Ex 10.1.2 a) $a_{n+1} - \frac{3}{2} a_n = 0 \quad , n \geq 0$

$$a_n = a_0 \left(\frac{3}{2}\right)^n$$

b) $4a_n - 5a_{n-1} = 0 \quad , n \geq 1$

$$a_n = \left(5/4\right)^n \cdot c$$

c) $3a_{n+1} - 4a_n = 0 \quad , n \geq 0$

$a_2 = 5$

$$a_n = a_0 \left(4/3\right)^n$$

and $a_1 = 5$ gives

$$5 = a_0 \cdot 4/3$$

So $a_0 = \frac{15}{4}$.

d) $2a_n - 3a_{n-1} = 0 \quad , n \geq 1$

$a_4 = 81$

$$a_n = c\left(3/2\right)^n$$

With $a_4 = 81$ we get

$$c \cdot \left(\frac{3}{2}\right)^4 = 81$$

$$c = 16.$$

Ex 10.2.4: Here, there is only one way to end a valid sequence of length $n-1$ to one of length n. For a sequence of length $n-2$ there is also one way to get a valid sequence of length n. We get

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1$$
$$a_2 = 2$$

The equation is $r^2 - r - 1 = 0$ which has roots $\frac{(1 \pm \sqrt{5})}{2}$, the general solution

is $A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$.

$a_2 = 2$

Since $a_1 = 1$, we have:

$$\begin{cases} A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1+\sqrt{5}}{2}\right) = 1 \\ A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1+\sqrt{5}}{2}\right)^2 = 2 \end{cases}$$

So we get $A = \frac{\sqrt{5}}{10} + \frac{1}{2}$, $B = -\frac{\sqrt{5}}{10} + \frac{1}{2}$

valid

<u>Ex 10.2.12</u>: Starting with a stack of length $n-1$ there is 3 ways to get a valid stack of length n by adding a chip that is not blue. This contributes $3a_{n-1}$.

If we have a stack of size $n-2$, we also get 3 possibilities to get a stack of size n that ends with a blue chip.

We get:

$$a_n = 3a_{n-1} + 3a_{n-2}.$$

The characteristic equation is

$r^2 - 3r - 3$, its roots are

$$\frac{3}{2} \pm \frac{\sqrt{21}}{2}.$$

The general solution is

$$A\left(\frac{3}{2} + \frac{\sqrt{21}}{2}\right)^n + B\left(\frac{3}{2} - \frac{\sqrt{21}}{2}\right)^n.$$

We have $a_1 = 4$, $a_2 = 15$ so

we can compute $A = 6 + \frac{10}{7}\sqrt{21}$

$B = 6 - \frac{10}{7}\sqrt{21}$.

<u>Ex 10.2.1h</u>: We have $63 = 7h$ so $h = 9$.

<u>Ex 10.2.2h</u>: There is only one way to end a valid $2 \times n-1$ chessboard. There are two ways to end a valid $2 \times n-2$ chessboard so we get: $a_n = a_{n-1} + 2a_{n-2}$.

The characteristic equation

$$r^2 - r - 2 = 0$$

The roots are 2 and -1 so the general solution is

$$A \cdot 2^n + B \cdot (-1)^n.$$

We also have $a_1 = 1$, $a_2 = 3$ thus the system:

$$\begin{cases} 2 \cdot A - B = 1 \\ 4 \cdot A + B = 3 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 6A = 4 \\ 2A - B = 1 \end{cases} \quad \begin{cases} A = \frac{2}{3} \\ B = 2A - 1 \\ \quad = \frac{1}{3}. \end{cases}$$

So $a_n = \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n.$

Ex 10.2.11 : a) There is one way to end a valid string of length $n-1$ by adding a 0. Starting with a string of length $n-2$ there is only one way to end the string with 1 which is by adding 01. We get

$$a_n = a_{n-1} + a_{n-2} .$$

The characteristic polynomial is $r^2 - r - 1 = 0$ with roots $\left(\frac{1 \pm \sqrt{5}}{2}\right)$.

The general solution:

$$A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n$$

and $a_1 = 2$, $a_2 = 3$

b) Starting with a length $n-1$ valid string we get a length n valid string by adding 0. Starting with a length $n-2$ string, if it starts with 1

it ended with 0 so we have to add 10.
If it starts with 0, we add 01.

We get the relation

$$b_n = b_{n-1} + b_{n-2}$$

with conditions $b_1 = 1$, $b_2 = 3$.

We can then compute b_n in the usual way.

Ex 10.2.13: Starting with a string of length $n-1$ we can get a valid string of length n by adding any of the numerical characters, this contributes $4a_{n-1}$. Starting with a string of length $n-2$ and ending with an alphabetic character. We get $4 \times 7 = 28$ possibilities to finish the string of length n. The relation is

$$a_n = 4a_{n-1} + 28a_{n-2},$$

The characteristic equation is

$$r^2 - 4r - 28 = 0$$

The roots are: $\frac{4 \pm \sqrt{16 + 4 \cdot 28}}{2}$

$$= 2 \pm 4\sqrt{2}.$$

The general solution is

$$A(2 + 4\sqrt{2})^n + B(2 - 4\sqrt{2})^n$$

and $a_1 = 11$, $a_2 = 16 + 2 \cdot 4 \cdot 7$
$= 72$.