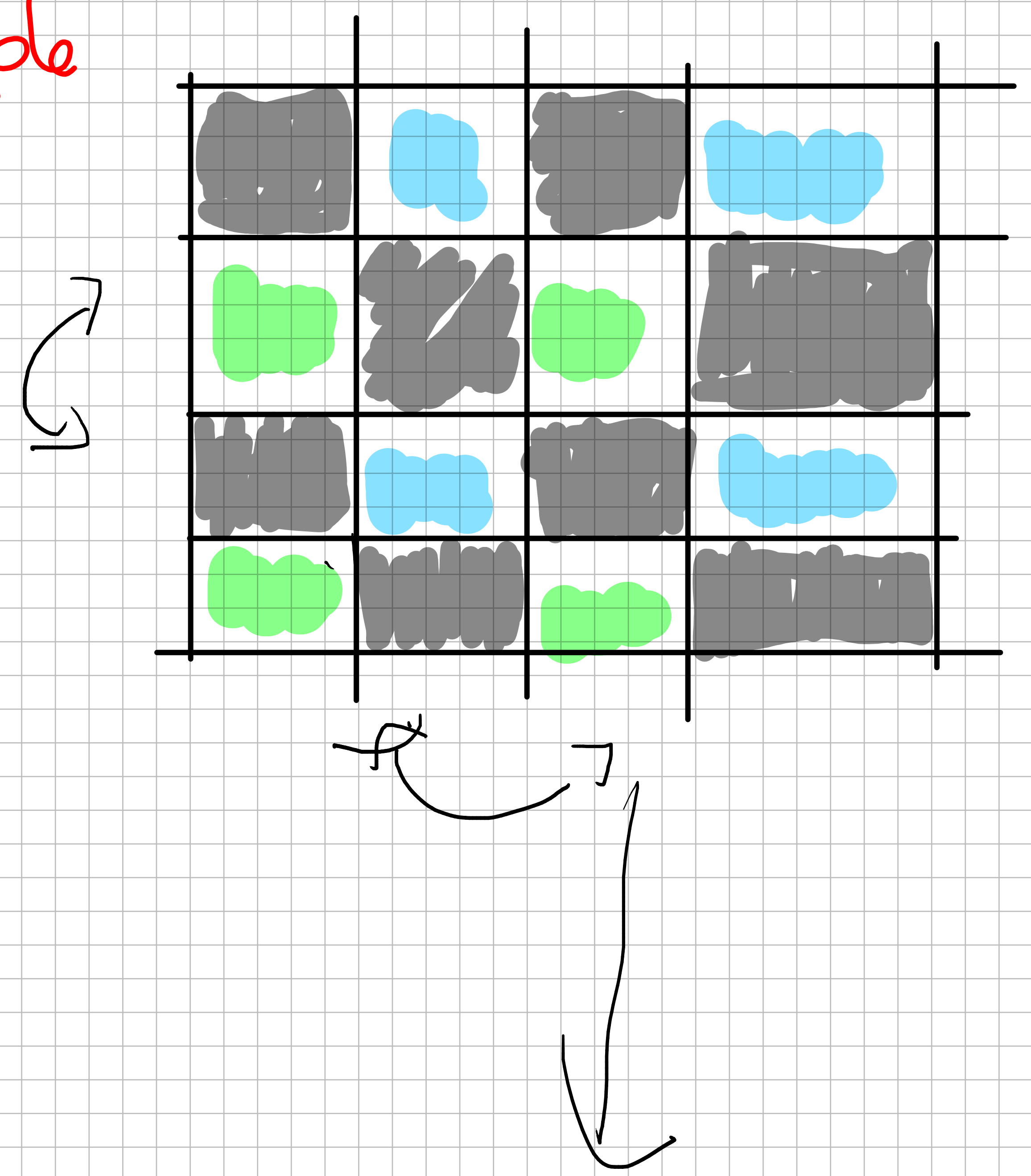
$$r_j(C_1) \cdot r_{k-j}(C_2)$$

RULE OF SUM in all cases

$$r_k(C) = \sum_{j=0}^{k} r_j(C_1)\, r_{k-j}(C_2)$$

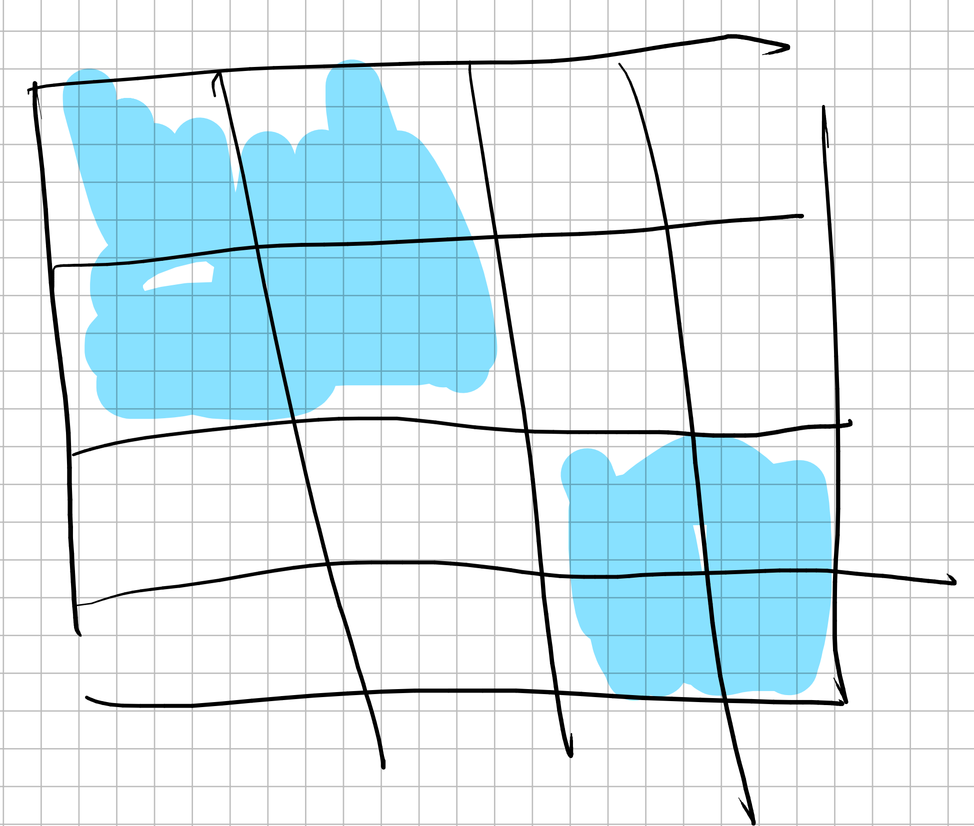
!!

Example



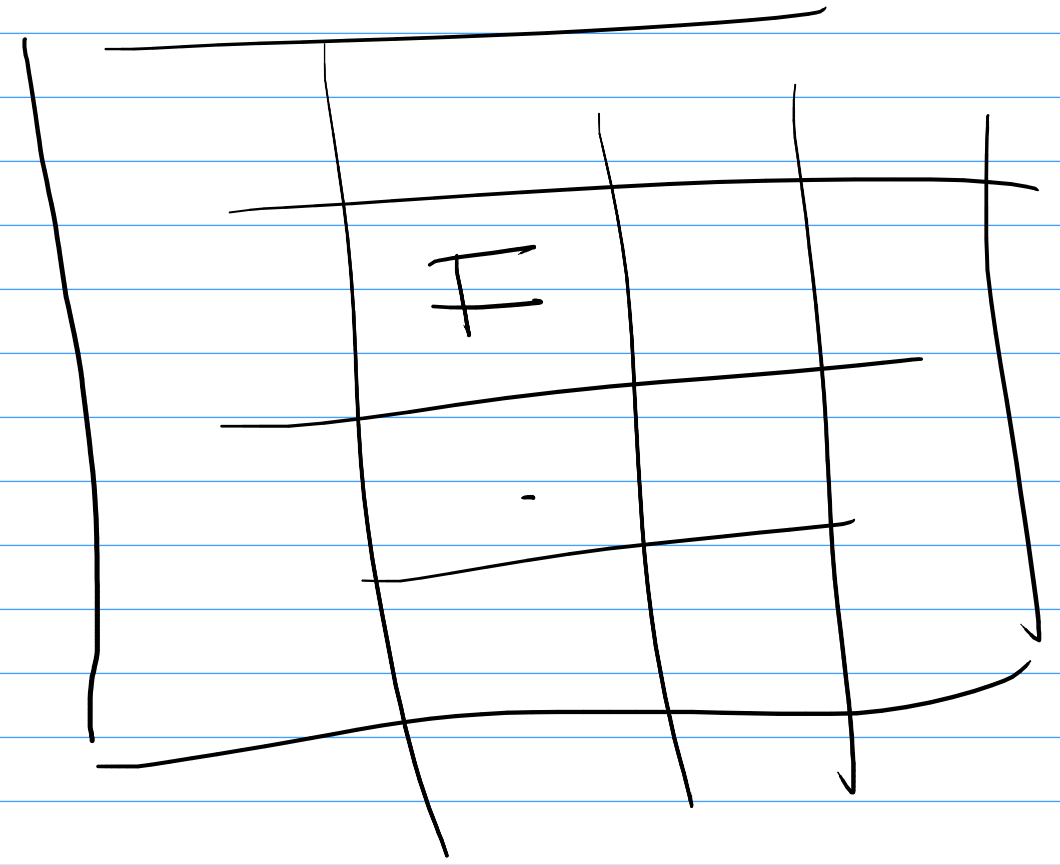
$r(C,x)$ does not change if we permute rows or columns of C

$$r(C,x) = (1 + 4x + 2x^2)^2$$



$$r(C,x) = (1 + 4x + 2x^2)^2$$

A RECURSIVE FORMULA



C chessboard

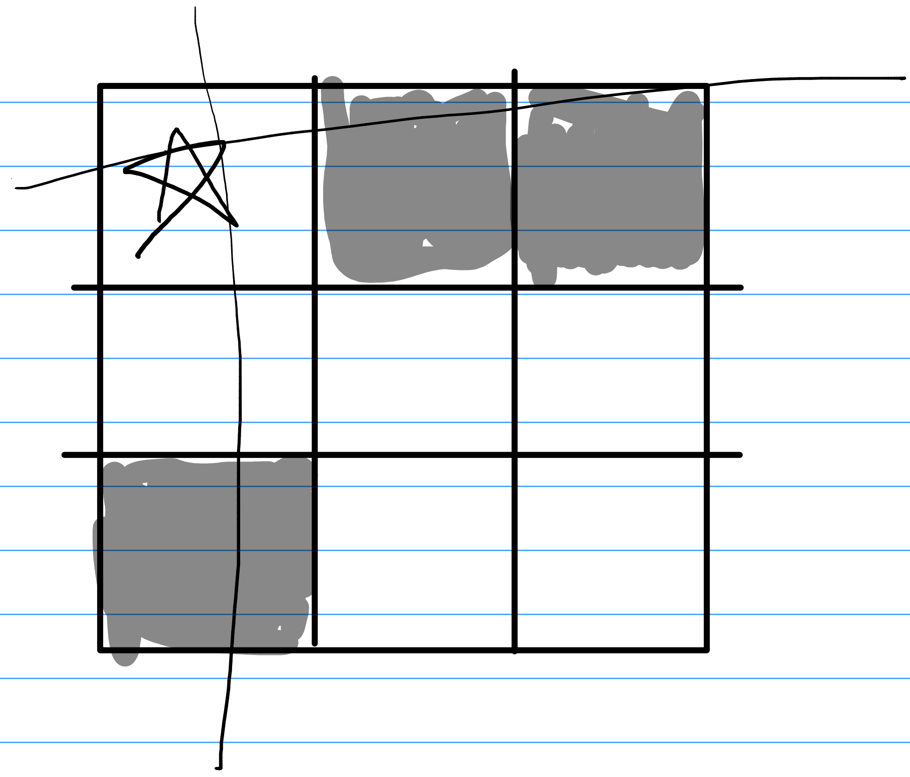
F allowed cell

C_s = chessboard obtaine by C by forbidding the row & col of F

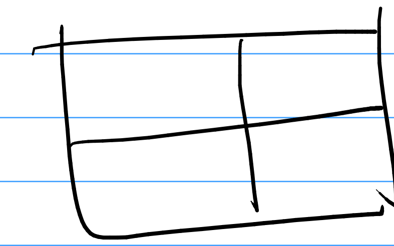
C_e = chessboard obtained ~~by~~ ~~C~~ from C by forbiding F

$$r(C,x) = x\, r(C_s, x) + r(C_e, x)$$

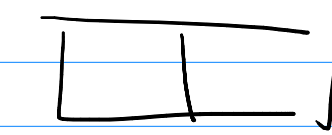
Example



C_s

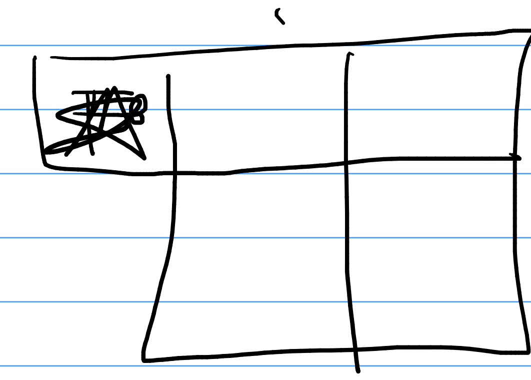


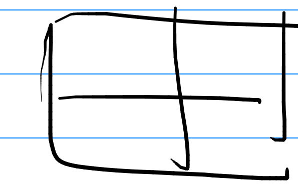
$1+4x+2x^2$



C_s'

C_e





C_e'

$$r(C,x) = x\,r(C_s, x) + r(C_e, x)$$

$$= x(1+4x+2x^2) + x(1+2x)$$

$$+ 1+4x+2x^2$$

$$= x + 4x^2 + 2x^3 + x + 2x^2 + 1$$

$$+ 4x + 2x^2 = 1 + 6x + 8x^2 + 2x^3$$

c

Proof We want to show that the $r(C,x)$ and $x\,r(C_s, x) + r(C_e, x)$ have the same coefficients

$$r_k(C) = r_{k-1}(C_s) + r_k(C_e)$$

We want to place k rooks in C & choose a cell F and there are two cases

① I place a rook in F

I need to place the remaining $k-1$ rooks in C minus the row & col of F

I place $k-1$ rooks in C_s

$r_{k-1}(C_s)$ ways to do that

② I place no rook in F

I place k rooks in the chessbord C
but I avoid F. It is like F is
forbidden

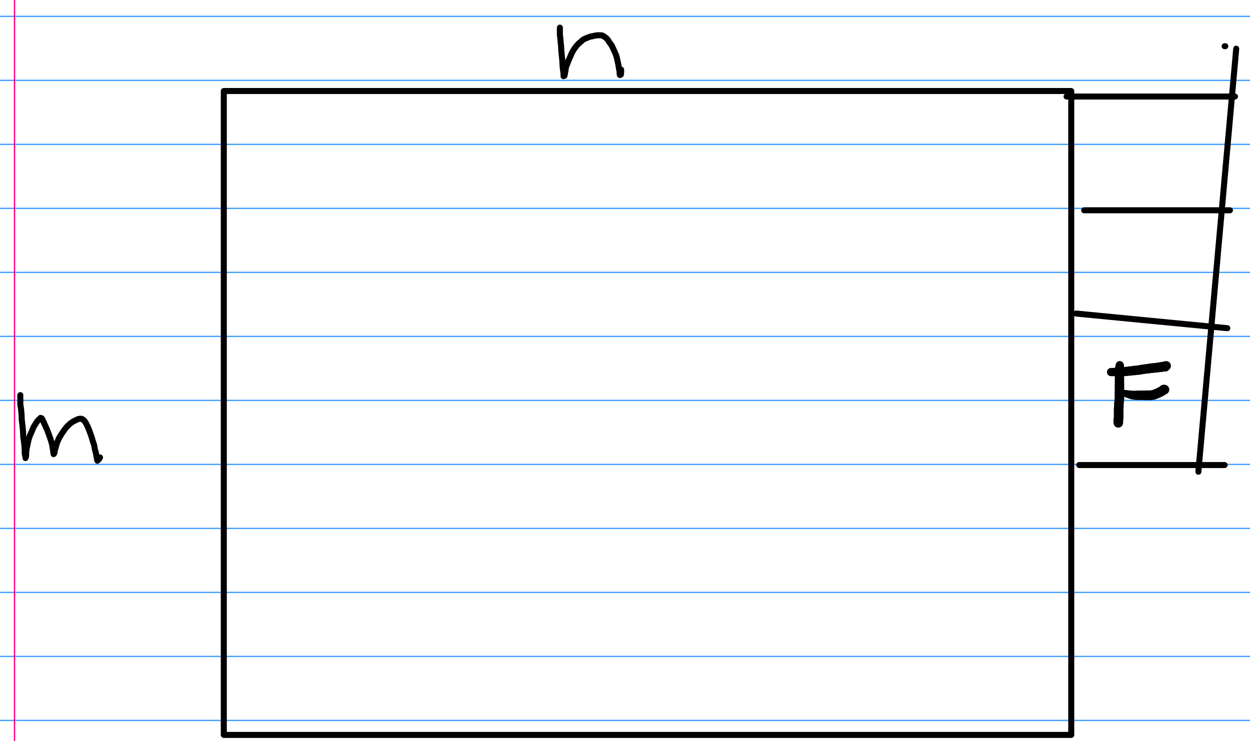
$r_k(C_e)$ ways to do it.

The two cases are disjoint
RULE OF SUM

$$r_k(C) = r_{k-1}(C_S) + r_k(C_e)$$

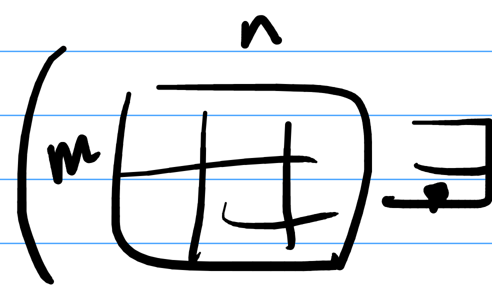
:)

Example (old exam)

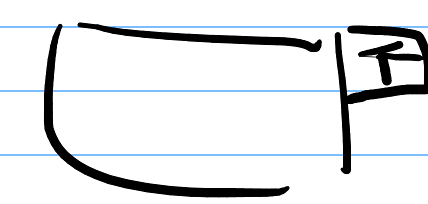


$$m > 3$$

$$r(C, x) = x\, r((m-1 \times n), x)$$

$$+\ r\Big($$  $$\Big)$$

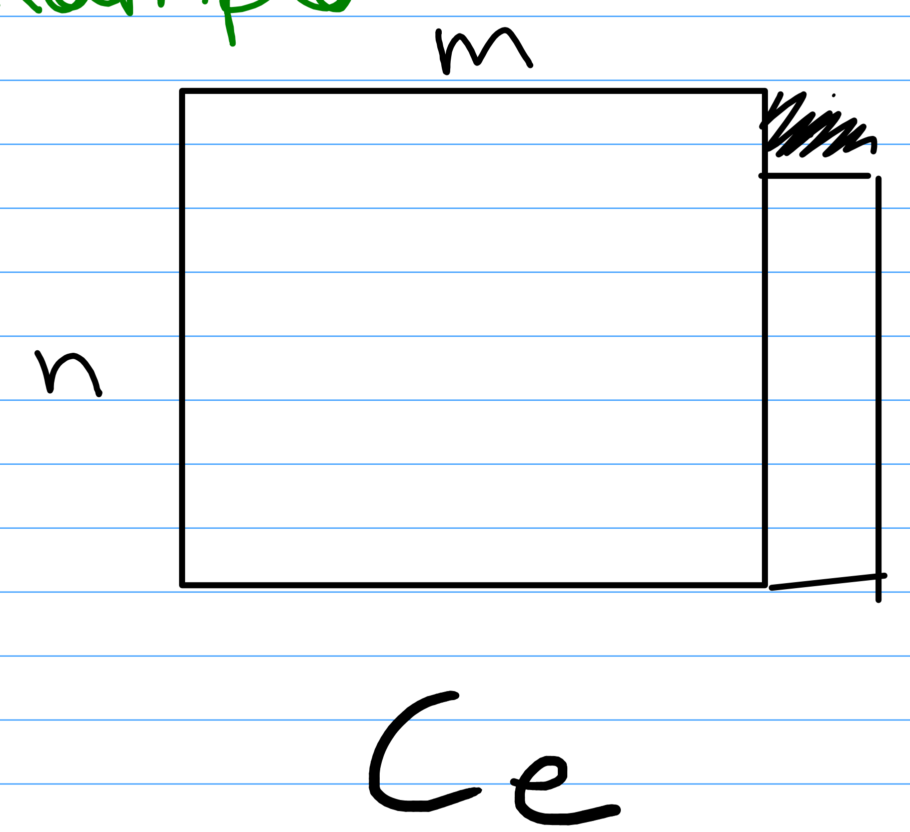
$$= x\, r((m-1) \times n, x) + x\, r((m-1) \times n, x)$$

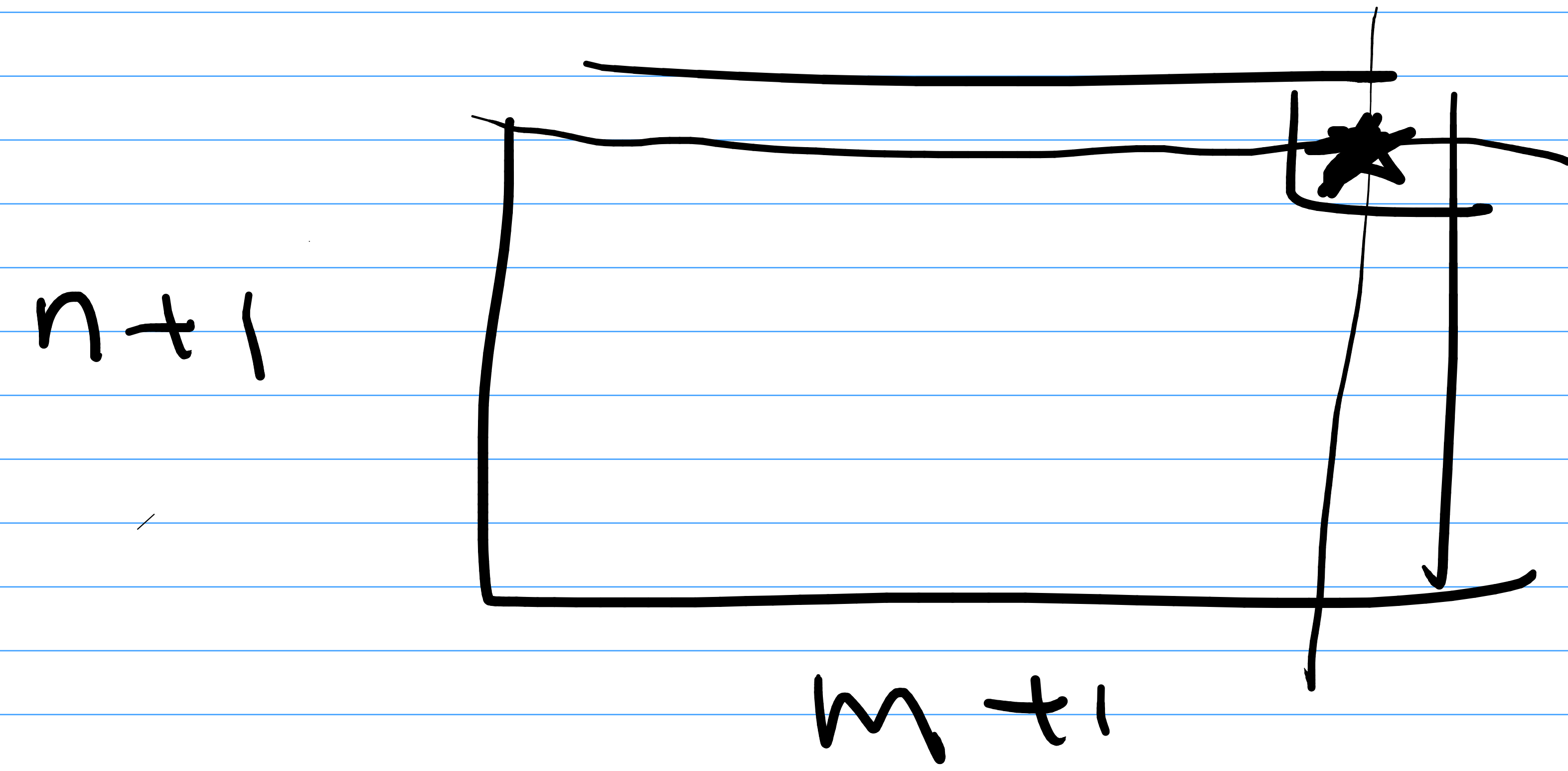
$$+\ r\Big($$  $$\Big)$$

$$= 2x\, r((m-1 \times n), x) + x\, r((m-1 \times n), x) + r(m \times n)$$

$$= 3x\, r((m-1 \times n), x) + r(m \times n)$$

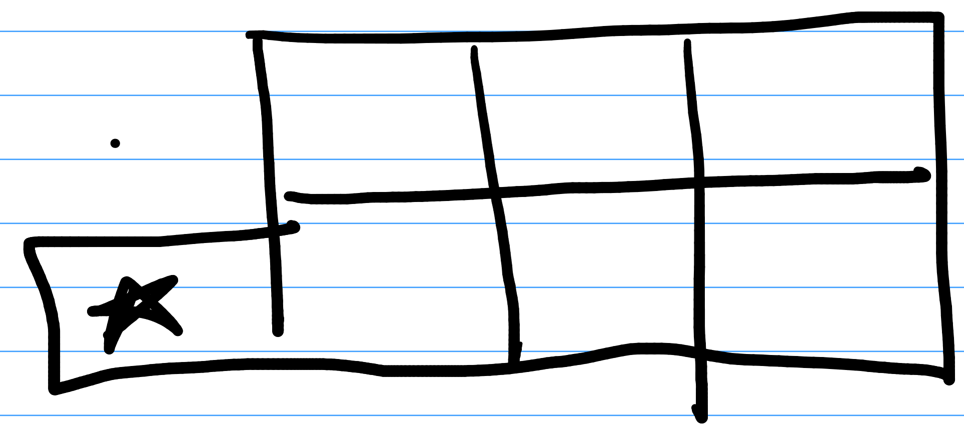
Example





$$r\left((m+1 \times n+1), x\right) = x\, r\left((m) \times (n)\right) + r(C_e, x)$$

$$r(C_e\, x) = r(m+1 \times n+1,\ x) - x\, r(m \times n, x)$$



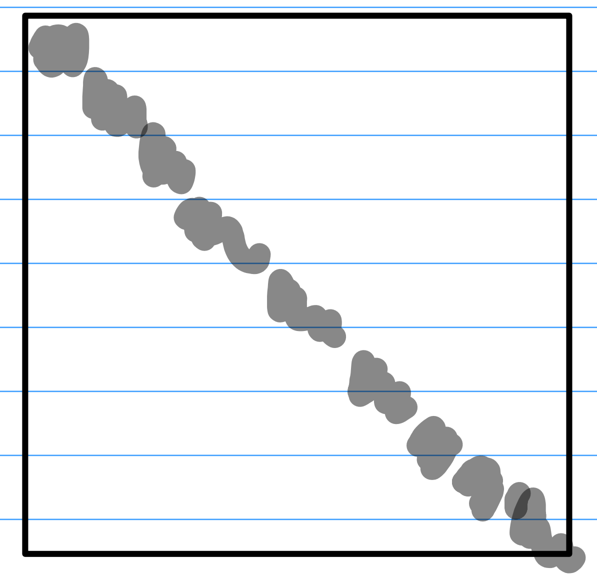
$1+3x$

$$r(C,x) = x\,r(\quad, x) + r(\quad, x)$$

$$= x(1+3x) + \sum_{k=0}^{2} \binom{2}{k}\binom{3}{k} k!\, x^k$$

$$= \underline{1 + 7x + 9x^2}$$

Connection with derangements (Generalizations of derangements)



Square chessbord with n rows / cols

rook placement $\iff$ $\sigma \in S_n$
of n rooks

derangements as a rook placement of n rook whic avoid the diagonal

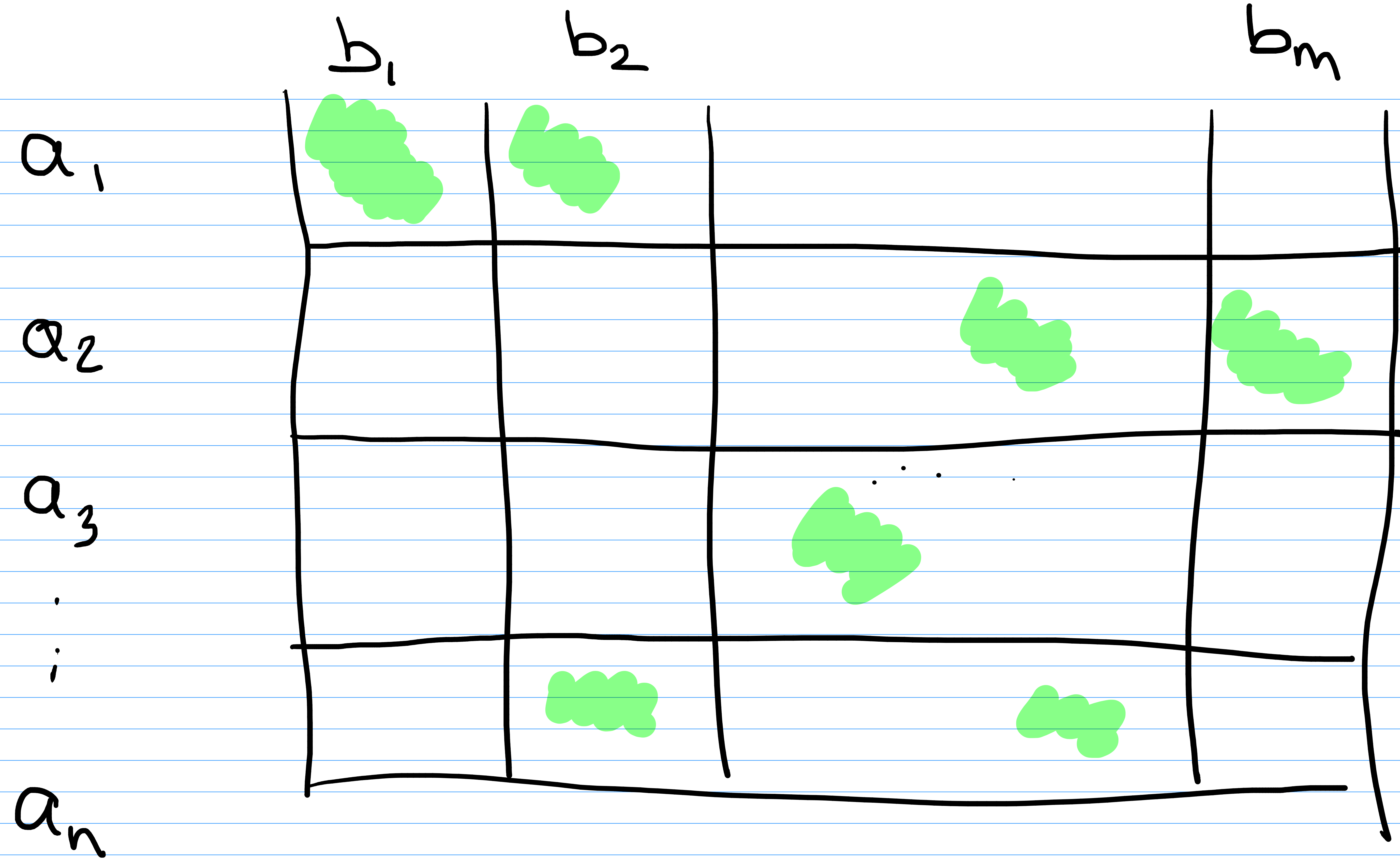
A B two finite set

$|A| = n$ $|B| = m$ $n \leq m$

there is a collection of subsets

$\{B_a\}_{a \in A}$ $B_a \subseteq B$

$? = \left| \{ \text{injective functions } f : A \longrightarrow B \mid f(a) \notin B_a \} \right|$



C

↓

I shade everything but not a B_a

$f: A \longrightarrow B$ injective $\iff$ place n rooks in C not in the cell (a, b) $b \in B_a$

$f(a) \notin B_a$

$$\sum (-1)^k r_k(C)\, P(n-k, m-k)$$

<u>why</u>: $S := \{ f : A \to B \text{ injective} \}$

$$A_i = \{ f \in S \mid f(a_i) \in B_{a_i} \}$$

want $\left| \bigcap A_i^c \right| = \sum_{k=1} (-1)^k |S_k| = \sum_k (-1)^k r_k(C) P(n-k, m-k)$

$$S_k = \{ f \in S \mid f(a_i) \in B_{a_i} \text{ for at last } k \text{ } i \}$$

$$= r_k(C) \underbrace{P(n-k, m-k)}_{\substack{\text{random} \\ \text{other} \\ \text{values}}}$$

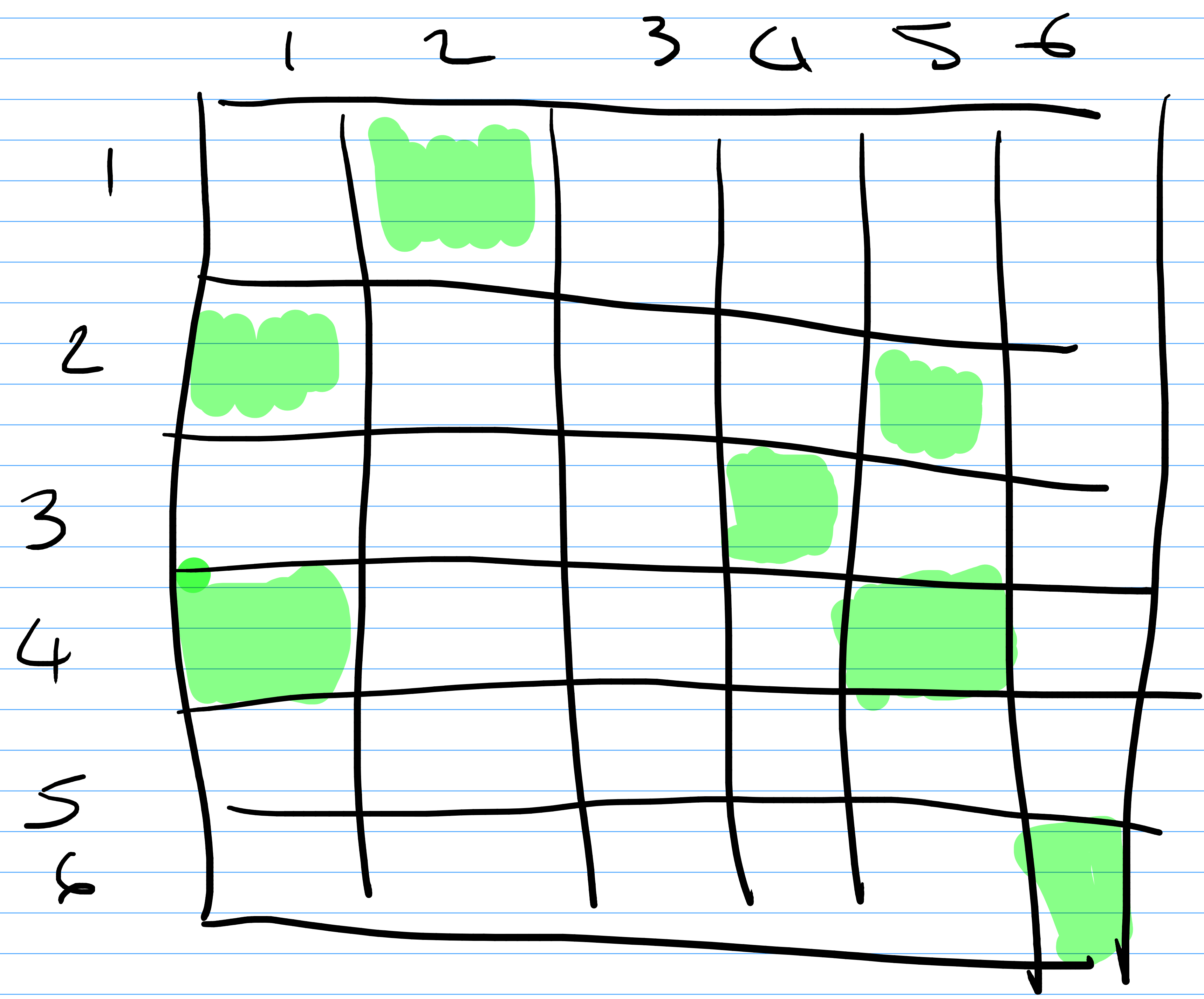
k values in B_a

trow two different D6 dices 6 times

what is the probability such that the result is not one of the following

(1,2) (2 1) (2 5) (3 4) (4,1) (4 5) (6 6)

" green chessboard)



$$r(C', x) = 1 + 7x + 17x^2 + 99x^3 + 10x^4 + 2x^5$$

Place 6 rooks in the chessbo

what is the probability that one result is forbidde

C_i = " the i of the first dice is paired with a forbidde one "

$$N(\bar{c}_1 \cdots \bar{c}_6) = \sum (-1)^k S_k = \sum (-1)^k r_k(C)(6-k)!$$

S_k: at least k are paired with a forbidden

$$6! - 7\cdot 5! + 17\cdot 4! - 19\cdot 3! + 10\cdot 2! + 2\cdot 1!$$

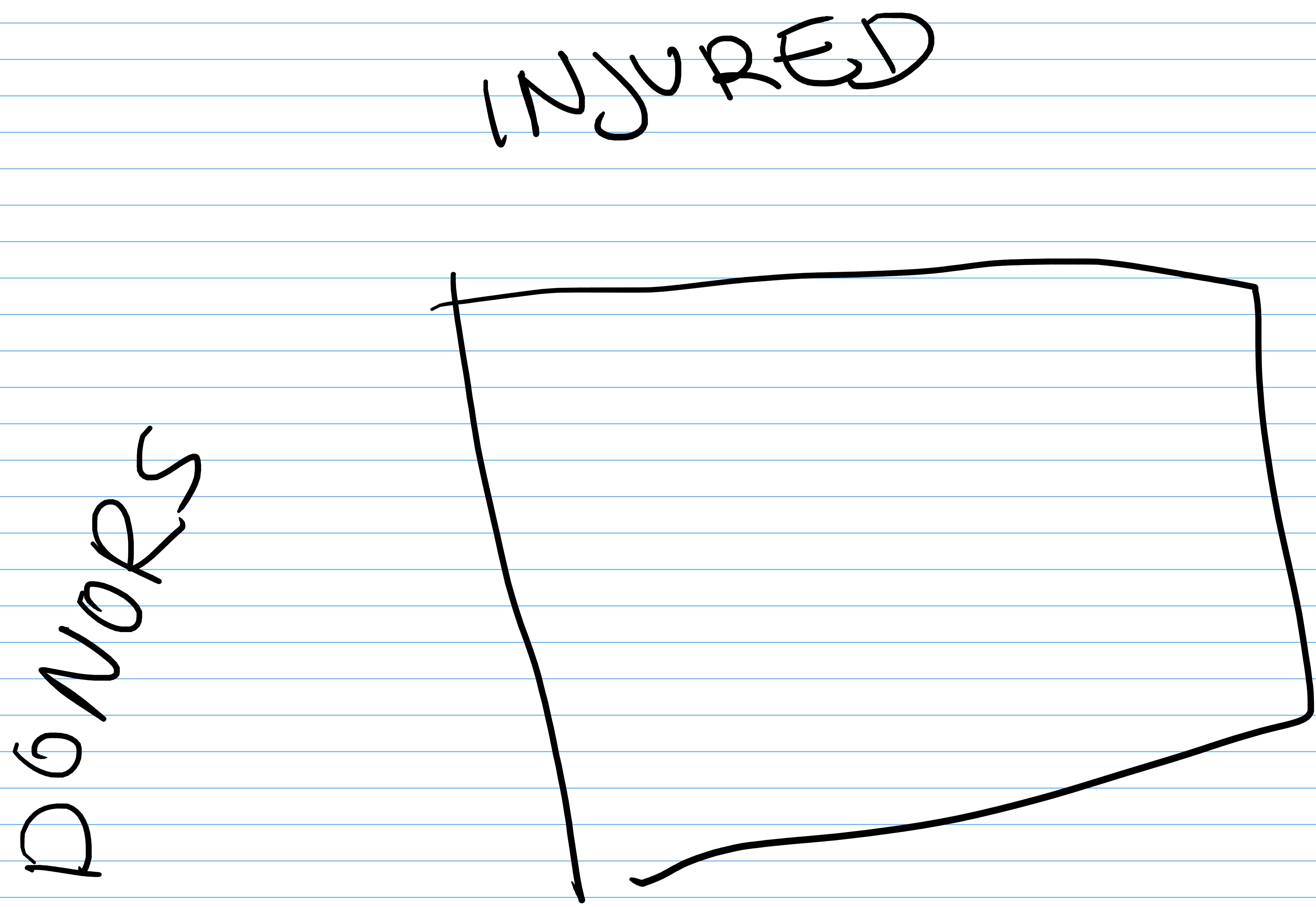
$$= 192.$$

$$\frac{192 \cdot 6!}{(29)^6}$$

# of cases

# of possible case

29 allowed pair 6 times.



forbidden cells
injured of wrong
blod type

In lecture 13
another way to
solve this problem
(Network flow)
→ Better Algorithms