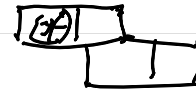
Ex 8.4.4: a) $r(C_1)(x)$



$$= x(1+2x) + 1 + 3x + x^2$$
$$= 1 + 4x + 3x^2$$

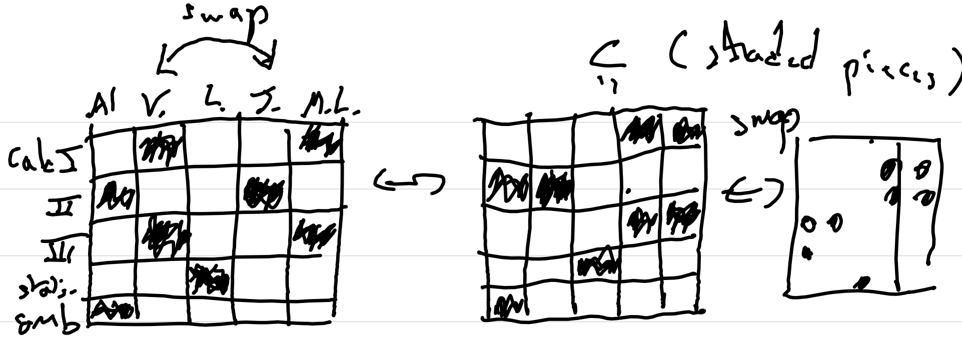
b) $r(C_2)(x) = (1+x)(1+3x)$

$$= 1 + 4x + 3x^3.$$

Ex 8.4.6: a) $$r_k(C) = \underbrace{\binom{m}{k}\binom{n}{k} k!}_{\text{Formula from the course}}$$

b) $$r(C, x) = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} k!\, x^k.$$

Ex 8.4.9:



The rook polynomial for C is:

$$r(C,x) = (1+2x)^3(1+x)^2$$
$$= (1+6x+6x^2+8x^3)(1+2x+x^2)$$
$$= 1+6x+6x^2+8x^3+2x+12x^2+12x^3$$
$$+16x^4+x^2+6x^3+6x^4+8x^5$$
$$= 1+8x+\ \ x^2+26x^3+24x^4+8x^5$$

$$r(C,x) = (1+3x+x^2)(1+x)$$
$$(1+4x+2x^2)$$

$$= (1+3x+x^2+x+3x^2+x^3)$$
$$(1+4x+2x^2)$$
$$= (1+4x+4x^2+x^3)(1+4x+2x^2)$$
$$= 1+4x+4x^2+x^3+4x+16x^2$$
$$+16x^3+4x^4+2x^2+8x^3$$
$$+8x^4+2x^5$$

$$r(C,x) = 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5$$

The number of possible assignments:

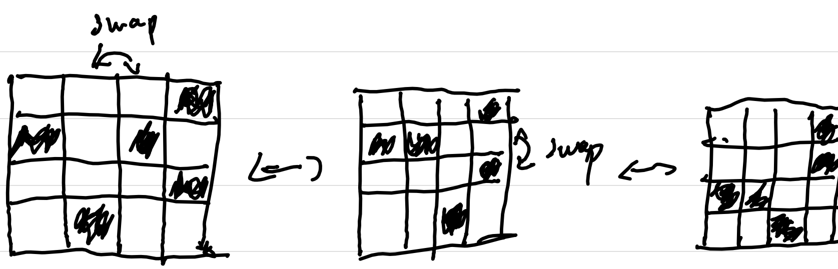
$$\sum_{k=0}^{5} (-1)^k \, r_k(C) \underbrace{\left(\frac{(5-k)!}{((5-k)-(5-k))!} \right)}_{= P(m-k,\, n-k)}$$

$$= 5! - 8 \cdot 4! + 22 \cdot 3! - 25 \cdot 2! + 12 \cdot 1! - 2 \cdot 0!$$

$$= 120 - 8 \cdot 24 + 22 \cdot 6 - 50 + 12 - 2$$

$$= (\ldots) = 20.$$

b)



The rook polynomial of C is:

$$r(C,x) = (1+2x)^2(1+x)$$

$$r(C, x) = (1 + 4x + 4x^2)(1 + x)$$

$$= 1 + 4x + 4x^2 + x + 4x^2 + 4x^3$$

$$= 1 + 5x + 8x^2 + 4x^3.$$

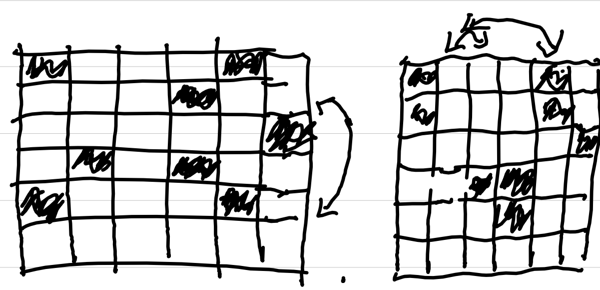
So the number of assignments is:

$$4! - 5 \cdot 3! + 8 \cdot 2! - 4 \cdot 1!$$

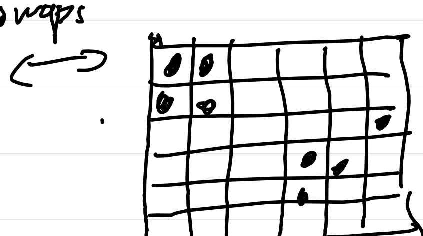
$$= 24 - 30 + 16 - 4 = 6$$

The probability is $6/20$.

Ex 8.4.10 :



2 swaps



The rook polynomial is

$$r(C,x) = 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5$$

The number of assignments:

$$N = 6! - 8 \cdot 5! + 22 \cdot 4! - 25 \cdot 3! + 12 \cdot 2! - 2 \cdot 1!$$

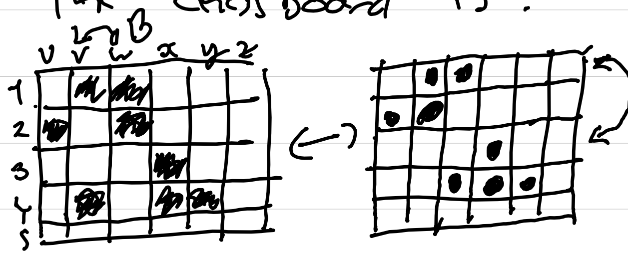
$N = (\ldots)$.

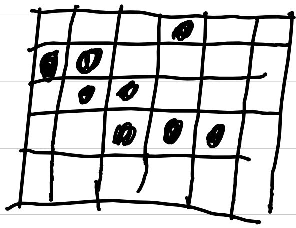
The set of all events is 28^6 (ordered sequence of 6 chairs in the non shaded chessboard).

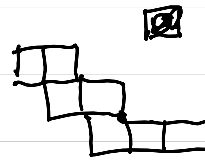
The probability is $\dfrac{6! \cdot N}{28^6}$.

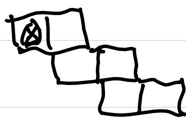
Ex 8.4.12: The chessboard is:

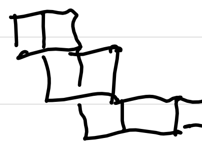
A



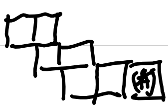
$\rightsquigarrow$ 

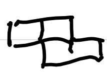
$C =$  ← remove

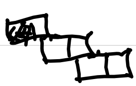
$C_1 =$ 

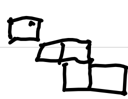
$C_2 =$ 

$$r(C,x) = x\, r(C_1,x) + r(C_2,x)$$

$C_2 =$  $\rightsquigarrow$ $r(C_2,x) = x\, r(C_3,x) + r(C_1,x)$

$C_3 =$  $\rightsquigarrow$ $r(C_3,x) = 1+4x+3x^2$

$C_1 =$  $\rightsquigarrow$ $r(C_1,x) = x\, r(C_3,x) + r(C_4,x)$

$C_4 =$  $\rightsquigarrow$ $r(C_4,x) = (1+x)\, r(C_3,x)$

$$= (1+x)(1+4x+3x^2)$$

$$= 1+4x+3x^2+x+4x^2+3x^3$$

$$= 1+5x+7x^2+3x^3$$

So we have:

$$r(C_1, x) = x + 4x^2 + 3x^3 + 1 + 5x + 7x^2 + 3x^3$$

$$= 1 + 6x + 11x^2 + 6x^3$$

$$r(C_2, x) = x + 4x^2 + 3x^3 + 1 + 6x + 11x^2 + 6x^3$$
$$= 1 + 7x + 15x^2 + 9x^3$$

$$r(C, x) = x + 6x^2 + 11x^3 + 6x^4 + 1 + 7x + 15x^2 + 9x^3$$
$$= 1 + 8x + 21x^2 + 20x^3 + 6x^4.$$

Thus the number of injective maps submitted to the conditions is:

$$6! - 8 \cdot 5! + 21 \cdot 4! - 20 \cdot 3! + 6 \cdot 2!$$

$$= 156.$$