Mm5023 lecture 6

Recursion I

Plan

- Linear first order recursion
- Second order recursion:

Strategy

The homogeneous problem

Example Find a non recursive formula for the n-th term of the following sequence

$a_1 = -1 \qquad a_2 = 2$

$$a_{n+1} = 3a_n - 2a_{n-1}$$

- ~~Guess a formula~~
- Prove it by induction

First order relations

a_{n+1} depends just of a_n

linear in a_n

A first order linear homogeneous recurrence relation with constant coefficient is a relation of the form

does not depend of n.

$$a_{n+1} = \alpha a_n \quad + f(n)$$

non homogeneous

a_{n+1} depends only of a_n

α does not depend from n.

a_{n+1} is a linear function of a_n

Example

$a_{n+1} = 3a_n + 4$ linear non homogeneous first order ct coefficient

Example

$a_{n+1} = na_n + \sqrt{n}$ linear 1st order non-homo.

the coeff are not ct

Example $a_{n+1} = 3a_n$ 1st linear hom ct coefficient

Boundary Conditions

$a_0 = C$ initial condition

$a_k = C'$ boundary condition

Prop: The geometric Progression $a_n = Ad^n$ is the only solution of the recurrence relation

$a_{n+1} = d a_n$ with initial condition $A = a_0$

Proof

(!) a_n and b_n are two solutions

if $b_n = 0$ $\rightarrow$ $d = 0$ $\Rightarrow$ $b_n = 0$ for every n, $a_n = 0$ for every n

$\rightarrow$ $b_{n-1} = 0$

if $d \neq 0$ $\Rightarrow$ b_n for every k

$\rightarrow$ for every $k \geq n$ induction

$\rightarrow$ $b_{n-1} = 0$ $\Rightarrow$ $b_{n-2} = 0$ ... $b_0 = 0 = A$

$a_1 = 0$ $a = d \cdot a_0$ by induction $a_n = 0$ $\forall n$

$b_n = a_n$

We can assume $b_n \neq 0 \implies d \neq 0$

$$\frac{a_n}{b_n} = \frac{\cancel{d}\, a_{n-1}}{\cancel{d}\, b_{n-1}} = \frac{\cancel{d}\, a_{n-2}}{\cancel{d}\, a_{n-2}} = \quad = \frac{a_0}{b_0} = \frac{A}{A} = 1$$

$a_n = b_n$ concluded the proof for unicity.

$a_n = A \cdot d^n \qquad a_0 = A$ ✓

$a_{n+1} = A d^{n+1} = d \cdot A d^n = d\, a_n$ the one provided is a solution #

Examples

$$\begin{cases} a_{n+1} = 7a_n \\ a_2 = 98 \end{cases}$$

$a_n = A \cdot 7^n$ We have to find. A

$98 = a_2 = A \cdot 7^2 = A \cdot 49$ $A = \frac{98}{49} = 2$

$$\boxed{a_n = 2 \cdot 7^n}$$

" C(n)

- a_n = # Combinations of n $x_1 + \ldots + x_k = n$ order counts
  $a_1 = 1$ of 2^{n-1}

if you have a combination of n then there are 2 cases

① the last summand is 1

combination of n−1 + 1

c(n−1)

② the last summand is not 1

combination of n−1 you add 1 to the last summand

C(n−1)

RULE OF SUM

$$C(n) = C(n-1) + C(n-1)$$
$$= 2\,C(n-1)$$

$$C(n) = C(0) \cdot 2^n$$

$$1 = C(1) = C(0) \cdot 2 \qquad C(0) = \frac{1}{2}$$

$$\boxed{C(n) = 2^{n-1}}$$

Non Homogeneous Example

$(x_1 \dots x_n)$ to be sorted. How many comparisons?

$x_n \; x_{n-1}$ if $x_n > x_{n-1}$ do nothing
otherwise swap.

$x_{n-1} \; x_{n-2}$ if $x_{n-1} > x_{n-2}$ do nothing
otherwise swap

$n-1 \implies x_1$ is the smallest element

Run again $(x_2 - x_n) \to x_2$ smalles

Q: How many comparison?

$a_n = a_{n-1} + n-1$ You cannot use the theorem to solve

$a_1 = 0$

6.2 Linear recurrence relations with ct coeff.

→ linear in the a_i

c_i are ct.

A linear recurrence relation with ct coeff. is a relation of the form

$a_{n+1} - d a_n = 0$

$$(*) \quad c_k a_{n+k} + c_{k-1} a_{n+k-1} + \dots + c_0 a_n = f(n)$$

with $c_i \in \mathbb{R}$ f a function $f: \mathbb{N} \to \mathbb{R}$

$c_i \in \mathbb{C}$ $\to \mathbb{C}$

The order of the relation is k

If $f \equiv 0$ then we say that the relation is homogeneous

Boundary / Initial conditions

$a_0 = \alpha_0$

$a_2 = \alpha_2$

$a_5 = \alpha_5$

$\vdots$

$a_0 = \alpha_0$

$a_1 = \alpha_1$

$\vdots$

$a_{k-1} = \alpha_{k-1}$

k fixed value

$\Rightarrow$ k initial / boundary condition for a recurrence of order k

$\Downarrow$

there is just one solution.

Goal: Give a non rec formula for a_n

Lemma Given a recurrence relation like (*) and a solution $(a_n^{(p)})_{n\in\mathbb{N}}$ then any solution can be written as

$$a_n = a_n^{(h)} + a_n^{(p)}$$

p is for particular

Where $a_n^{(h)}$ is a solution of the homogeneous relation.

$$(*) : c_k a_{n+k} + c_{k-1} a_{n+k-1} + \quad + c_0 a_n = f(n)$$

Proof

Suppose that b_n is a solution of the recursion relation

$$(*) : C_k a_{n+k} + C_{k-1} a_{n+k-1} + \quad + C_0 a_n = f(n)$$

$a_n^{(h)} := b_n - a_n^{(p)}$ is a solution of the corresponding homogeneous relation

$$(H): C_k a_{n+k} + C_{k-1} a_{n+k-1} + \quad + C_0 a_n = 0$$

$$C_k \left(b_{n+k} - a_{n+k}^{(p)}\right) + \quad + C_0 \left(b_n - a_n^{(p)}\right) = f(n) - f(n)$$
$$= 0$$

$$b_n = a_n^{(h)} + a_n^{(p)} \quad \#$$

Strategy

1) Find a (particular) solution of the non homo problem (Next time)

2) Find the general solution for the homo problem (TODAY)

1) + 2) all the solution regardless of the boundary values

3) If needed impose boundary values (Today / Next time)
(H) (NH)

↳ find the unique solution for the recursion with boundary

The general solution

The solutions of the homogeneous problem form a vector space. (Exercise)

(H) . $c_k a_{n+k} + c_{k-1} a_{n+k-1} + \quad + c_0 a_n = 0$.

What is the dimension?

$$\mathbb{C}^k \longrightarrow \text{Solution}$$

$(x_0 \ldots x_{k-1}) \longrightarrow$ Solution of the recursion with boundary condition $a_i = x_i$

this is linear and injective.

injectivity for $k=2$

$b_n = a_n$

$b_0 = x_0$ $\quad a_0 = y_0$ $\quad \Rightarrow \quad x_0 = y_0$

$b_1 = x_1$ $\quad a_1 = y_1$ $\quad\quad y_1 = y_1$

dim Solution $\geq k$

the map is surjective (show that a solution exist)

dim Solution $= k$.

Suppose that we find k linearly ind soluto

$a_n^1 \quad \dots \quad a_n^k \quad \Rightarrow$ basis

that any solution can be expressed in a unique way

as a linear combination

$A_1 a_n^1 + \qquad + A_k a_n^k$ → the general solution of the hom prob

of the a_n^i's

To find the general solution:

$\Longleftrightarrow$ to find k linearly indep solutions

6.3 Homogeneous relations (order 2)

(H) $c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_0 a_n = 0$

find the general sol.

Def The characteristic equation of the relation is

$$c_k \lambda^k + c_{k-1} \lambda^{k-1} + \quad + c_1 \lambda + c_0 = 0$$

$$a_{n+j} \longleftrightarrow \lambda^j$$

Example (order 1)

$a_{n+1} = \alpha a_n$

$$a_{n+1} - \alpha a_n = 0$$

$$\lambda - \alpha = 0$$

$$\boxed{\lambda = \alpha}$$

Example

$a_{n+2} = a_{n+1} + a_n$

$$a_{n+2} - a_{n+1} - a_n = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

Prop: If $a_n = Ar^n$ is a solution of a recursive relation for any A

$$\sum_1 c_k a_{n+k} = 0$$

then r is a root of the characteristic equation.

If r is a root of the char equation

$a_n = r^n$ is a solution

How many possible r are there

the char equation has deg k

$\leq k$ distinct solution ($r \in \mathbb{C}$)

if Exactly k (no sol with higher mult)

$r_1^n \; \dots \; r_k^n$ are candidate for a basis.

$\sum A_i r_i^n \quad A_i \in \mathbb{C}$

Proof Prop

$$\sum_{j=0}^{k} c_j \cdot a_{n+j} = 0 \qquad a_{n+k} = A r^{n+k}$$

$$\sum_{j=0}^{K} c_j A r^{n+j} = 0 \qquad \text{for every } n$$

$$\| $$

$$A r^n \sum_{j=0}^{n} c_j r^j = 0$$

$n = 0$

$A \neq 0$ → [illegible]

$$\sum_{j=0} c_j r^j = 0$$

r is a solution of the [illegible] equation

If r is a root with mult j

$n^i r^n$ is a solution

for every $i = 0 \ldots j-1$

$\longrightarrow$ j solution for r

they are linearly independent !

After the break

proof of Thm.

particular case $k = 2$.

Theorem Given an order 2 homogeneous linear recurrence relation with constant coefficie[nts] and char equation

(*) $c_2\lambda^2 + c_1\lambda + c_0 = 0$ $\left(c_2 a_{n+2} + c_1 a_{n+1} + c_0 a_n = 0\right)$

Then there are 3 cases

① (*) has two distinct real roots r_1 and r_2 in this case we can write the general solution

Order 2 relation (now with ct coeff) $c_i \in \mathbb{R}$

the char equation

$$c_2 \lambda^2 + c_1 \lambda + c_0 = 0$$ deg 2

- 2 distinct roots
  - 2 real roots
  - 2 complex roots z_1, z_2

    $$z_1 = \overline{z_2}$$
- a single root with mult 2

  the general solution

  $$A_1 r^n + A_2 n r^n$$

General solution

$$: A_1 r_1^n + A_2 r_2^n$$

two complex solution De Moivre formulas for computing

$$A_1 z_1^n + A_2 z_2^n$$

$$z_1 = \mathrm{Re}(z_1) + i\,\mathrm{Im}(z_1)$$

$$z_2 = \mathrm{Re}(z_1) - i\,\mathrm{Im}(z_2)$$

$$\alpha = \arctan\left(\frac{\mathrm{Re}(z_1)}{\mathrm{Im}(z_1)}\right) \qquad \alpha = \arg(z_1)$$

$$\therefore \rho = \sqrt{\mathrm{Re}(z_1)^2 + \mathrm{Im}(z_1)^2}$$

$$z_1 = \rho\,(\cos\alpha + i\sin\alpha)$$

$$z_2 = \rho\,(\cos\alpha - i\sin\alpha)$$

$$z_1^n = \rho^n\,(\cos n\alpha + i\sin n\alpha)$$

$$z_2^n = \rho^n\,(\cos n\alpha - i\sin n\alpha)$$

Rewrite $A_1 z_1^n + A_2 z_2^n = \rho^n\left(A_1'\cos(n\alpha) + A_2'\sin(n\alpha)\right)$

to the recurrence relation as

$$a_n = A_1 r_1^n + A_2 r_2^n \quad \text{for } A_i \in \mathbb{R}.$$

② (*) can have one double real root r

In this case the general solution of the recurrence relation is

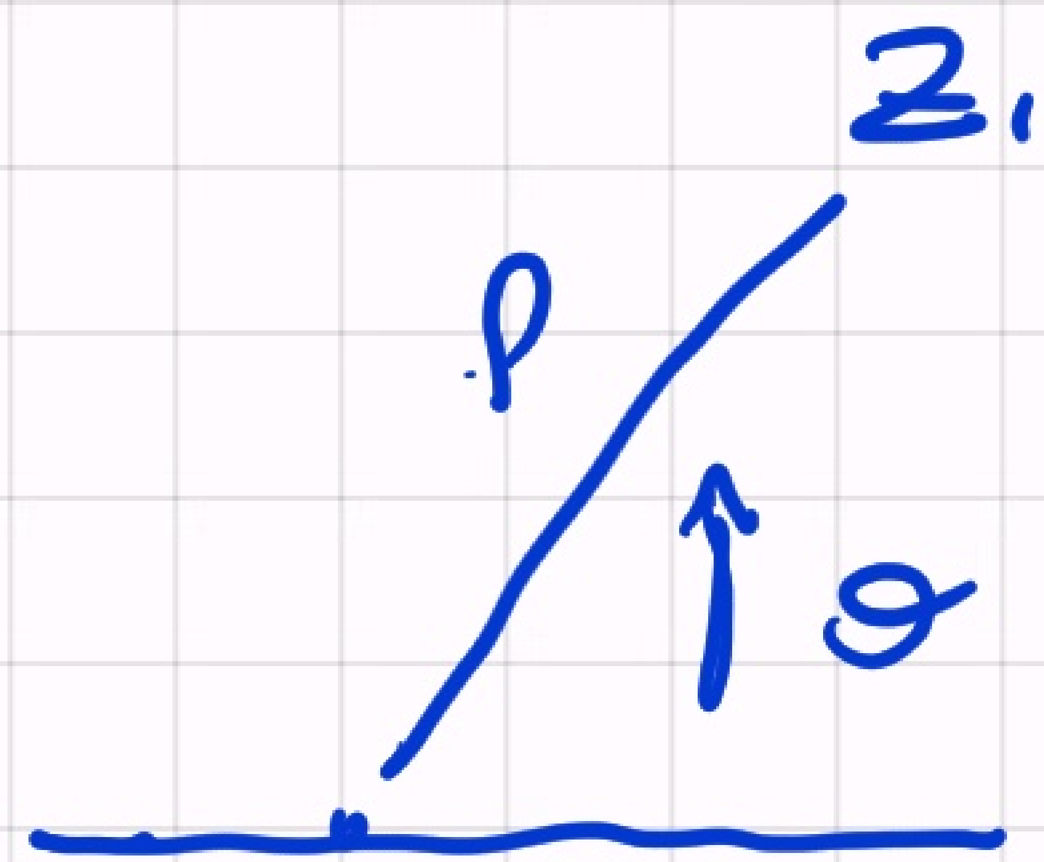
$$a_n = A_1 r^n + n A_2 r^n = r^n (A_1 + n A_2)$$

for $A_i \in \mathbb{R}$.

③ (★) has two complex roots z_1 and z_2.

Observe that $z_1 = \overline{z_2}$

We can write



$z_1 = \rho \cos\theta + i \rho \sin\theta$

for some $\theta \in \mathbb{R}$ $\rho \in \mathbb{R}^+$

(unique if $\theta \in [0, 2\pi)$)

$z_2 = \rho \sin\theta - i \rho \sin\theta$

The general solution for the recurrence relation is

$$a_n = \rho^n \left(A_1 \cos n\theta + A_1 \sin n\theta\right)$$

for A_1 and $A_2 \in \mathbb{R}$.

You can merge 1 and 3 by using the formula for powers of complex number.

$$A_1 z_1^n + A_2 z_2^n = A_1 \rho^n \underbrace{(\cos n\theta + i \sin(n\theta))}_{}\ \ e^{i\theta}$$

$$+ A_2 \rho^n (\cos n\theta - i \sin n\theta) e^{i -\theta}$$

change of variable & trig = you got the same result.

Example (Fibonacci)

Closed formula for the Fibonacci numbers

$F_0 = 0 \qquad F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$

$F_{n+2} = F_{n+1} + F_n$

The characteristic equation is

$\lambda^2 - \lambda - 1 = 0$

the roots are $\lambda_\pm = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1-\sqrt{5}}{2}, \ \frac{1+\sqrt{5}}{2}$

The general solution is

$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

We have to impose initial conditions

$$\begin{cases} 1 = a_0 = A + B \\ 1 = a_1 = A \cdot \left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \end{cases}$$

$$A = 1 - B$$

$$1 = (1-B)\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = \frac{1+\sqrt{5}}{2} - B\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$1 - \frac{1+\sqrt{5}}{2} = -B\left(\frac{2\sqrt{5}}{2}\right)$$

$$B = \left(\frac{1+\sqrt{5}-2}{2}\right)\frac{1}{\sqrt{5}} = \left(\frac{\sqrt{5}-1}{2}\right)\frac{1}{\sqrt{5}} = \frac{5-\sqrt{5}}{10}$$

$$A = 1 - B = 1 - \left(\frac{\sqrt{5}-1}{2\sqrt{5}} \right) = \frac{2\sqrt{5} - \sqrt{5} + 1}{2\sqrt{5}}$$

$$= \frac{\sqrt{5}+1}{2\sqrt{5}} = \frac{5 + \sqrt{5}}{10}$$

Formula for the Fibonacci number is

$$\frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n} + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)$$

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}$$

$F_0 = 1$ $F_1 = 1$ F_n F_{n+1}

$$\begin{cases} 0 = A + B \\ 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \end{cases}$$

$$A = -B$$

$$1 = B\left(\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right)$$

$$1 = B\left(-\sqrt{5}\right)$$

$$B = -\frac{1}{\sqrt{5}} \qquad A = \frac{1}{\sqrt{5}}$$

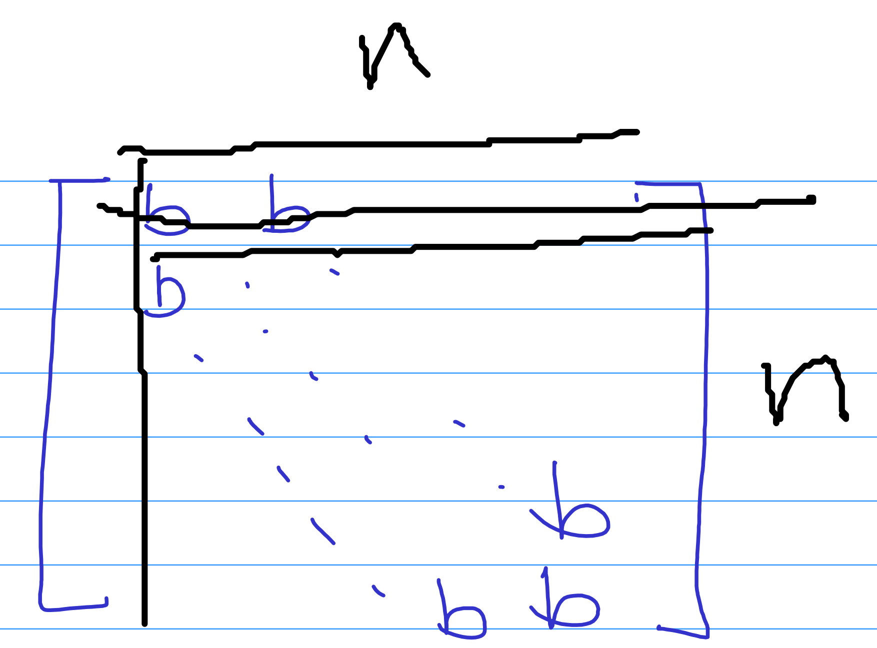
$$\underline{F_n} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

they model growth

↑ the golden section is behind Fibonacci

Example

det



$n = 1$ $\quad (b) \quad$ $a_1 = b$

$n = 2$ $\quad \begin{pmatrix} b & b \\ b & b \end{pmatrix} \quad$ $a_2 = 0$

recursive formula

$$a_{n+1} = b \det \begin{pmatrix} b & b & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & b & b \end{pmatrix} - b \det \begin{pmatrix} b & b & & & \\ 0 & b & b & b & \\ & b & & & \\ \vdots & & & & b \\ 0 & 0 & & b & b \end{pmatrix}$$

$$= b\, a_n - b^2 a_{n-1}.$$

$$a_{n+2} - b\, a_{n+1} + b^2 a_n = 0$$

$$\lambda^2 - b\lambda + b^2 = 0$$

$$\lambda_\pm = \frac{b \pm \sqrt{b^2 - 4b^2}}{2} = \frac{b + i\sqrt{3}\, b}{2}, \quad \frac{b - i\sqrt{3}\, b}{2}$$

$$z_1 = b\left(\frac{1 + i\sqrt{3}}{2}\right) = b\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z_2 = b\left(\cos\frac{\pi}{3} - i\sin\pi/3\right)$$

the general solution

$$b^n\left(A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right)$$

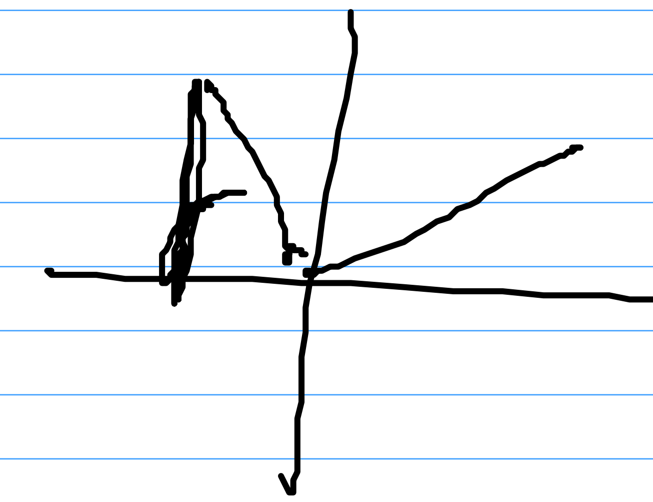
$b = 0$

$\longrightarrow$ easy

$b \neq 0$

$$\begin{cases} 1 = A\cos\frac{\pi}{3} + B\sin\frac{\pi}{3} \\ 0 = b^2\left(A\cos\frac{2\pi}{3} + B\sin\frac{2\pi}{3}\right) \end{cases}$$

$$\begin{cases} -\frac{1}{2}A + B\frac{\sqrt{3}}{2} = 0 \\ \frac{1}{2}A + B\frac{\sqrt{3}}{2} = 1 \end{cases}$$



$$\begin{cases} -\frac{1}{2}A + B\frac{\sqrt{3}}{2} = 0 \\ 0A + B\sqrt{3} = 1 \end{cases}$$

$$B = \frac{1}{\sqrt{3}}$$

$$-\frac{1}{2}A + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} = 0$$

$$-\frac{1}{2}A = -\frac{1}{2} \qquad A = 1$$

The solution $\ldots \; b^n\left(1 \cdot \cos\frac{n\pi}{3} + \frac{\sqrt{3}}{3}\sin\frac{n\pi}{3}\right)$

Next time find $a_n^{(p)}$

The general ~~solution of the~~ non hom probl.

$a_n^{(p)}$ + (general solution of the hom)

↓

depends on k parameters

Solution: find the values of the parameters that satisfy the initial conditions