Notes from today lecture

- Show that if G is simple $|G|=168$ then $G \hookrightarrow A_8$. Compute the number of elements of order 7 of G

$|G| = 168 = 2^3 \cdot 3 \cdot 7$

$n_7 \equiv 1 \mod 7$ $\quad n_7 \mid 2^3 \cdot 3$; If G is simple then $n_7 = 8$

$G \curvearrowright Syl_7(G)$ by conjugation. We take the permutation rep. of G

$\varphi : G \longrightarrow S_8$

Since G is simple $\ker\varphi = id$. G has no subgroup of index 2 $\Rightarrow G \hookrightarrow A_8$

$|\{\text{elements of order } 7\}| = 8 \cdot 6 = 48$

- A group of order 63 has one subgroup of order 21

$63 = 3^2 \cdot 7$ $\quad n_7 \equiv 1 \mod 7$ $\quad n_7 \mid 9$ $\quad \Rightarrow n_7 = 1$

if $n_3 = 1$ then $G \simeq P \times Q$ $\quad |P| = 9$ $\quad |Q| = 7$

let $x \in P$ of order 3 $\quad y \in Q$ of order 7 $\quad xy = yx$ and (xy) has

order 21 $\quad \langle xy \rangle$ is the group we are looking for

$n_3 = 7$ $\quad |G/Q| = 9$ $\quad \exists\, xQ$ in G/Q of order 3

$\pi : G \longrightarrow G/Q$ $\quad \pi^{-1}(\langle xQ \rangle) = H$

$|H| = |\langle xQ \rangle| \cdot |Q| = 21$

. Show that if $|G| = p \cdot q \cdot r$ primes $p > q > r$ $\Rightarrow G$ is not simple

- $n_p \equiv 1 \mod p$ $\quad n_p \mid q \cdot r$ $\quad \Rightarrow n_p = q \cdot r$ $\quad r \geq 2 \quad q \geq 3 \quad p \geq 5$
- We have $(p-1)q \cdot r$ elements of order p
- $n_q \equiv 1 \mod q$ $\quad n_q \mid p \cdot r$ $\quad n_q \geq p$ $\quad$ if not 1
- $n_r \equiv 1 \mod r$ $\cdot\ n_q \mid q \cdot p$ $\quad \Rightarrow n_r \geq q$ $\quad$ if not 1

$|G| \geq (p-1)qr + (q-1)n_q + (r-1)n_r \geq (p-1)qr + (q-1)p + (r-1)q$

$= pqr - \cancel{qr} + qp - p + \cancel{qr} - q$

$= pqr + qp - p - q \geq pqr + 7$

$qp - p - q = p(q-1) - q \geq 5(q-1) - q = 4q - 5 \geq 12 - 5 = 7$

so one of them is 1

Every group of order $3 \cdot 5 \cdot 17$ is abelian

$n_{17} = 1$ $\quad P \in Syl_{17}(G) \Rightarrow P \trianglelefteq G$ $\quad G \curvearrowright P$ by conjugation

$G \longrightarrow Aut(P) \simeq \mathbb{Z}/16\mathbb{Z}$

This is trivial: $x \in G$ has order relatively prime with 2

the order of $\varphi(x)$ is relative prime with 2 but $\varphi(x)$ lies in a 2-group

$\Rightarrow \quad g \in G \quad p \in P \quad gpg^{-1} = p \quad \Rightarrow \quad P \subset Z(G)$

G/P is cyclic $\Rightarrow$ G is abelian.

Classify all groups of order $2^2 \cdot 503$

$502 = 2 \cdot 251$

$P \in Syl_{503}(G)$ then $P \trianglelefteq G \qquad G \curvearrowright P$ by conjugation

$\varphi : G \longrightarrow Aut(P) \cong \mathbb{Z}/502$

$\Rightarrow$ the rest uses semidirect product that we do not know of !!

There is no simple group of order 96

$96 = 2^5 \cdot 3 \qquad n_2 = 3 \qquad$ let $P, Q \in Syl_2(G) \qquad [P : P\cap Q] \overset{\shortparallel}{=} [Q : P\cap Q] = 2$

$P\cap Q \trianglelefteq P \qquad P\cap Q \trianglelefteq Q \quad \Rightarrow \quad N_G(P\cap Q) \supseteq P, \supseteq Q \quad \Rightarrow \quad \supseteq P.Q$

$$|N_G(P\cap Q)| \geq |P.Q| = \frac{2^5 \cdot 2^5}{2^4} = 2^6 \qquad \mid 2^5 \cdot 3$$

$$\Rightarrow N_G(P\cap Q) = G$$