Ex 9.2.8: By the binomial formula:

$$(1+x+x^2)(1+x^n) = (1+x+x^2)\sum_{k=0}^{n}\binom{n}{k}x^k$$

$$= 1 + \binom{n}{1}x + \binom{n}{0}x + \left(\binom{n}{2}+\binom{n}{1}+\binom{n}{0}\right)x^2$$

$$+ \cdots \left(\binom{n}{k}+\binom{n}{k+1}+\binom{n}{k+2}\right)x^{k+2} +$$

$$\left(\binom{n}{n}+\binom{n}{n-1}\right)x^{n+1} + \binom{n}{n}x^{n+2}$$

$$= \sum_{k=-2}^{n}\left(\binom{n}{k}+\binom{n}{k+1}+\binom{n}{k+2}\right)x^{k+2}$$

a) If $n<5$ then 0.
If $n=5$ then 1.
If $n=6$ then 7.
If $n\geq 7$ then $\binom{n}{5}+\binom{n}{6}+\binom{n}{7}$.

b) If $n<6$ then 0.
If $n=6$ then 1.
If $n=7$ then 8.

If $n \geq 8$ then $\binom{n}{6} + \binom{n}{7} + \binom{n}{8}$.

c) Check the formula.

Exg. 2.10: a) The generating function for one line is:

$$x^3 \frac{1}{1-x}$$

For four lines we get:

$$\left(\frac{x^3}{1-x}\right)^4 = \frac{x^{12}}{(1-x)^4}.$$

The coefficient of x^{24} is given the coefficient for x^{12} in $\frac{1}{(1-x)^4}$ which is $\binom{15}{3}$.

b) The generating function for one line $x^3 + x^4 + \dots + x^9$.

For four lines we get:

$$(x^3+x^4+\dots+x^9)^4 = x^{12}(1+x+\dots+x^6)^4$$
$$= x^{12}\left(\frac{1-x^7}{1-x}\right)^4$$
$$= x^{12}\frac{(1-4x^7+6x^{14}-4x^{21}+x^{28})}{(1-x)^4}$$

We get $\binom{15}{3} - 4\cdot\binom{8}{3}$.

Ex 3.2.16: The generating functions for hamburgers are:

Jane: $\frac{x}{1-x}$, Others: $\frac{x^2}{1-x}$

The generating function for the products

$$\frac{x^7}{(1-x)^4}$$

The coefficient in x^{12} is: $\binom{8}{3}$.

The generating function for hot dogs is:

James: $\dfrac{x^3}{1-x}$ others: $1+x+x^2+x^3+x^4+x^5 = \dfrac{1-x^6}{1-x}$

The product is given by:

$$\frac{x^3(1-x^6)^3}{(1-x)^4} = \frac{x^3(1-3x^6+3x^{12}-x^{18})}{(1-x)^4}$$

$$\binom{n+r-1}{r-1}$$

$$= 3\binom{4}{3} - 3\binom{10}{3} + \binom{16}{3}.$$

The end result is: $\binom{6}{3}\left(3\binom{4}{3}-3\binom{10}{3}+\binom{16}{3}\right)$

Ex 9.3.2: The generating function for the number of partition of integers is

$$\prod_{i\geq 1} \sum_{k\geq 0} x^{ik}$$

a) For even summands we get:

$$\prod_{\substack{i\geq 1\\ \text{even}}} \sum_{k\geq 0} x^{ik} = \prod_{i\geq 1} \frac{1}{1-x^{2i}}$$

b) $$\prod_{\substack{i\geq 1\\ \text{even}}} (1+x^i)$$

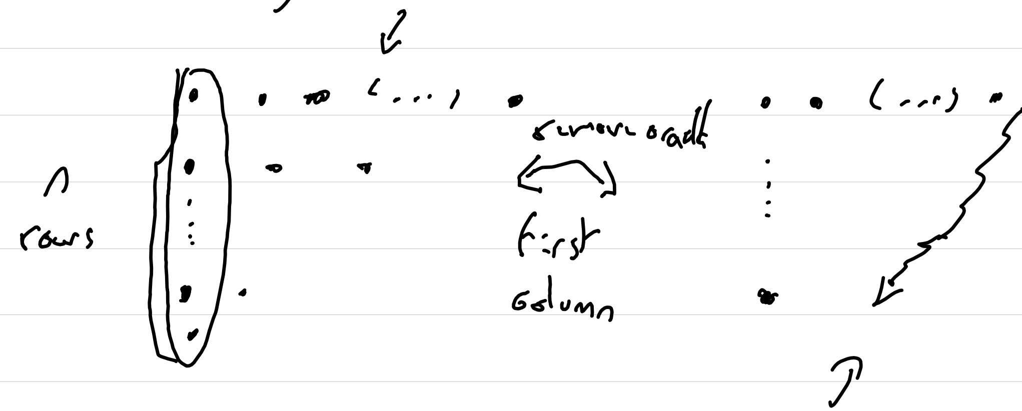
c) $$\prod_{\substack{i\geq 1\\ \text{odd}}} (1+x^i)$$

Ex 9.3.3: There is 7 ways to write 6 as a sum of 1's 2's and 3's.

Ex 9.3.5: a) $\prod_{i\geq 1} \left(\frac{1}{1-x^{2i}}\right)$

b) $\prod_{i\geq 1} \left(\frac{1}{1-x^{2i}}\right)$

Ex 9.3.10 : A Ferrers graph for $2n$ with n summand is of the following form:



Ferrers graph for partition of n.

Ex 9.2.30 : The generating function we obtain (with interpretation) of the condition is:

$$\frac{1}{(1-x)^2} \cdot \left(\frac{x^2}{1-x}\right)^6 = \frac{x^{12}}{(1-x)^8}$$

The number we are looking for is the coeff. in x^{40} which is $\binom{44}{7}$.

In order to compute this generating function we follow example 3.17(a) of the book.

There is a bijection between subsets of 7 non-consecutive integers in $\{1, 2, \dots, 50\}$ and integer solutions to

$$(*) \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 49$$

subject to the conditions $x_1, x_8 \geq 0$ and $x_2, \dots, x_7 \geq 2$.

This bijection is done in the following way:

$$S = \{n_1, \dots, n_7\} \longmapsto n_1 - 1, n_2 - n_1, \dots, 50 - n_7$$

$$S = \{1 + x_1, 1 + x_1 + x_2, \dots, 1 + x_1 + \dots + x_7\} \longleftarrow x_1, \dots, x_8$$

The generating function to the solutions to $(*)$ is the product of generating function for each x_i subject to its conditions:

$$x_1, x_8 : \frac{1}{1-x} \qquad x_2, \dots, x_7 : \frac{x^2}{1-x}.$$