Mm5023 lecture 15

Find two mutually orthogonal latin square in standard form of order 7

$a_{ij} = x_i + j$

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

$a_{ij} = x_i + 2j$

0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3

Max flow min cut.

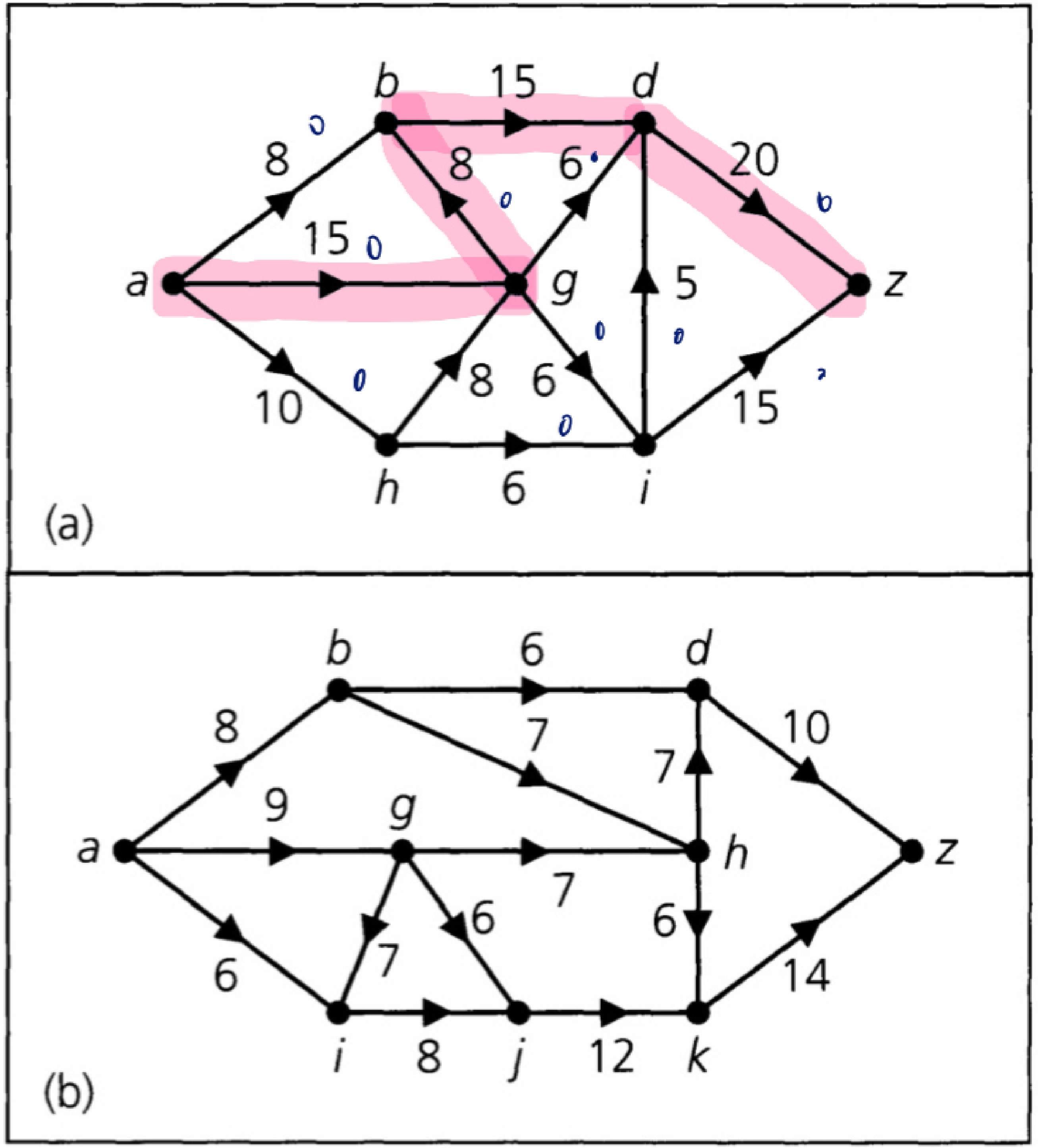


Figure 13.21

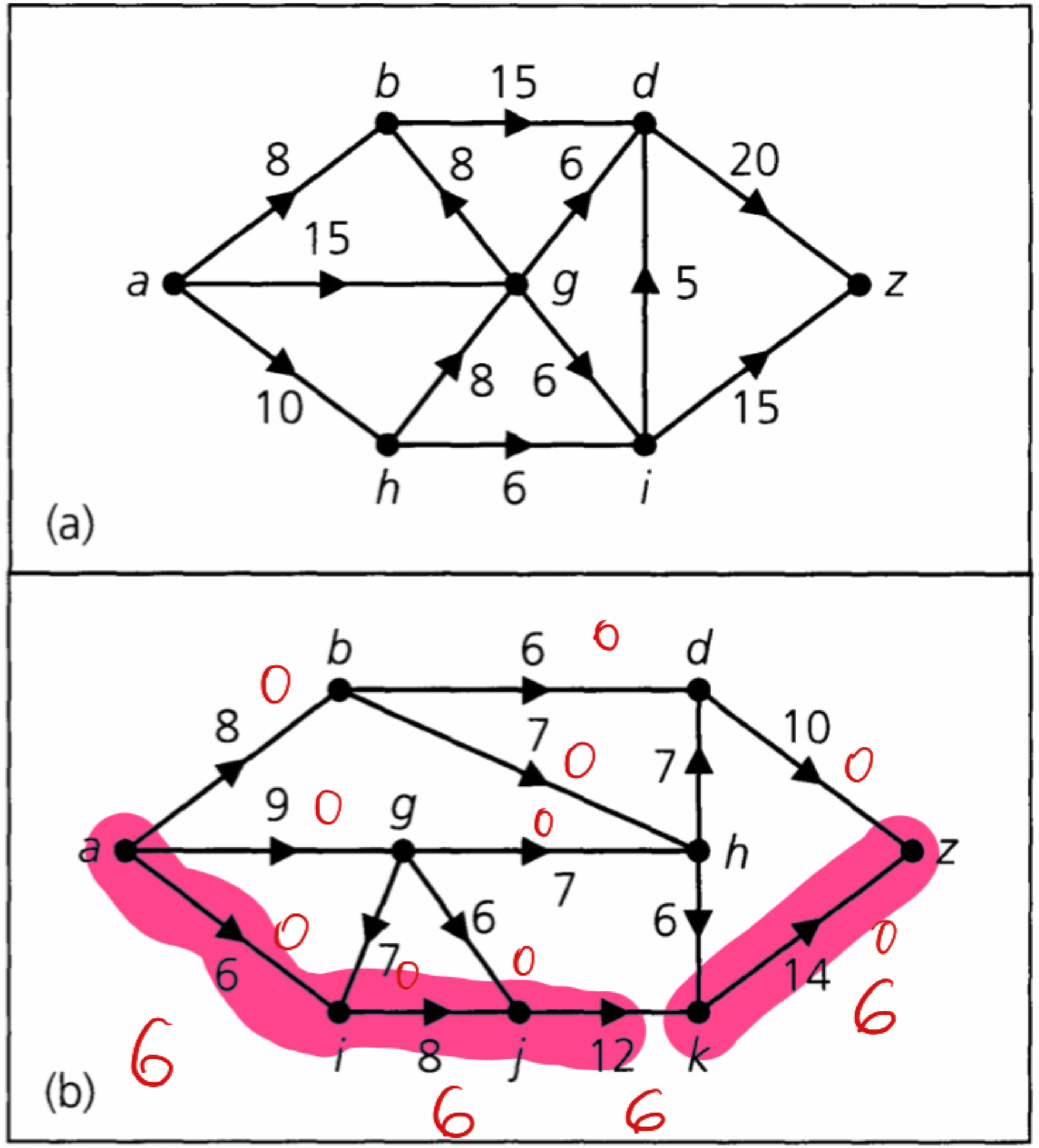
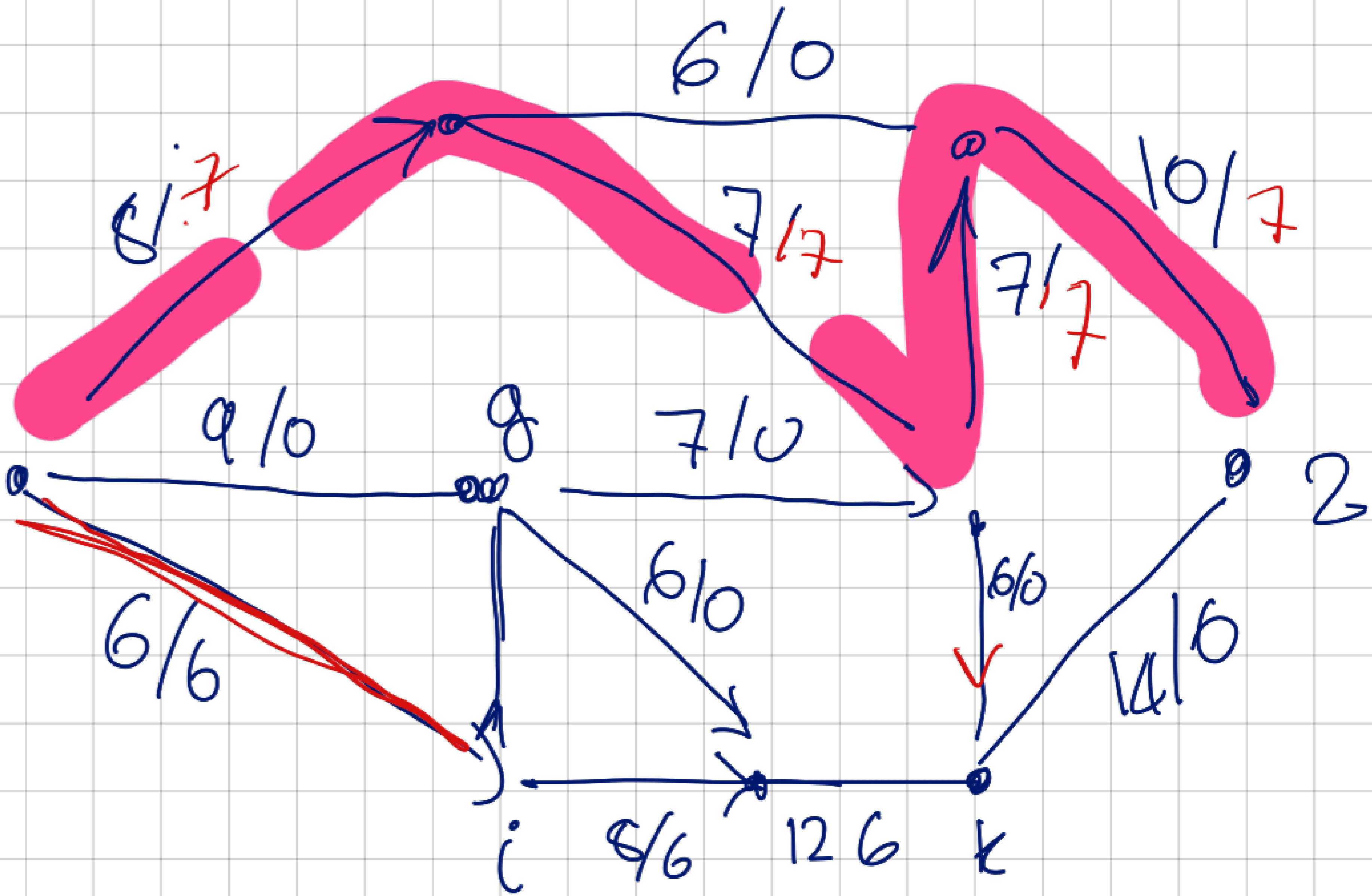
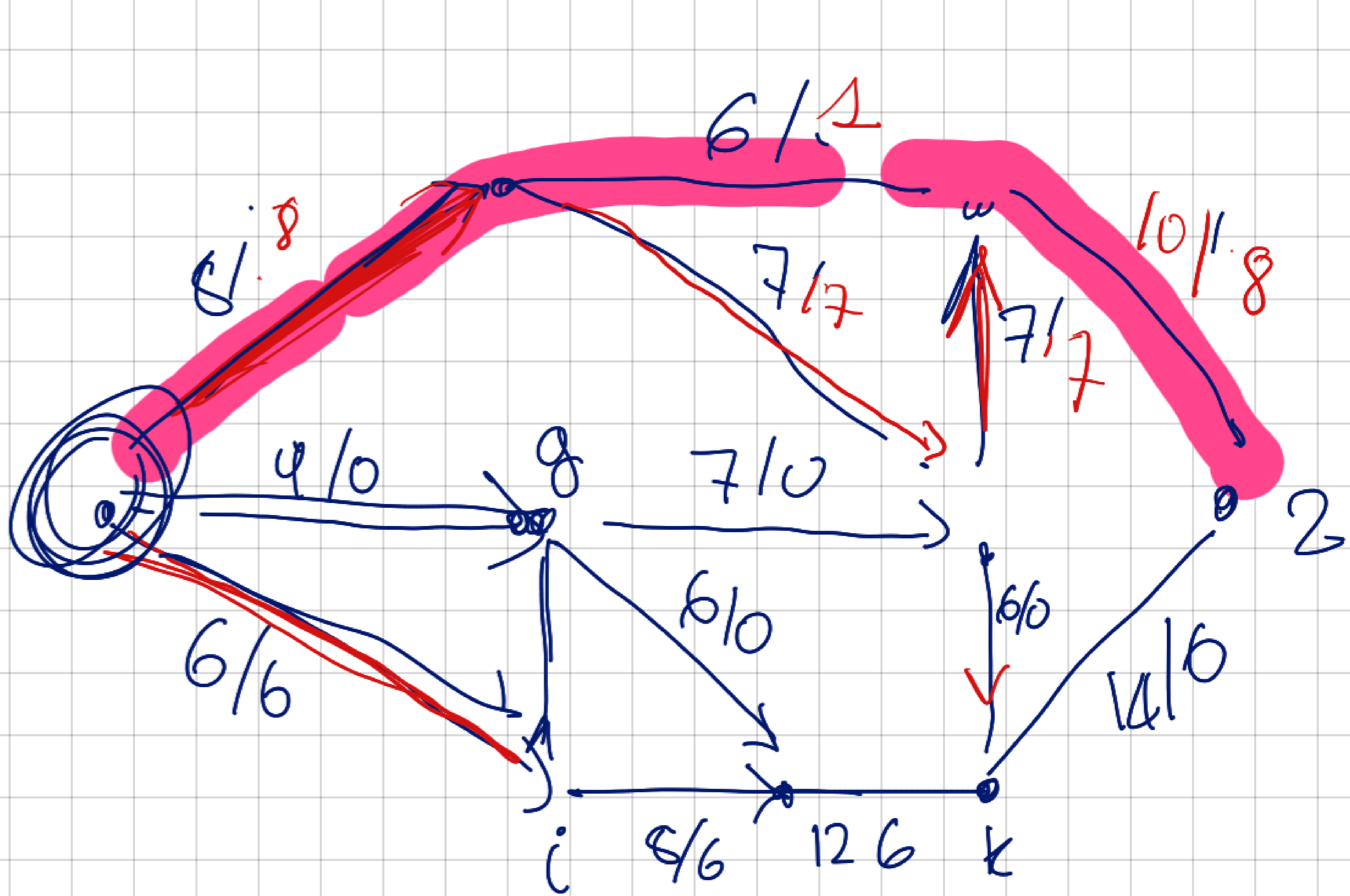


Figure 13.21

$\Delta_p = 6$



$\Delta = 7$



$\Delta = 1$

$P = \{a\}$

$P^c = V \setminus \{a\}$

Once you find a flow without flow-augmenting path there is a minimum cut associated to it

$$P = \{ v \in V \mid \text{there is a path } (a\ v_2 \ \dots \ v_{n-1}, v) \} \ni a$$

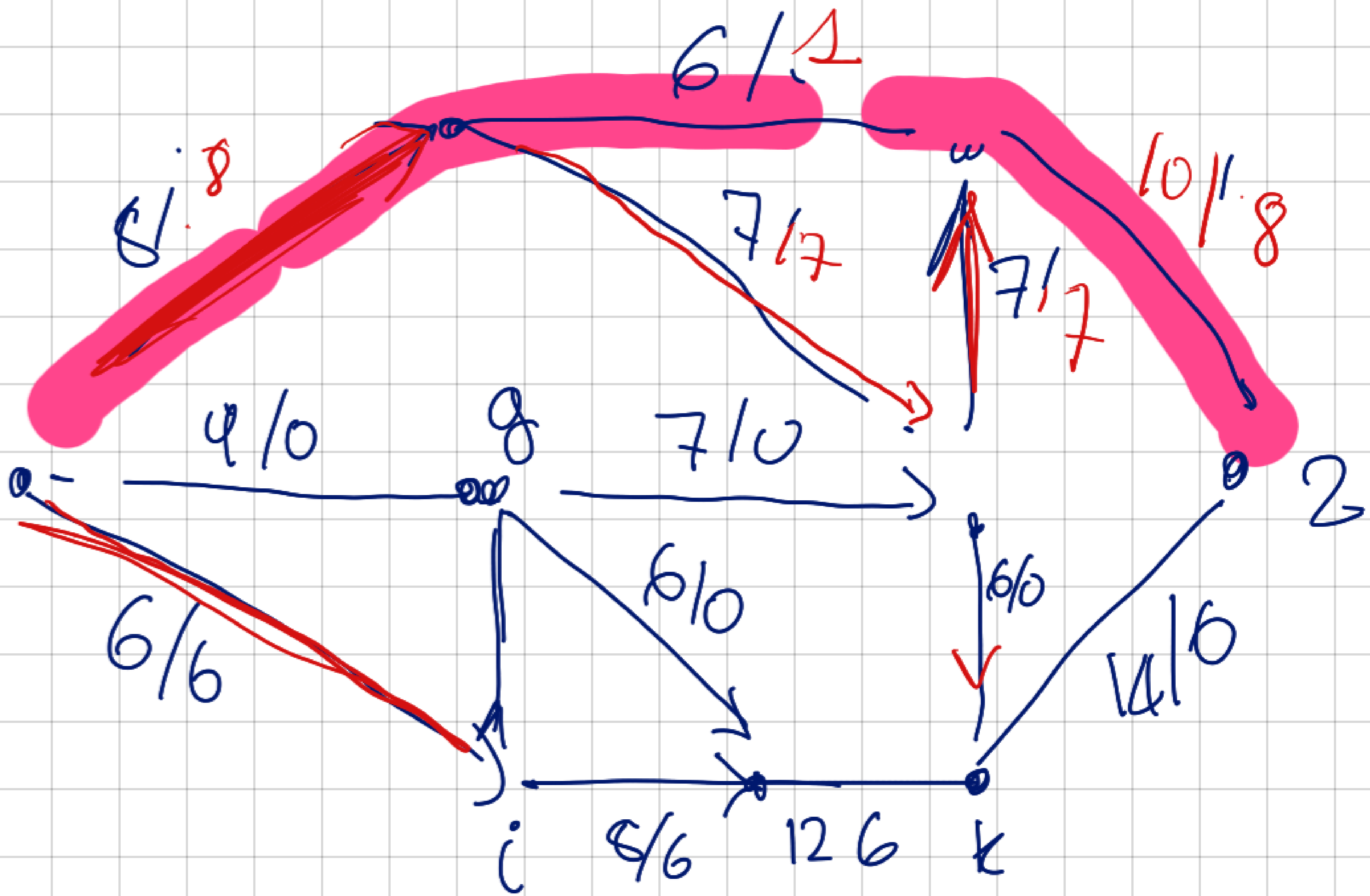
such that for all

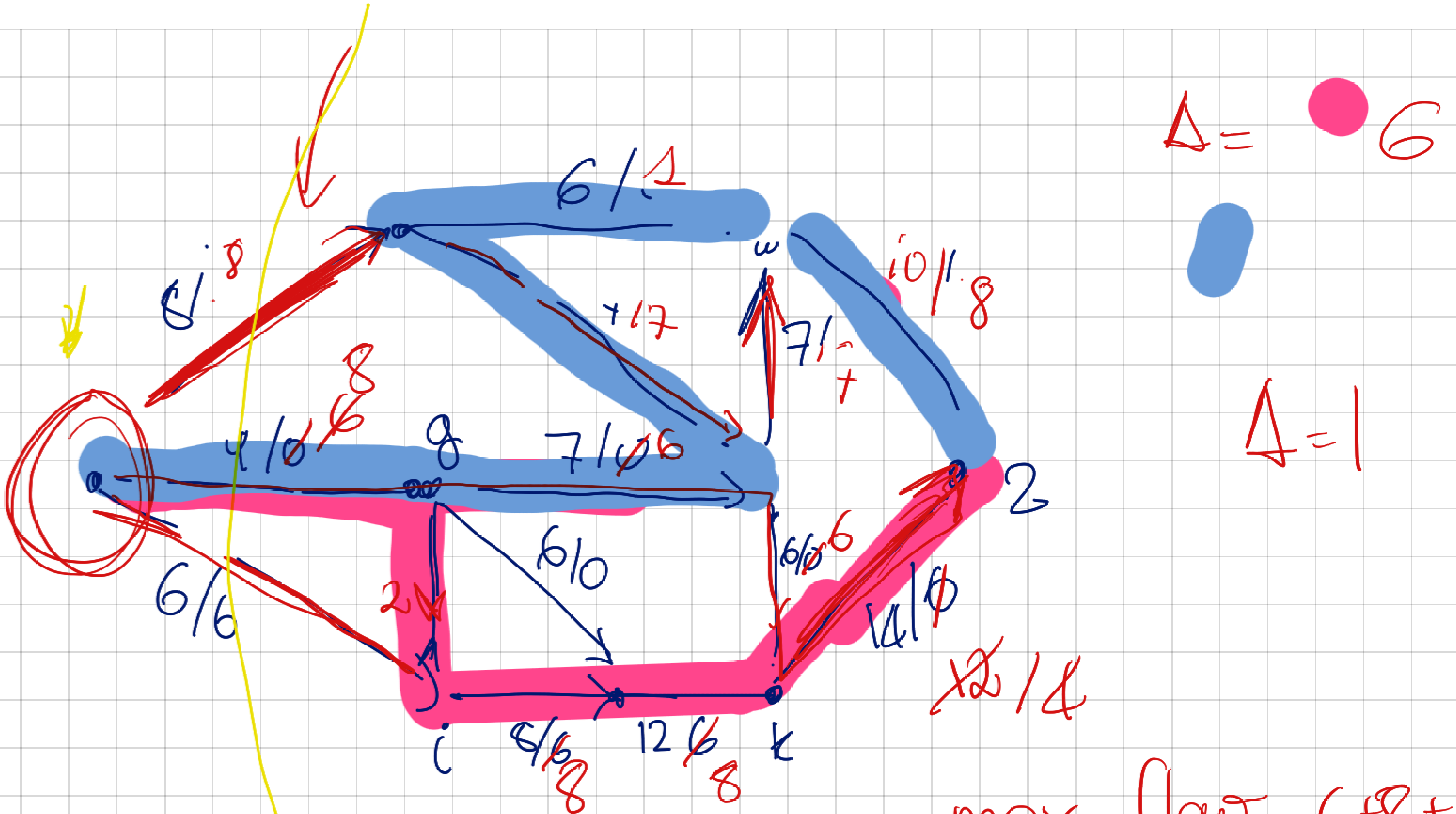
forward edges $c(e) - f(e) > 0$

for all backward edges $f(e) > 0$

if $z \in P$ then I have a flow augmenting chain.

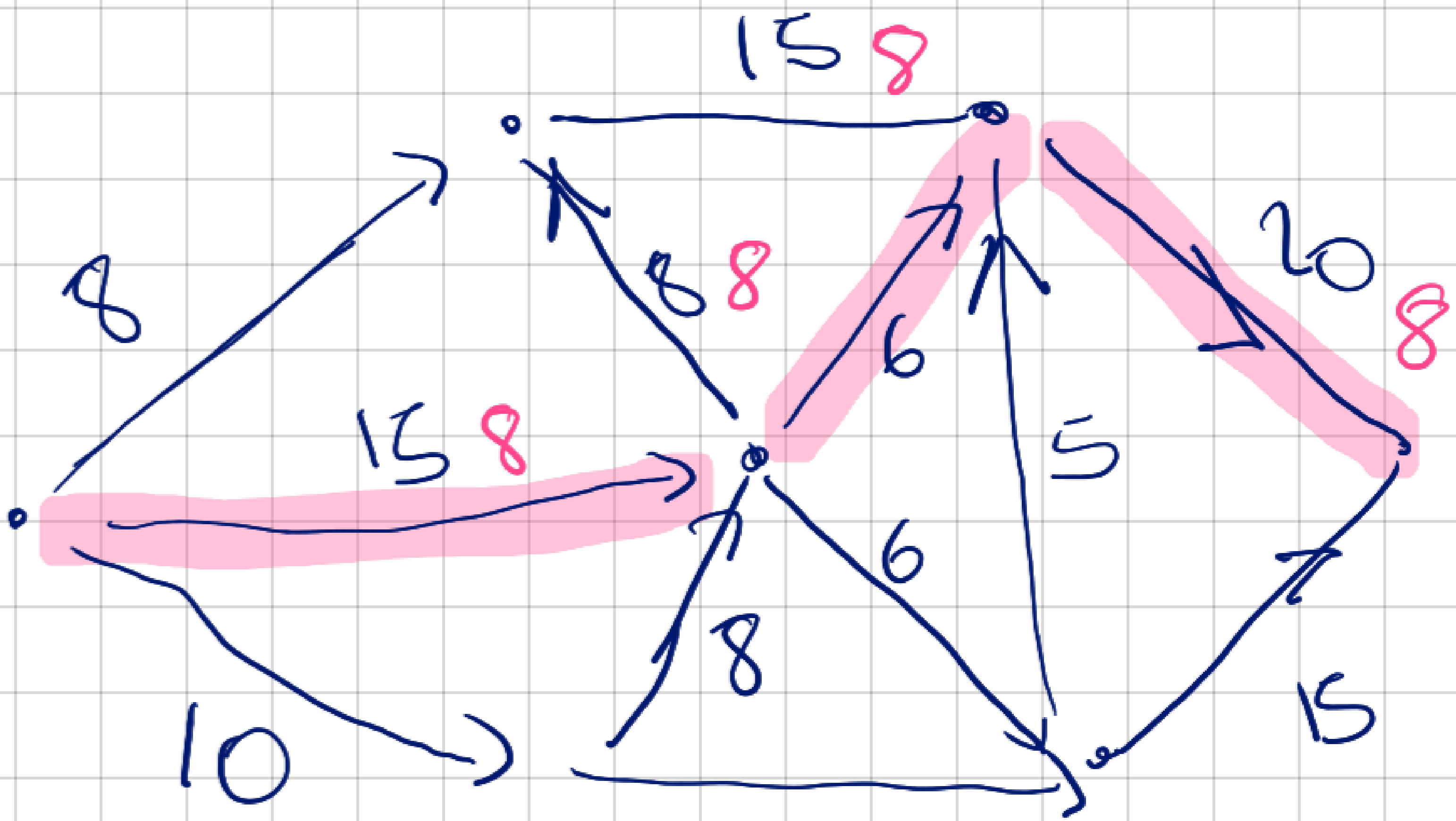
$\Delta = 1$



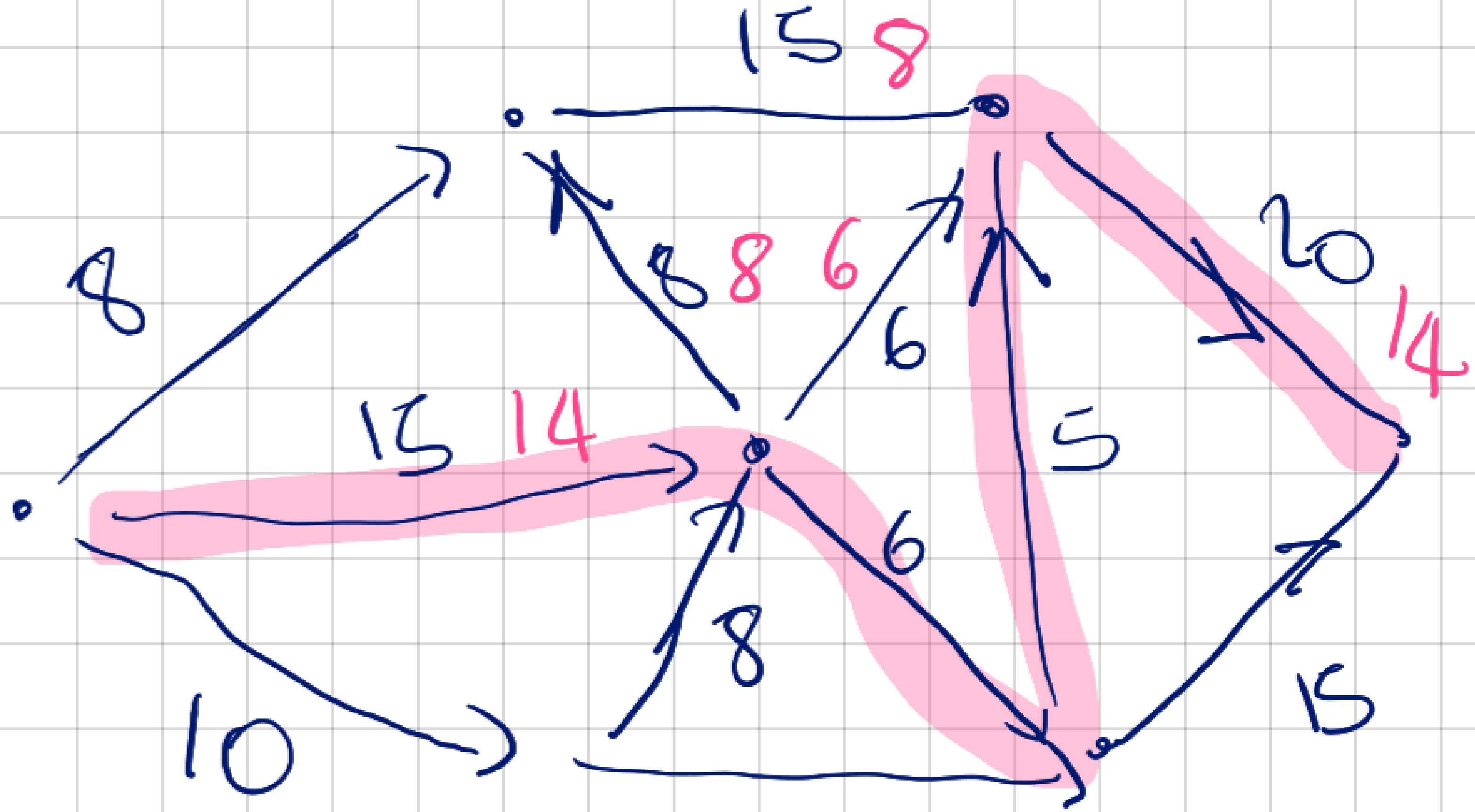


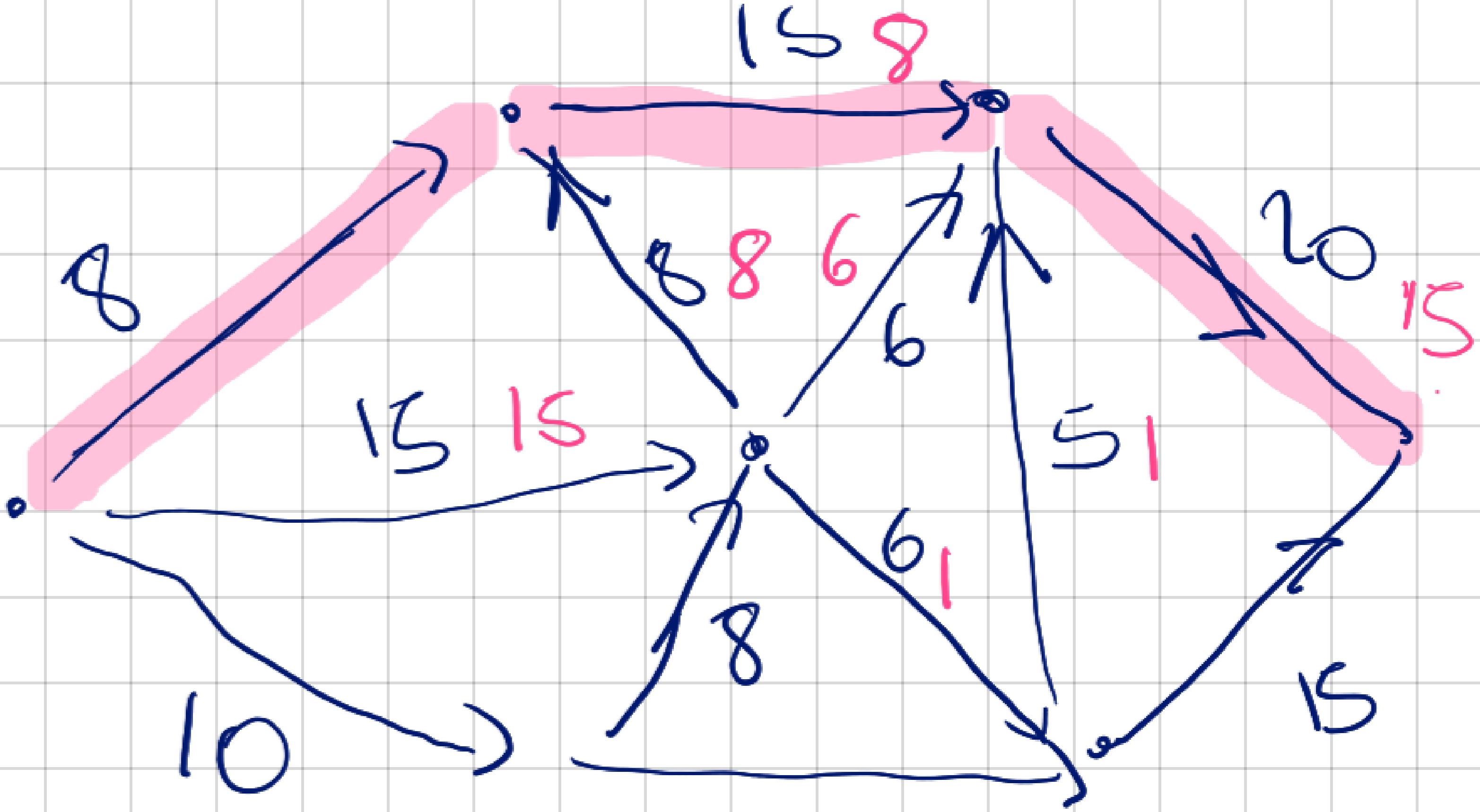
cut is $P = \{a\}$ P^c everyth.

max flow 6+8+9

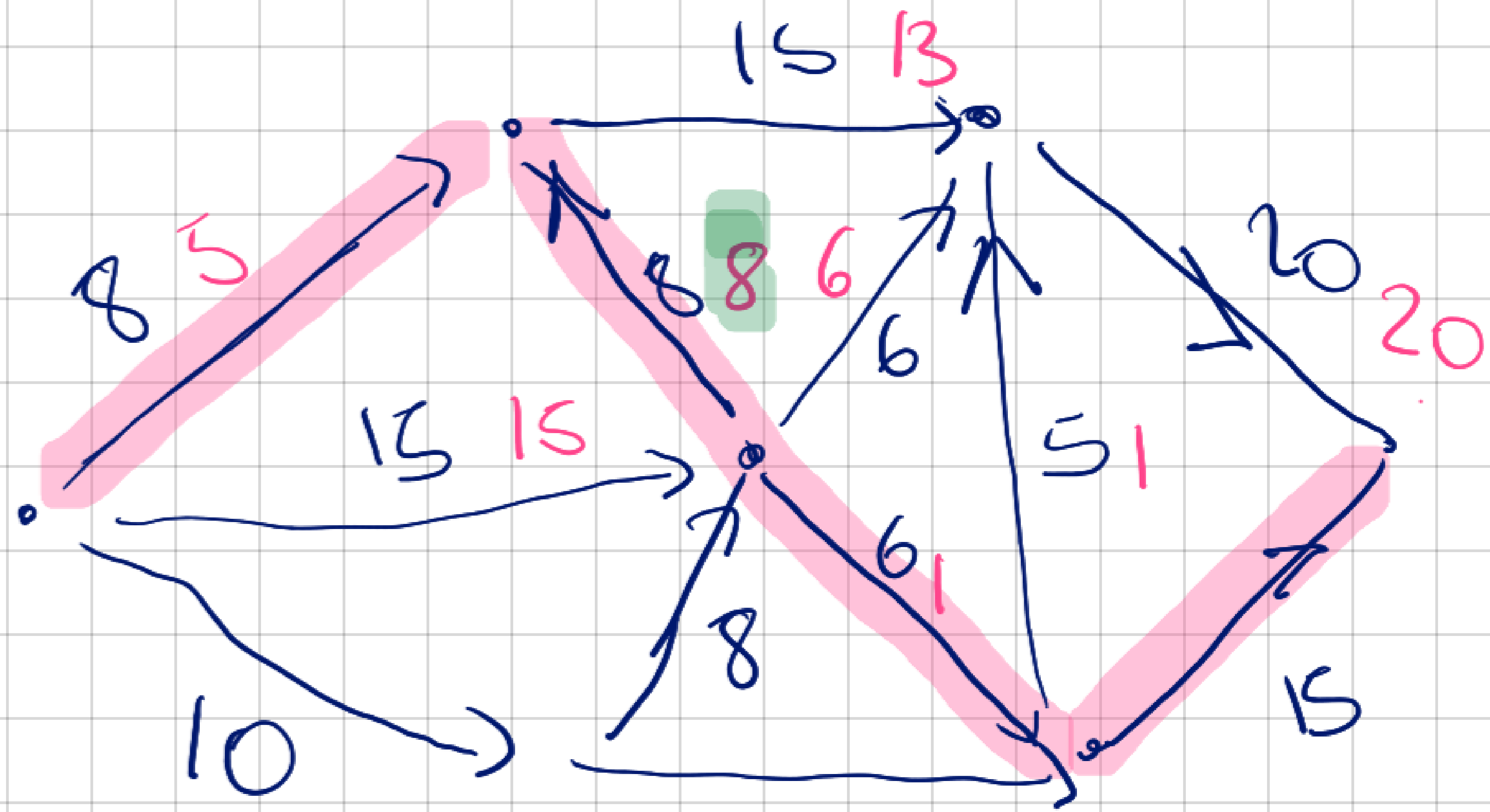


15 8
8
8 8
6
20
8
15 8
5
6
8
10
15

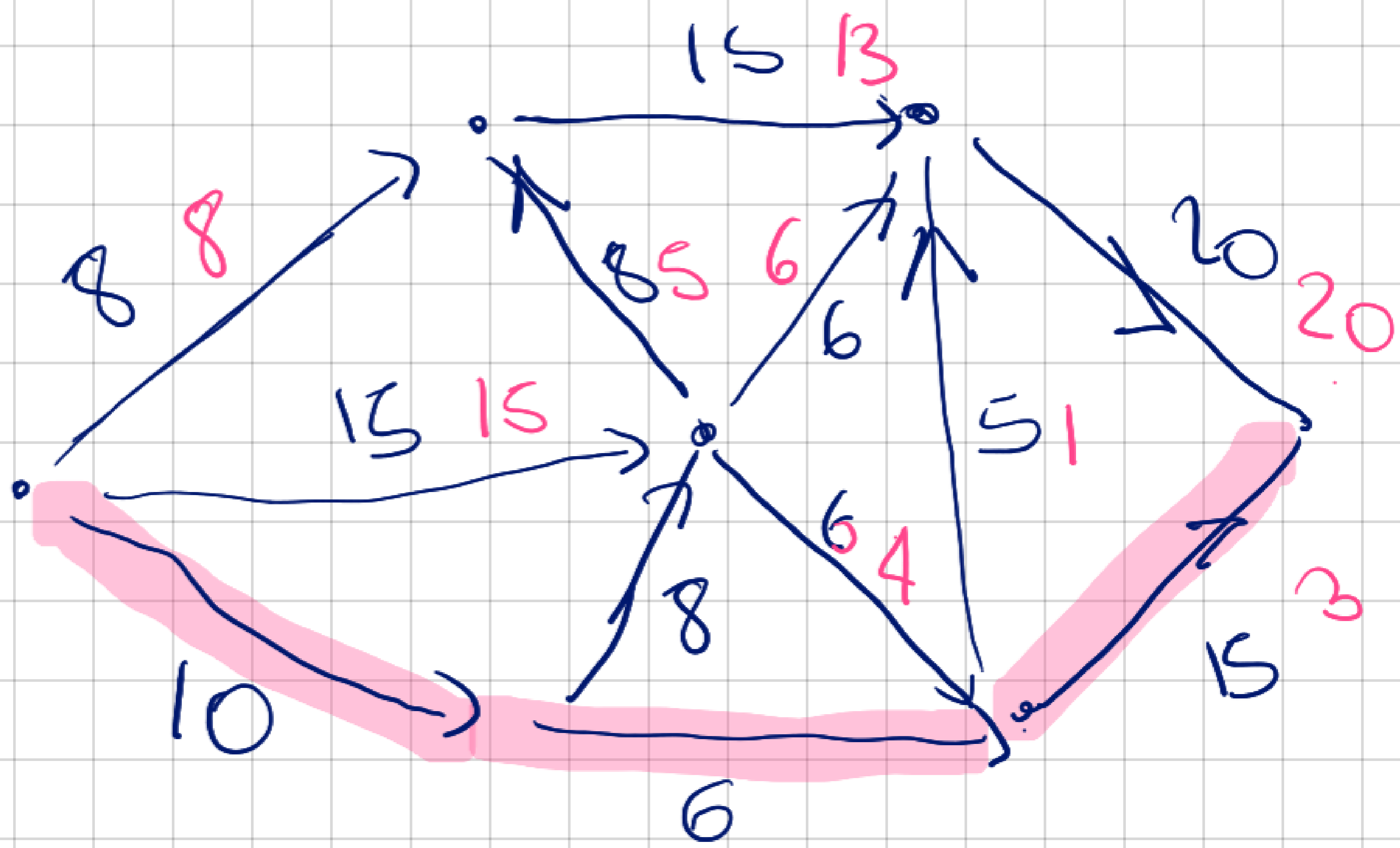

15 8
8
8 8 6
6
20
14
15 14
5
6
8
10
15



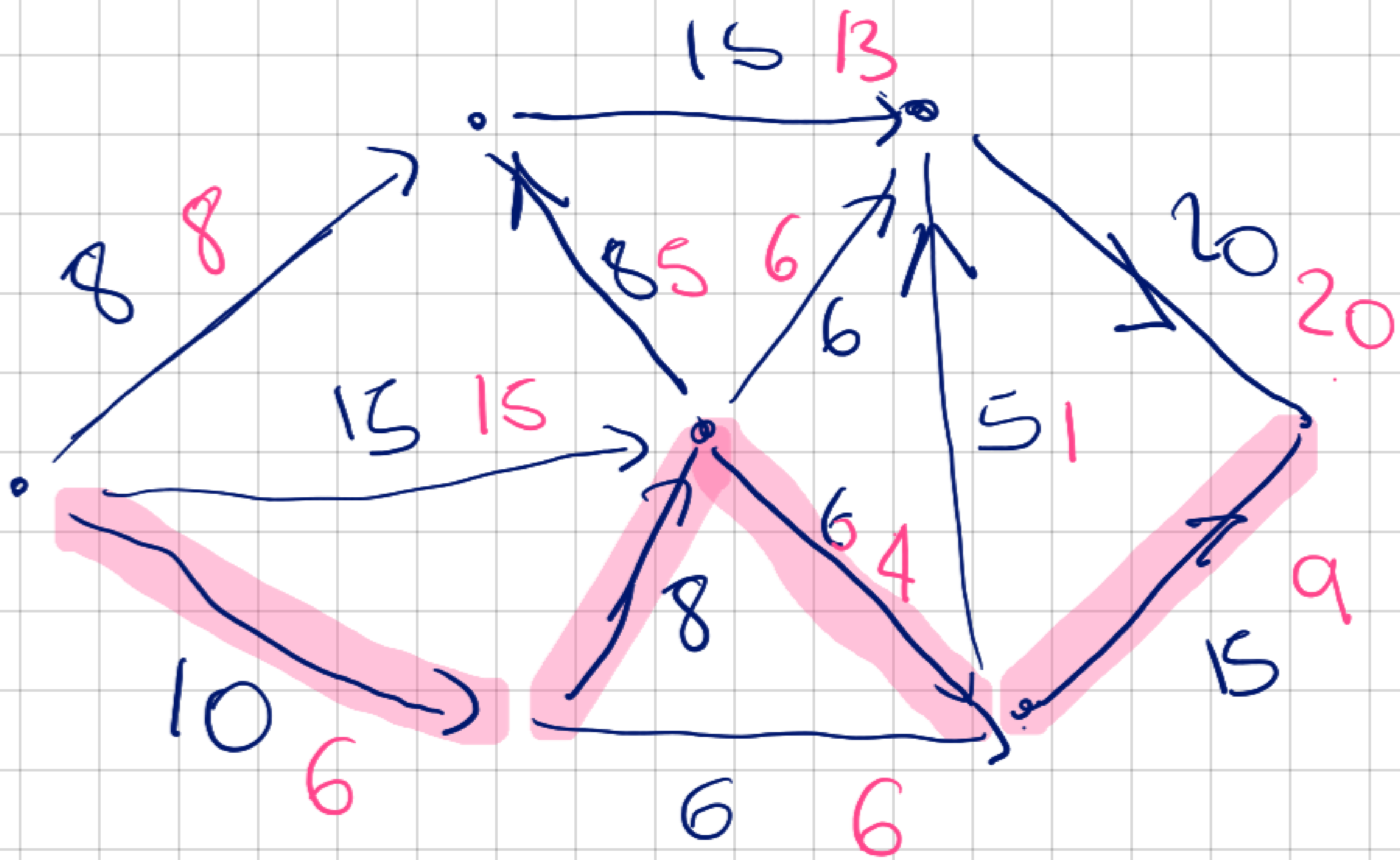
8
15 8
8 8 6
6
20 15
15 15
51
61
8
10
15



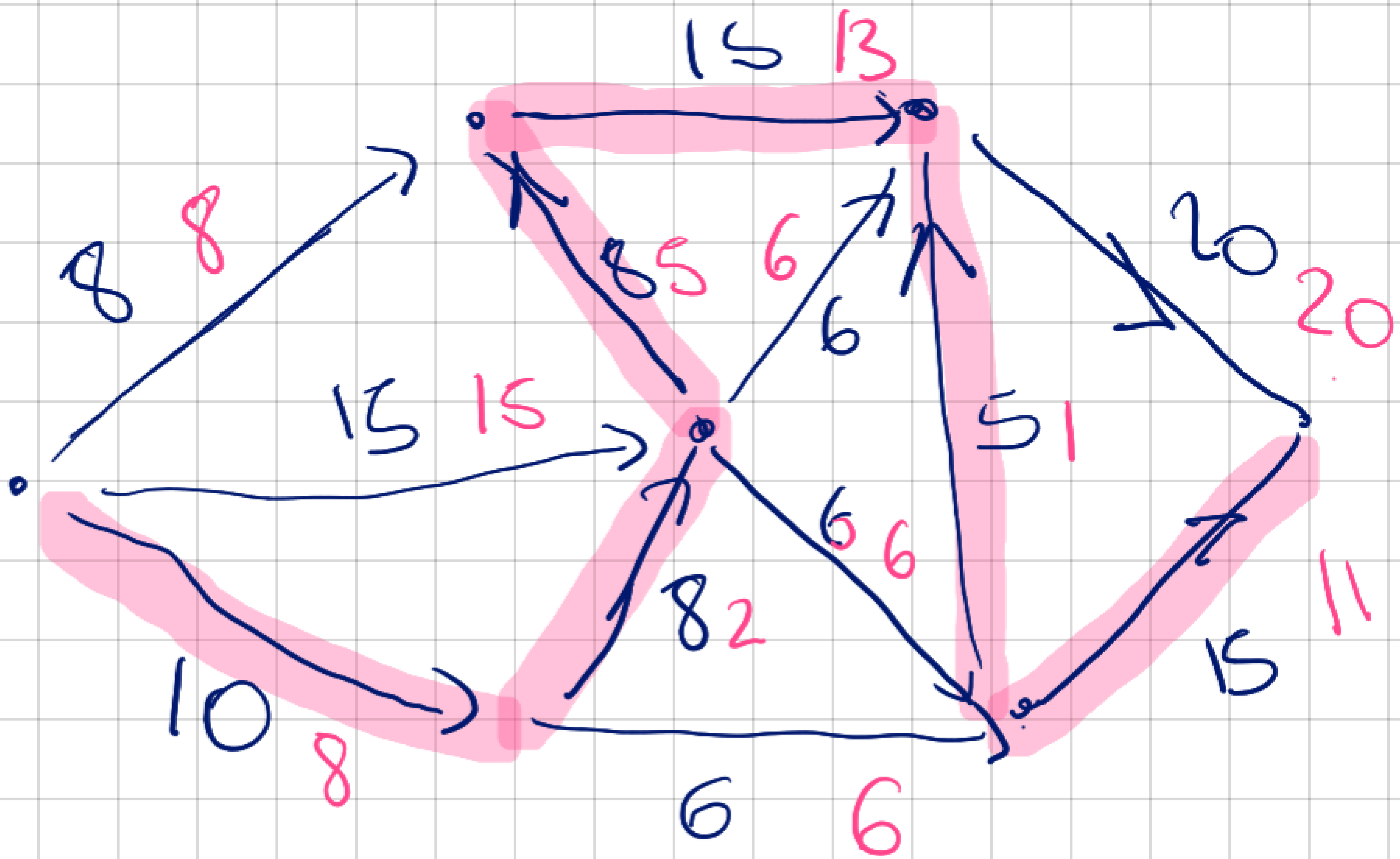
15 13
8 5
8 8 6
6
20
20
15 15
5 1
6 1
8
10
15

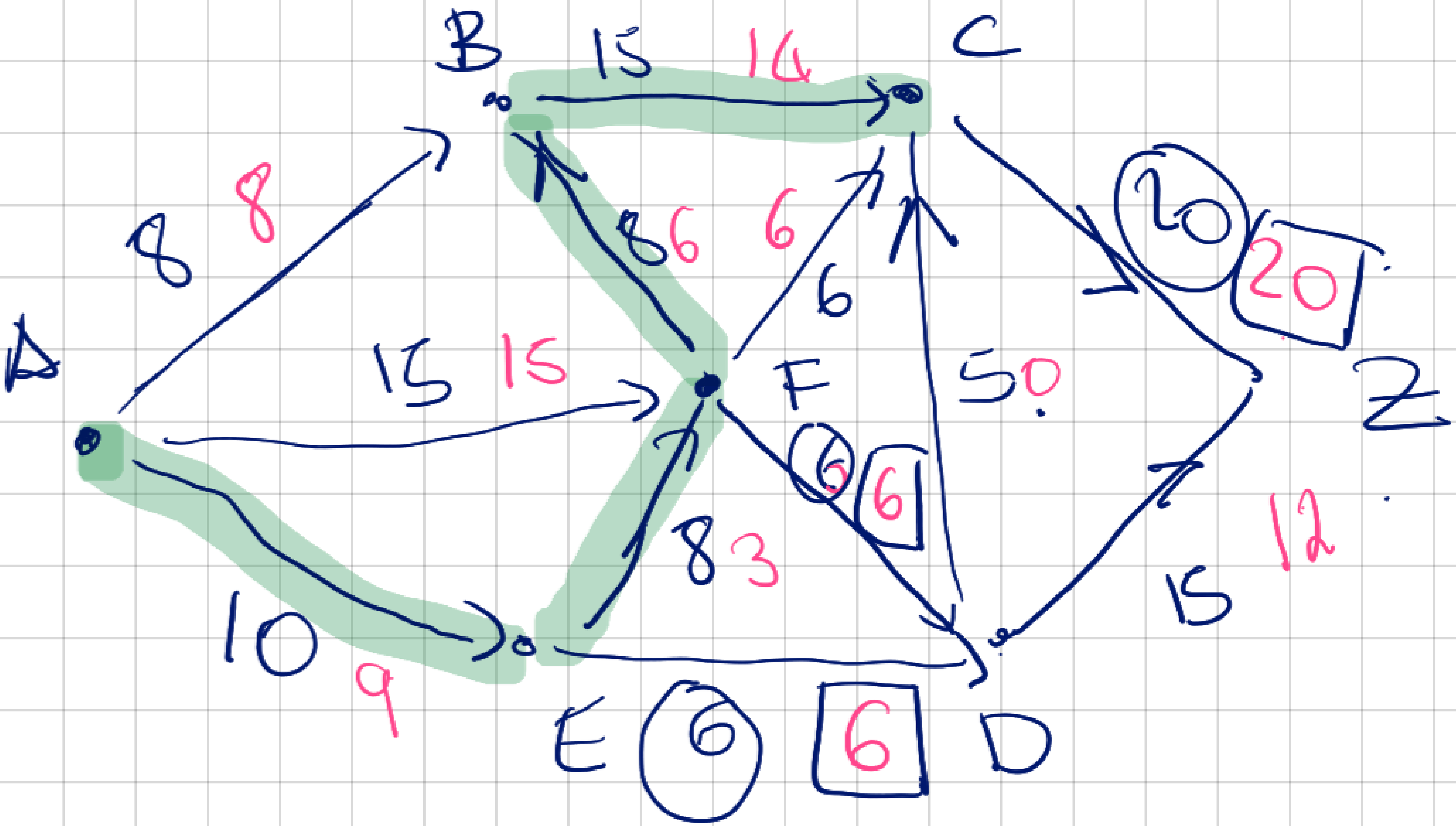


15 13
8 8
8 5 6
6
20 20
15 15
5 1
6 4
8
15 3
10
6



15 13
8 8
85 6
6
20
20
15 15
51
6 4
8
9
15
10
6
6 6


15 13
8 8
85 6
6
20
20
15 15
51
66
82
15
11
10
8
6 6

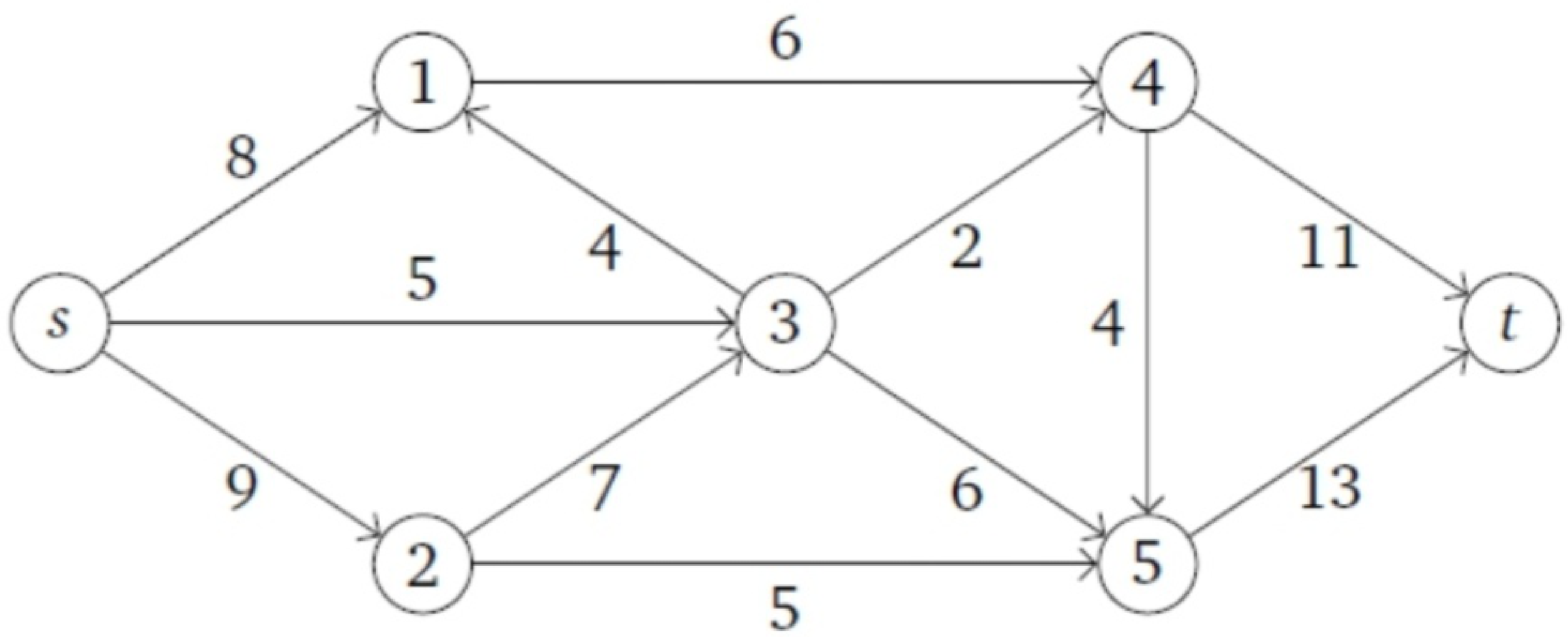


$$\mathrm{Val}(f) = 8 + 15 + 9 = 32$$
$$= 20 + 12$$

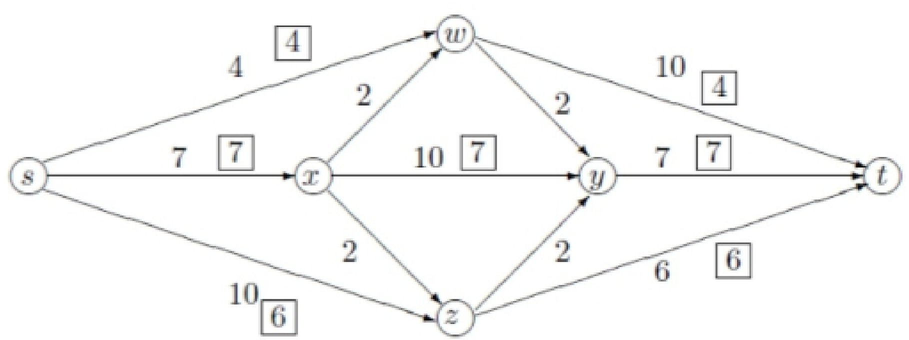
$P = \{A, B, F, E, C\}$ $P^c = \{D, Z\}$

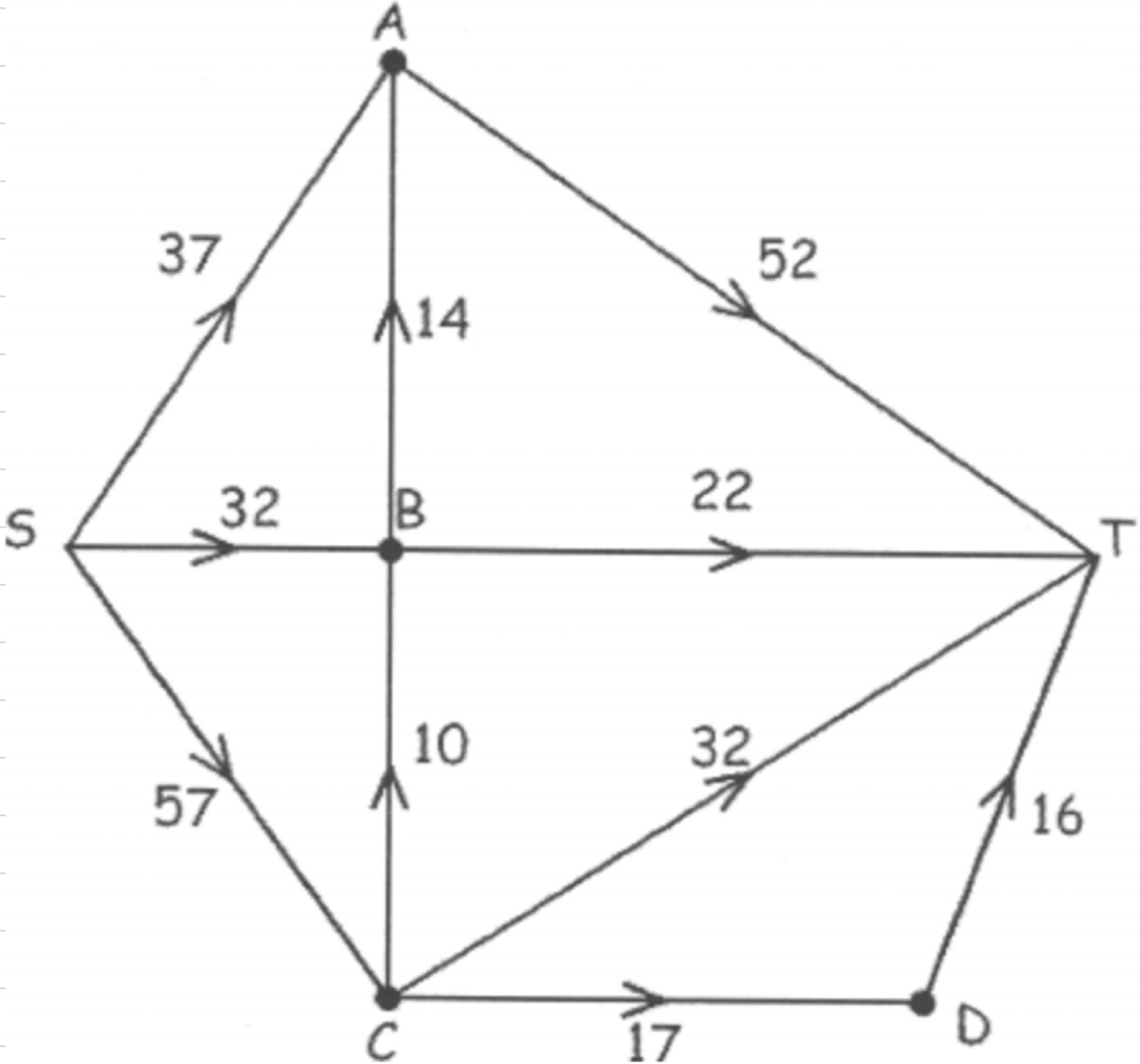
The capacity of the cut

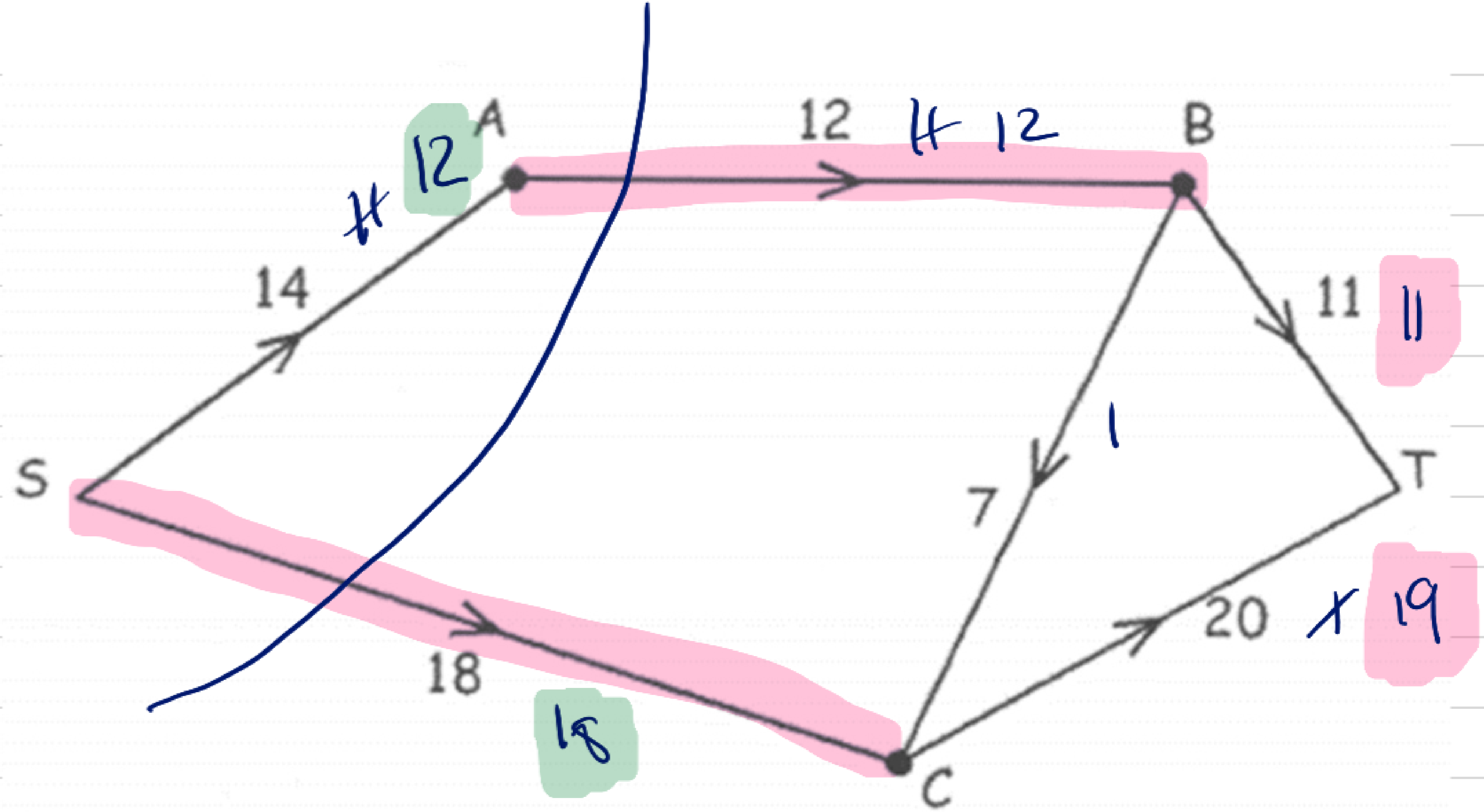
$$c(P, P^c) = 6 + 6 + 20 = 32$$



The figure below shows a flow network on which an st flow is shown. The capacity of each edge appears as a label next to the edge, and the numbers in boxes give the amount of flow sent on each edge. (Edges without boxed numbers have no flow being sent on them.) What is the value of this flow? Is this a maximum st flow in this graph? If not, find a maximum st flow. Find a minimum st cut. (Specify which vertices belong to the sets of the cut.)




A
37
52
14
S
32
B
22
T
10
32
57
16
C
17
D



$$Val(f) = 12 + 18 = 11 + 19 = \boxed{30}$$

$$C(P, P^c) = 12 + 18 = \boxed{30}$$

$\Rightarrow$ the flow is maximal & the cut is minimal.

Chromatic Polynomials

- $G = G_1 \cup \ldots \cup G_n$ connected cp then
- $\chi_G(x) = \prod \chi_{G_i}(x)$

collapsed e

- $\chi_{G-e} - \chi_{G'_e} = \chi_G$

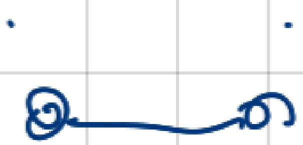
- $G = G_1 \cup G_2 \qquad G_1 \cap G_2 = K_n$

$$\chi_G = \frac{\chi_{G_1} \cdot \chi_{G_2}}{\chi_{K_n}}.$$

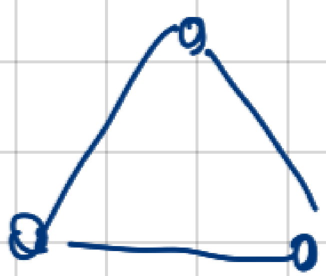
The chromatic polynomial of K_4

$\chi_G(\lambda)$ is a polynomial such that

$\chi_G(n) = \#$ of ways to color the vertices of G properly with n colors



K_2 $\quad n \quad \binom{n}{2}\cdot 2$



K_3 $\quad n \quad \binom{n}{3}\cdot 3!$

K_j n colors I have to choose j of them

$$\chi_{K_j}(n) = \binom{n}{j} j! = n(n-1)(n-2)$$

$$\underbrace{x(x-1)\ldots(x-j+1)}$$

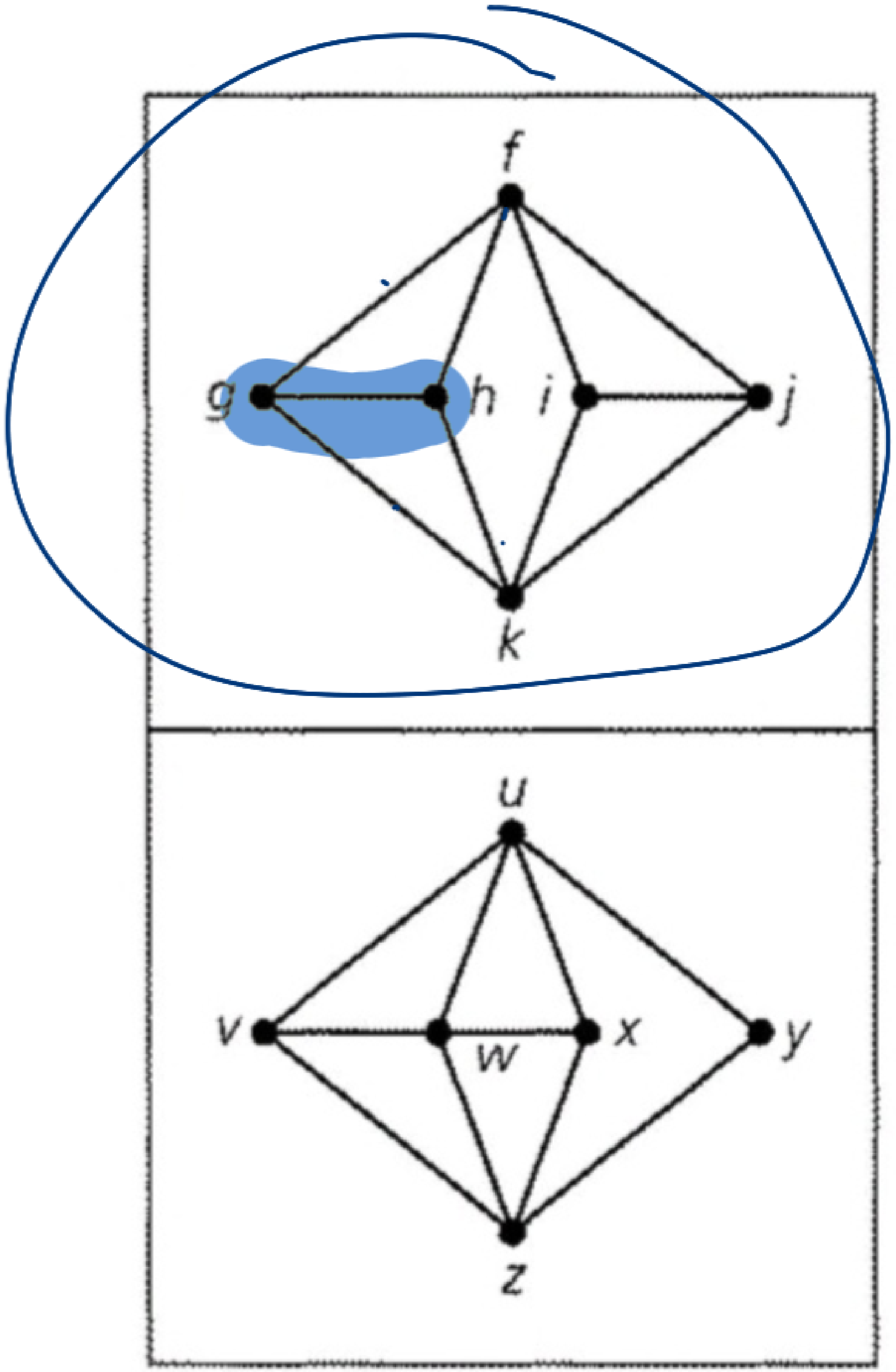
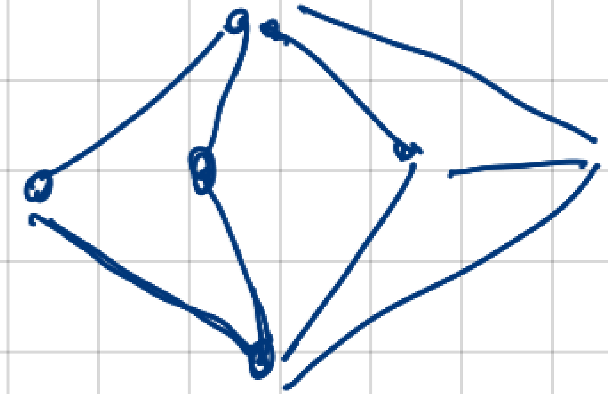
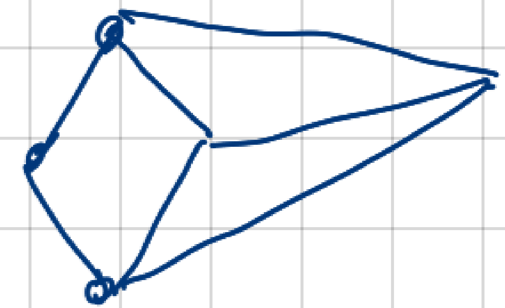


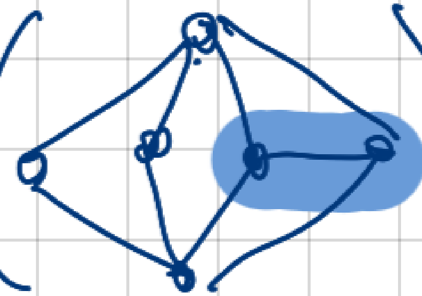
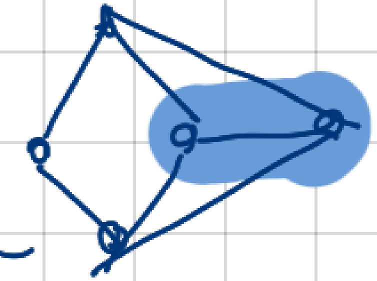
Figure 11.93

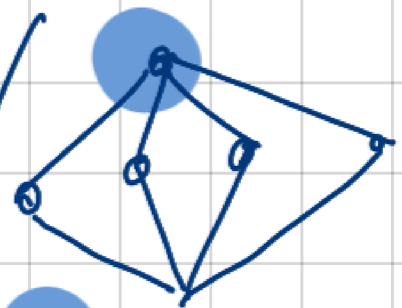
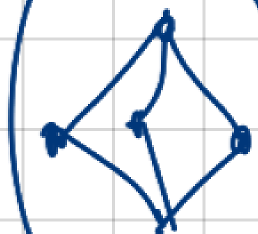
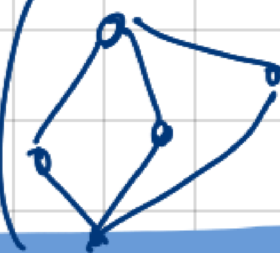
G-e

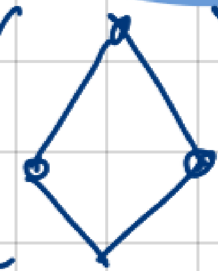


Ge

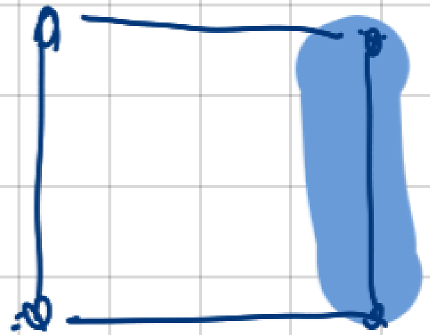


$\chi(G) = \chi($  $) - \chi($  $)$

$= \chi($  $) - \chi($  $) - \chi($  $)$

$+ \chi($  $)$

$=$



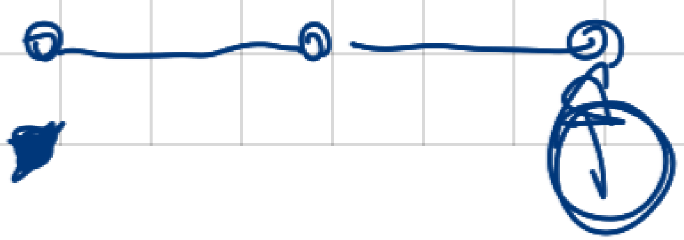
$$\binom{n}{4}4! \qquad \binom{n}{3} + \cdot$$

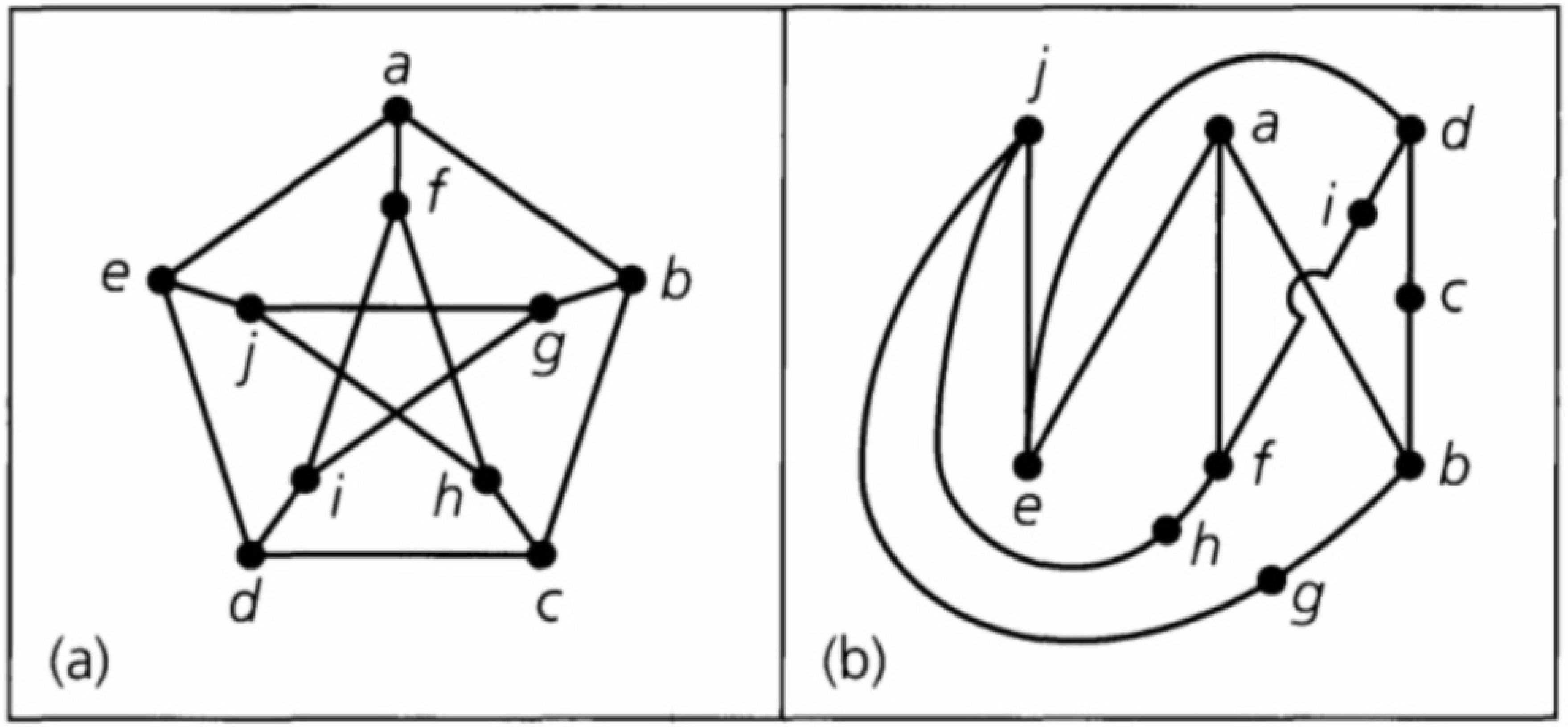
$$\chi(\sqcup) = \chi(\square) - \chi(\triangle) \qquad x^2(x-1)^2 - x(x-1)^2$$

$$= \chi(\llcorner\,\cdot) - \chi(\llcorner) - \chi(\triangle) - x(x-1)(x-2)$$

λ

$$\binom{n}{3}3! + \binom{n}{2}2$$





$n^n(n-1)^2$

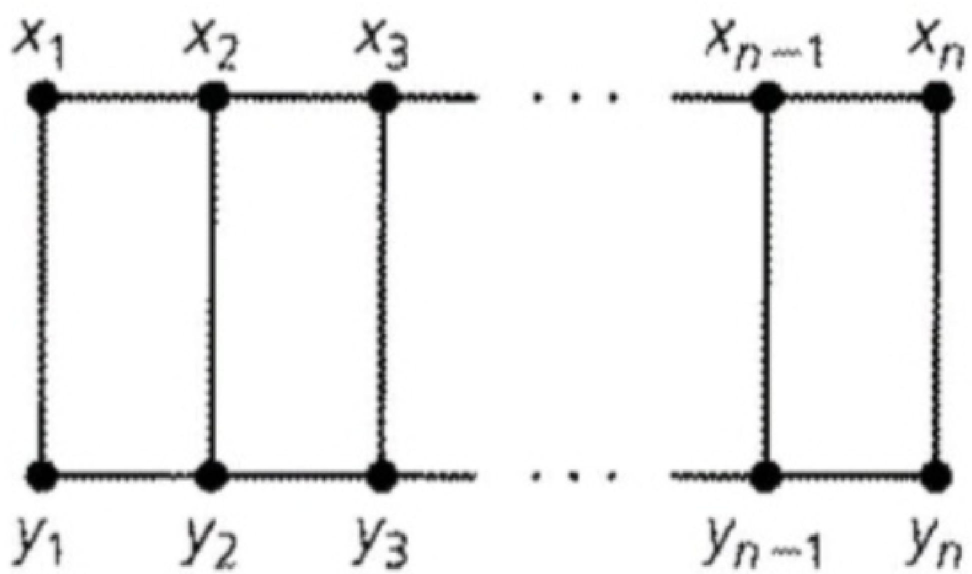
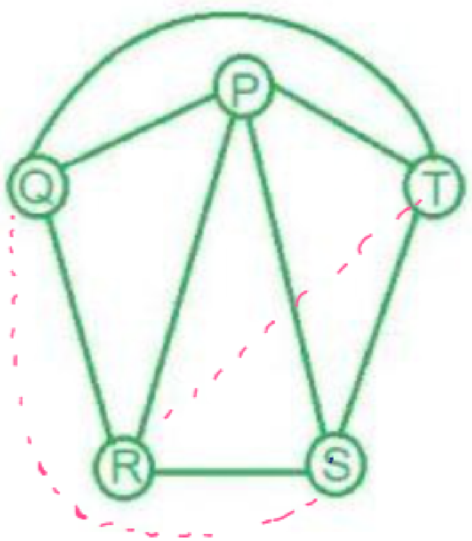


Figure 11.94

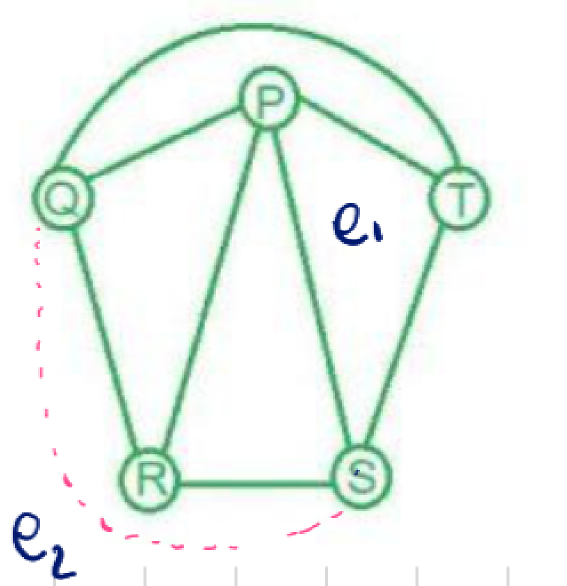


In order to color K_n with λ colors

$\rightarrow$ you chose n of λ colors. you multiply by $n!$

$$\binom{\lambda}{n} n! = \frac{\lambda}{(\lambda-n)!}$$

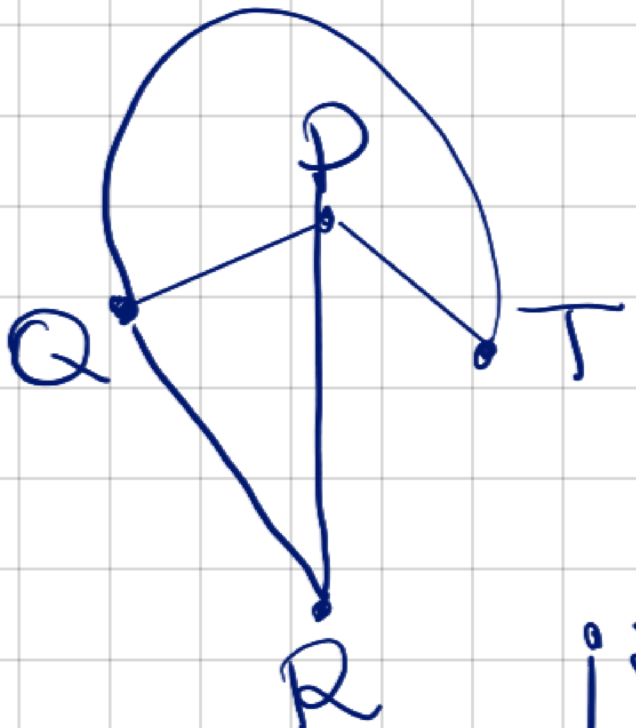
$$= \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$$

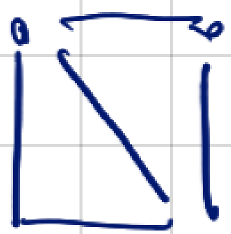


$$\chi_{K_5} = \chi_{K_n - e_1} - \chi_{{K_n}_{e_1}}$$

$$= \chi_{K_5 - e_1} - \chi_{K_4}$$

$$= \boxed{\chi_{K_5 - e_1 - e_2}} - \underset{K_4 - e}{\chi_{(K_5 - e_1)_{e_2}}} - \chi_{K_4}$$





$$\chi_6 = \chi_{K_5} + \chi_{K_4 - e_1} + \chi_{K_4}$$

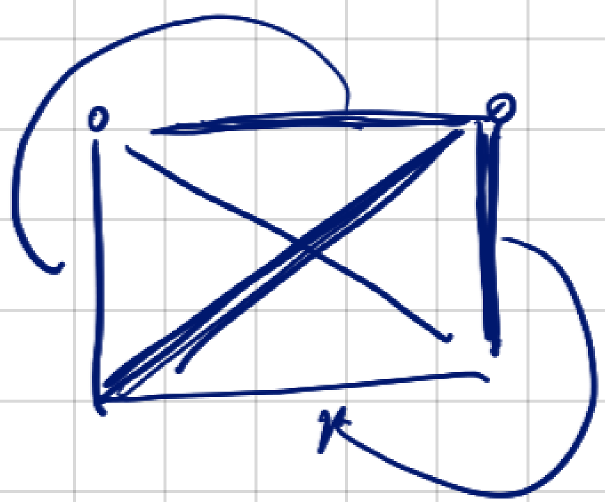
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

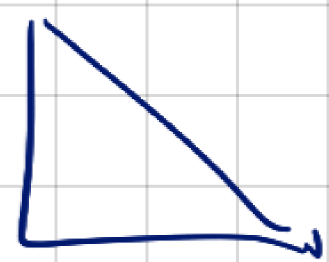
$$+ \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$

$$+ \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

Remember: this is not a generating function => in how many ways you can color it with k color $\Rightarrow$ plug $k \to \lambda$

$$\chi_{K_4} = \chi_{K_4 - e_1} - \chi_{K_4 e}$$





$$\chi_{K_4} = \chi_{K_4 - e_1} - \chi_{K_3}$$

$$\chi_{K_4 - e_1} = \chi_{K_4} + \chi_{K_3}$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$

Hamilton Cycles

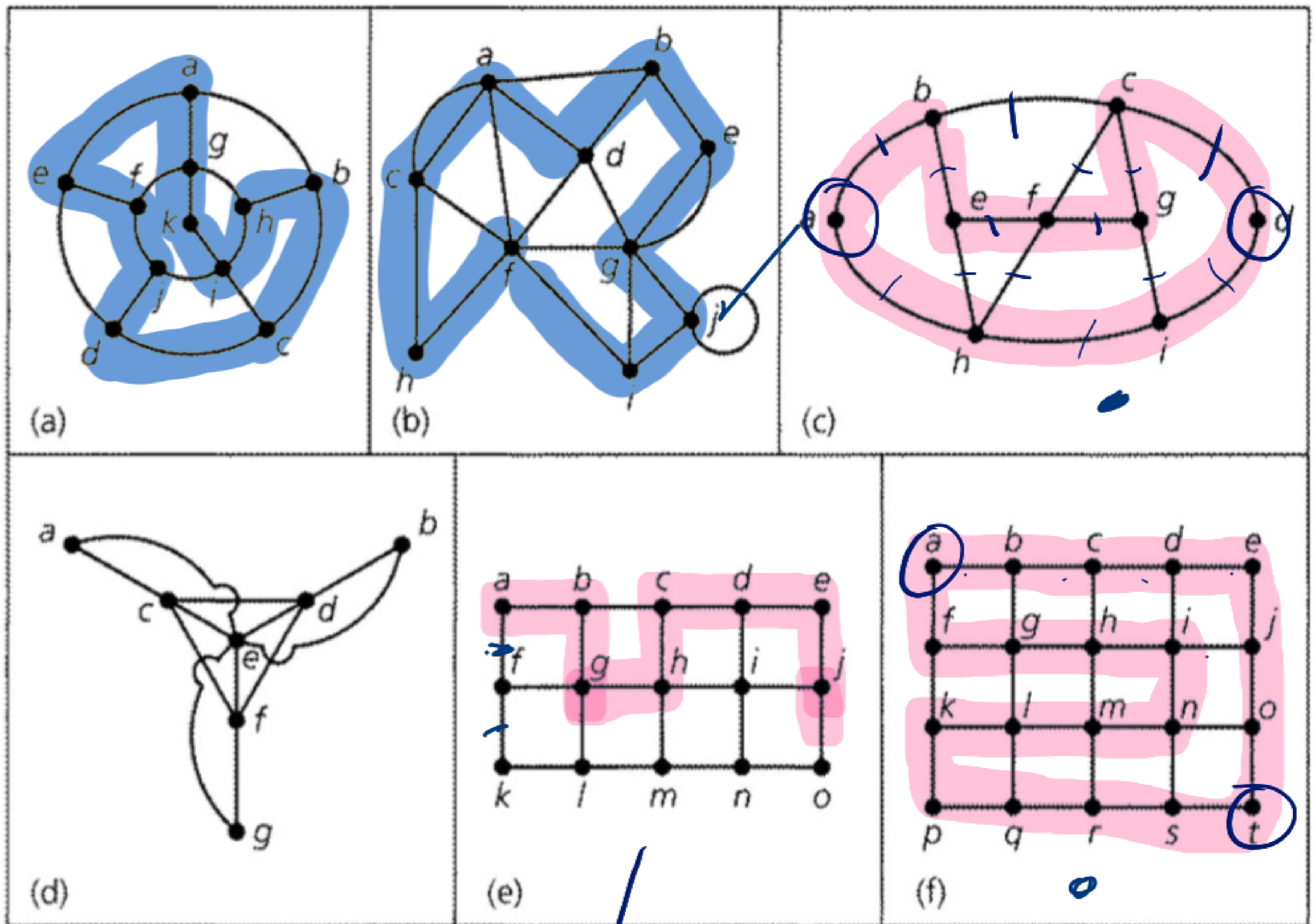


Figure 11.84

Not an HC

Recall

$\deg v + \deg w \geq |V|-1$
fn $v \not\sim w$

- $\deg(v) + \deg(w) \geq |V|$
- $|E(G)| \geq \binom{|V|-1}{2} + 2$

$\Downarrow$ $\cancel{\Uparrow}$

G has an HC

(a) $|V| = 11$

$$\binom{|V|-1}{2} + 2 = \frac{10(9)}{2} + 2 = 47$$

this graph has not 47 edges

$4 + 4 = 8 < 10 = 11 - 1$

(b) $|V| = 10$

$$\binom{|V|-1}{2} + 2 = \frac{9 \cdot 8}{2} + 2 = 38 \quad \checkmark$$

$$6 + 6 = 12 > 10$$

not ok

$$7nm - n - m + 7 \overset{?}{\geq} (nm)^2$$

$n = 3$

$m = 5$

$$7 \cdot 15 - 8 + 7 \overset{?}{\geq} 15^2$$

NO $\wedge$

$7 \cdot 15$

$\wedge$

$15 \cdot 15$

max deg is 4

$4+4=8$ ← really few vertices available.

$n \times m$ $\quad |E(G)|$ $\quad m-1$ edges in each row

$$n(m-1) + (n-1)m \geq \binom{nm-1}{2} + 2 \quad ?$$

$$nm - n + nm - m \geq \frac{(nm-1)(nm-2)}{2} + 2$$

$$4nm - n - m \geq (nm)^2 - nm - 2nm + 3 + 4$$

(f) $|V| = 20$

$\deg(a) + \deg(t) = 4 < 20$

$|E(G)| = 16 + 15 = 31$

$\binom{19}{2} = \frac{19 \cdot \cancel{18}}{\cancel{2}} = 19 \cdot 9$

(c) - has 9 vertices

$$\deg(a) + \deg(d) = 4 < |V|$$

- $E(G) = 14$ edges

$$\binom{9-1}{2} = \frac{8!}{2!\,6!} = \frac{8 \cdot 7}{2} = 4 \cdot 7 = 28$$

$$14 < 28 + 2 = 30$$

Planar Graphs

$G = (V, E)$ connected & planar

$$v - e + r = 2$$

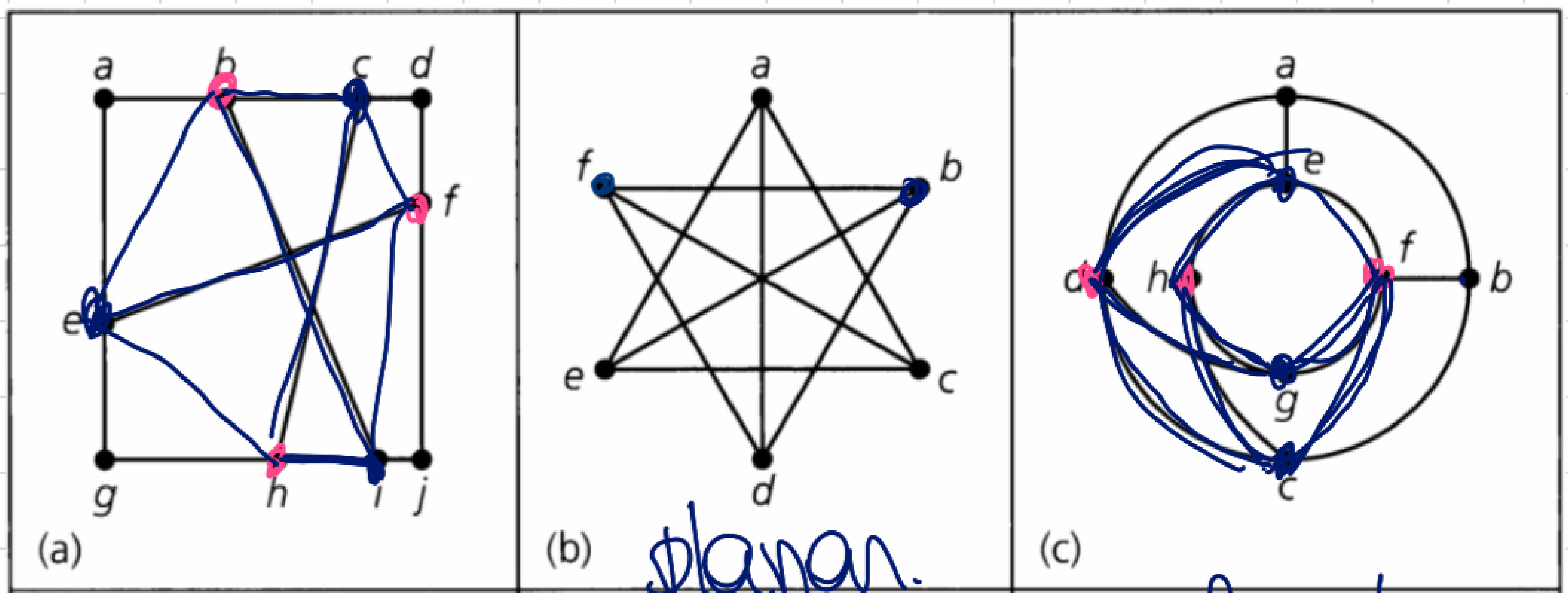
necessary condition.

$e \leq 3v - 6$

bip $\quad e \leq 2v - 4$

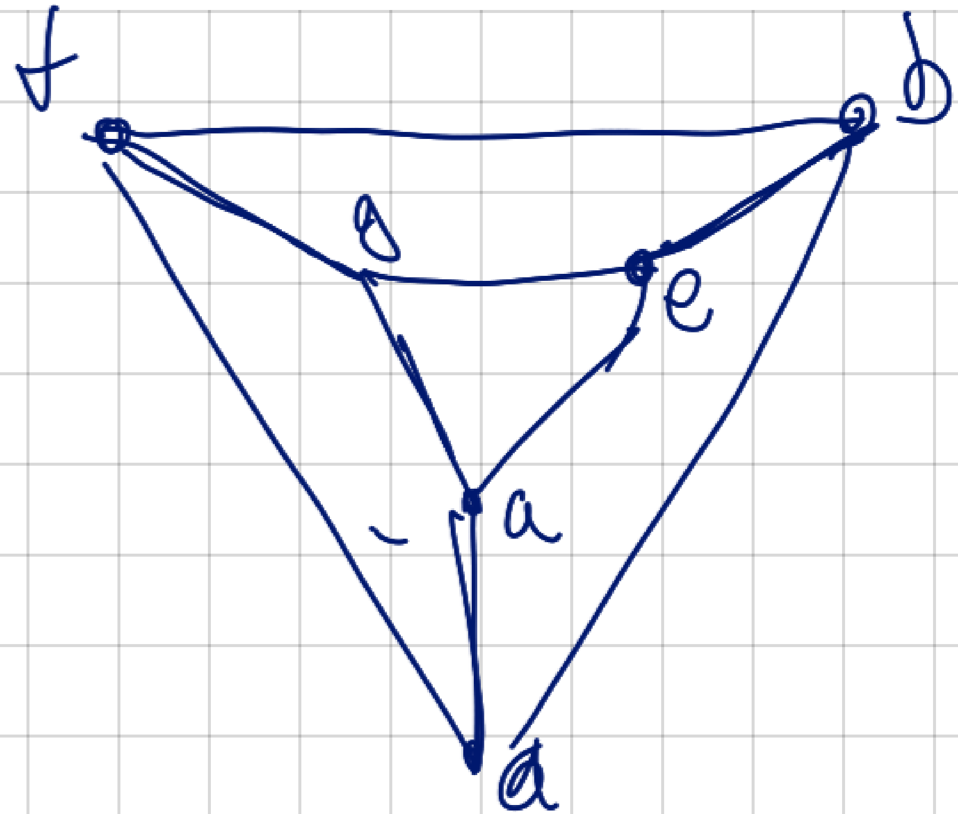
G non planar $\Longleftrightarrow$ has a subgraph iso to K_5 or $K_{3,3}$

homeomorph



(a) (b) (c)

not planar. planar. not planar.


f
b
g
e
a
d

(b) 6 vertices

9 edges $< 18 - 4$.

it can be plane.

(c) 8 vertices
10 edges

10 $\quad$ 24 - 6 $\quad$ 22

$<$

It could be planar

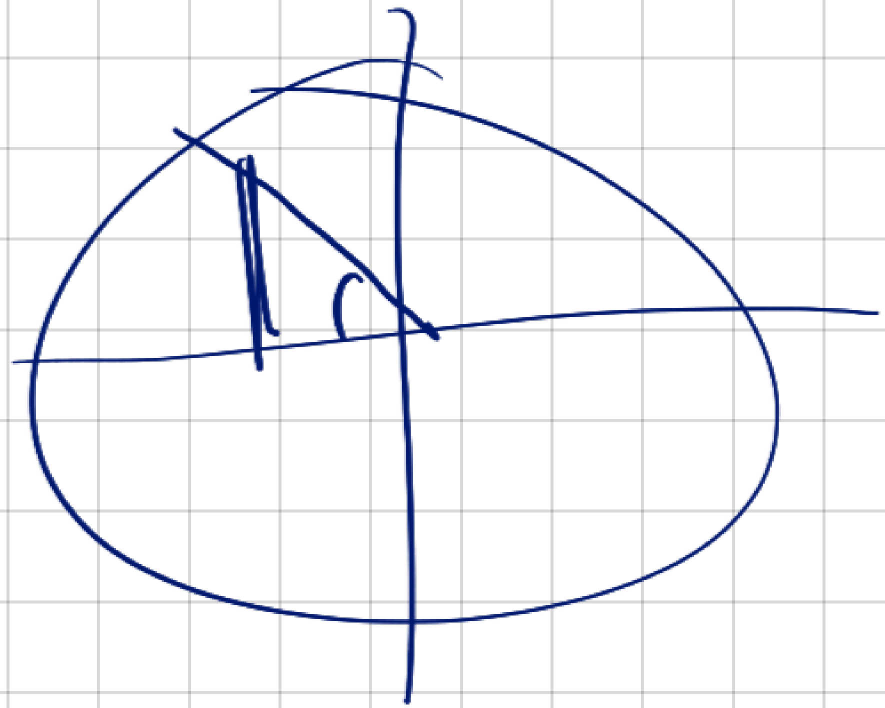
(a) $|V| = 10$

$|E| = 13$

$$3v - 6 = 30 - 6 = 24$$

$13 < 24 \Rightarrow$ it can be planar

$$\frac{-1+i\sqrt{3}}{2} = \cos\left(2\pi/3\right) + i\sin\left(\frac{2\pi}{3}\right)$$



$$1^n\left(A\cos\left(n\frac{2\pi}{3}\right) + B\sin\left(n\frac{2\pi}{3}\right)\right)$$

General solution of hom problem.

<u>Check</u> 1 is not a solution of the char equation

Recursion

$$a_{n+2} + a_{n+1} + a_n = 3 \cdot 1^n \qquad a_0 = 1$$

$$a_1 = 1$$

$$x^2 + x + 1$$

roots $\dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm i\sqrt{2}}{2}$

Solution of the homogeneous problem.

Educate guess $a_n^{(p)} = A.$

$$3A = 3 \Rightarrow A = 1$$

$$a_{n+2}^{(p)} + a_{n+1}^{(p)} + a_n^{(p)} = B$$

General solution

$$A\cos\left(n\frac{2\pi}{3}\right) + B\sin\left(n\frac{2\pi}{3}\right) + 1$$

$$\cos(2\pi/3) \; \cancel{\ } \; \frac{\sqrt{3}}{2}\sin\left(n\frac{2\pi}{3}\right) + 1$$

$$A\cos(0) + B\cancel{\sin(0)} + 1 = 1$$

$$A = 1$$

$$\cos\left(\frac{2\pi}{3}\right) + B\sin\left(\frac{2\pi}{3}\right) \cancel{+1} = \cancel{1}$$

$$B \; \frac{\sqrt{3}}{2} = -\frac{1}{2}$$

$$B = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

With the method of gf :

$$a_{n+2} + a_{n+1} + a_n = 3$$

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} a_{n+1} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 3$$

$$f(x) - a_0 - a_1 + x\left(f(x) - a_0\right) + x^2\left(f(x)\right) = 3$$

$$f(x)\left(1 + x + x^2\right) - 2 - x = 3$$

$$f(x) = \frac{5-x}{1+x+x^2}$$

?

Mc Laurin expansion

Rook Polynomials

formula

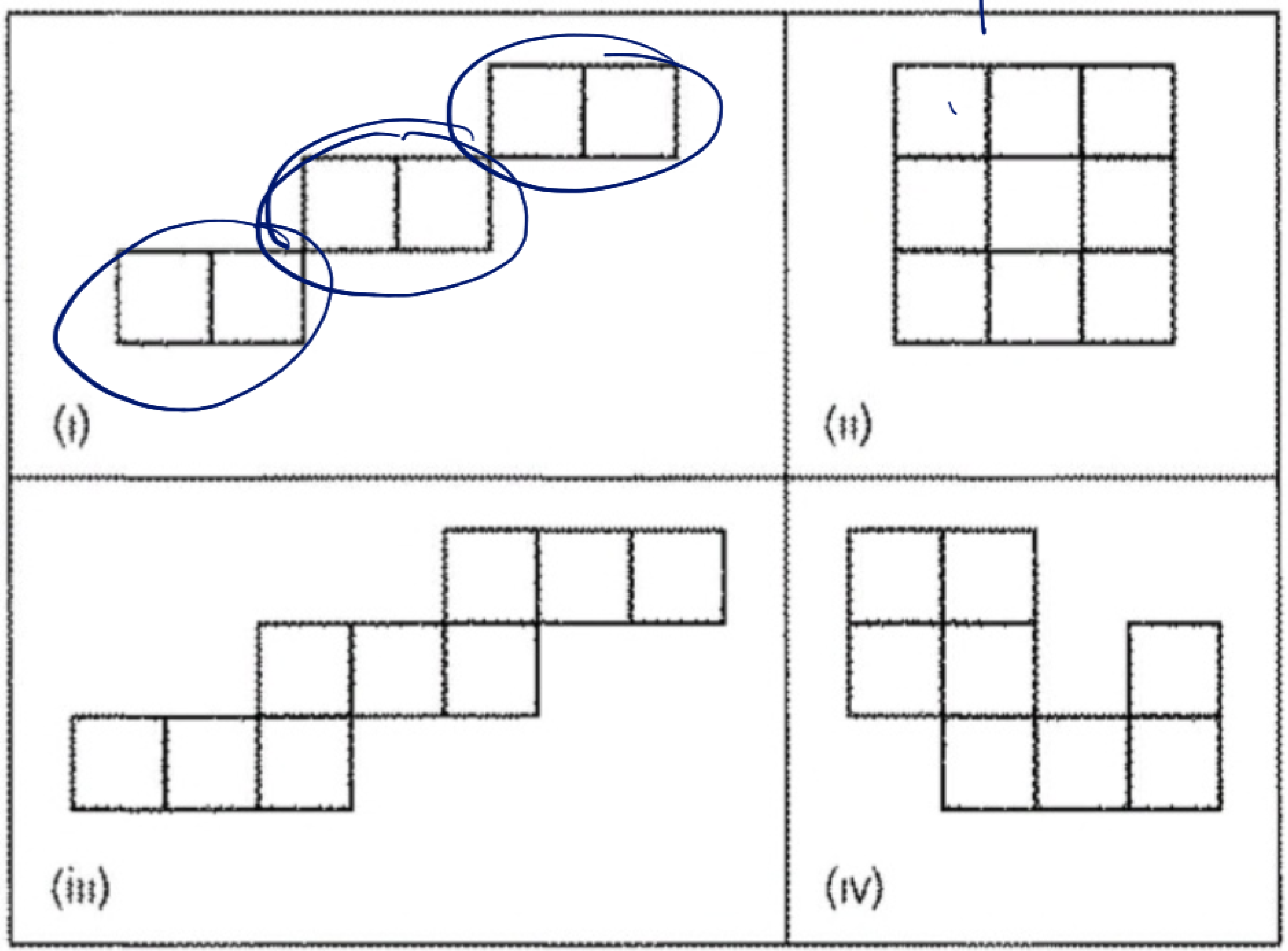
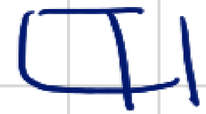


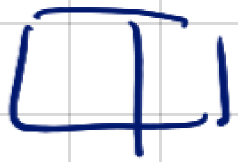
Figure 8.13

- $r(C, x) = r(C', x) + x \cdot r(C'', x)$

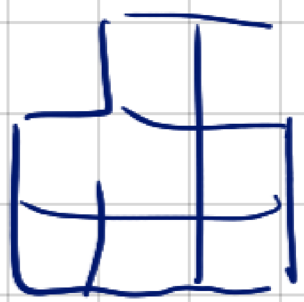
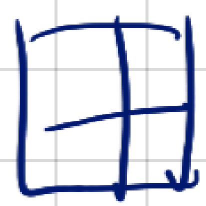
- $C_1 \cup C_2$ disj.

$r(C, x) = r(C_1, x) \cdot r(C_2, x)$

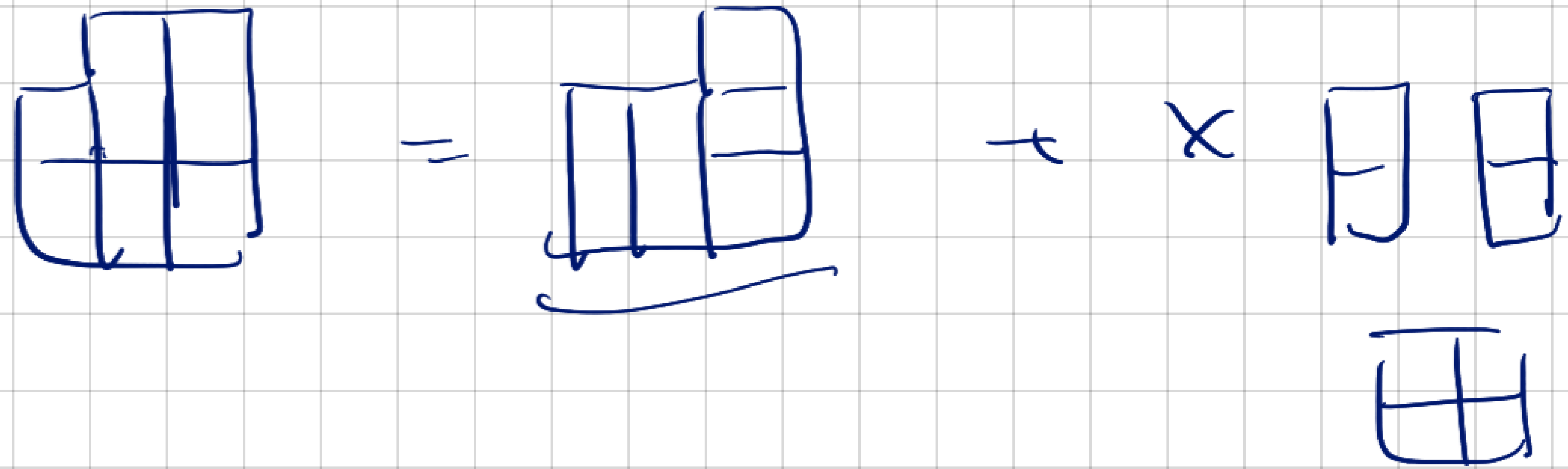
(a) $r(C_a) = r\Big($  $, x\Big)^3$

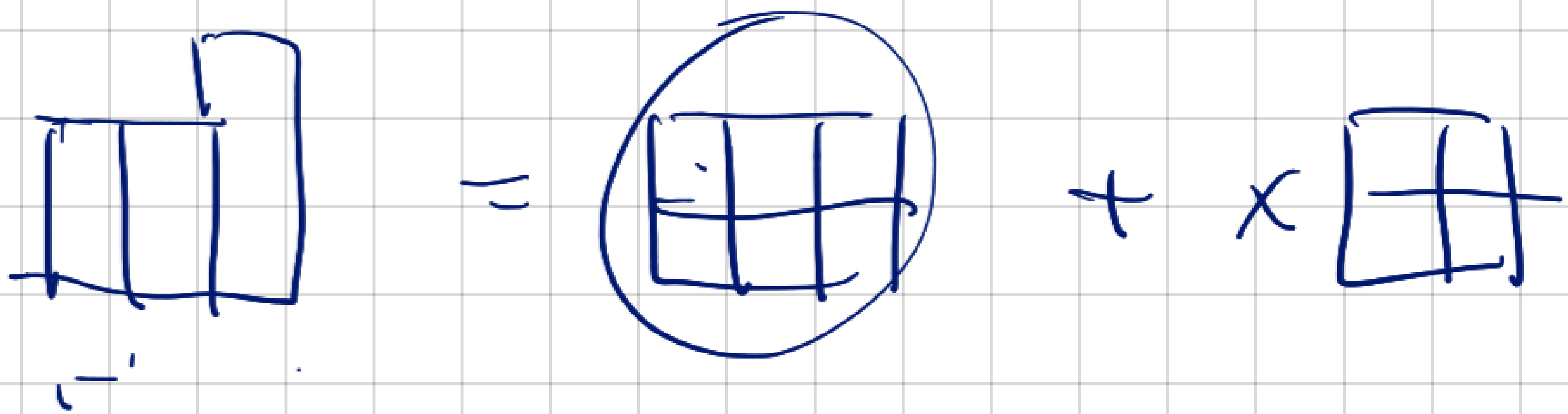
 $1+2x$

$= (1+2x)^3$

(b)  $+\ x$ 

 $= 1+4x+2x^2$





$$1+6x+3x^2+x(1+4x+2x^2)$$
$$+x(1+4x+2x^2)$$
$$+x(1+4x+2x^2)$$

Generating function

- Compute the generating function of

  n^3

- Compute te exponential generating function of $n \cdot 2^{n+1}$

$$\sum \frac{n \cdot 2^{n+1} \cdot x^n}{n!} = 2 \sum \frac{n}{n!} (2x)^n$$

$$2\sum_{1} \frac{n}{n!}(y)^n$$

$$= 2\left(0 + \sum_{1}^{\infty} \frac{n}{n!}(y)^n\right)$$

$$= 2\left(0 + y\sum_{1}^{\infty} \frac{n}{n!}(y)^{n-1}\right) \quad e^y$$

$$= 2\left(0 + y\frac{d}{dy}\left(\sum_{1}^{\infty} \frac{1}{n!}y^n\right)\right.$$

$$2\left(ye^{y}\right)$$

$$= 2(2x)e^{2x}$$

Rook polynomial & matchig

① Compute the rook polynomial of the chessboard where the forbidden places are the "forbidden pawns"

② Compute the rook polynomial of the chessboard in which

forbidden places are allowed pain
~~Then~~ inclusion exclusion.

k rook placing on the "wrong chessboard" an assignment in which k jobs go to the wrong person.

the coefficient of the rook polynomial of the "wrong chessbo" give you in how many ways you can assign to k persons the wrong jobs

S all the placement in the grid

A_i = Job i is given to a right person

$|A_1 \cap \cap A_n| \longleftarrow$

the needs goody of the wrong chessb gives you

$$|A_{i_1}^c \cap \cdots \cap A_{i_k}^c|$$