Ex 10.3.2: We set

$$a_n - a_{n-1} = n^2$$

We set $an^3 + bn^2 + cn$ as a particular solution. Substitute in the equation to get:

$$an^3 + bn^2 + cn - \left(a(n-1)^3 + b(n-1)^2 + c(n-1)\right) = n^2$$

$$\Leftrightarrow \quad 3an^2 + (2b - 3a)n + (a - b + c) = n^2$$

We get $a = \frac{1}{3}$; $b = \frac{1}{2}$, $c = \frac{1}{6}$.

The general solution is of the form:

$\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + C$. Since $a_1 = 1$

we get $C = 0$.

Ex 10.3.d : Let a_n be the number of valid strings of length n. The strings that end with 3 can not have 0's. There are 3^{n-1} such valid strings.
A string that does not end with 3, can have any valid length $n-1$ string before the last digit. There are $3a_{n-1}$ strings of length n obtained this way.

We get: $$a_n - 3a_{n-1} = 3^{n-1}$$

We set $a_n^{(p)} = cn3^n$. Substituting in the equation we have:

$$cn3^n - 3c(n-1)3^{n-1} = 3^{n-1}$$
$$\Longleftrightarrow \quad c \cdot 3^{n-1} = 3^{n-1}$$

So that $c = 1$. Now the general solution is of the form
$(n+C)3^n$.

Since $a_1 = 4$ we have

$$(1+c)\cdot 3 = 4$$

so that $c = \frac{1}{3}$.

Ex 10.3.12: Let a_n be the number of sequences of length n with an odd number of 1's. Let b_n the number of sequences of length n with an even number of 1's.

We have $a_n + b_n = 4^n$.

Starting with a string of length $n-1$ with an even number of 1's, by adding 1 at the end we get a string of length n with an odd number of 1's. If we start with a string of length $n-1$ with an odd number of 1's we can add any digit but 1 to get a valid string of length n.

We get: $a_{n+1} = b_n + 3a_n$

$$a_{n+1} = 2a_n + 4^n$$

$$a_{n+1} - 2a_n = 4^n$$

We set $a_n^{(p)} = c\,4^n$. By substituting we get:

$$c\,4^{n+1} - 2c\,4^n = 4^n$$

$$c\,4^n(4-2) = 4^n$$

so that $c = \frac{1}{2}$.

The general solution is of the form

$$\frac{4^n}{2} + d\,2^n,$$

Since $a_1 = 1$ we have

$2 + 2d = 1$ so $d = -\frac{1}{2}$.

Ex 10.4.1 : a) $a_{n+1} - a_n = 3^n, n \geq 0,$
$a_0 = 1$

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=0}^{\infty} 3^n x^n$$

$$(f(x) - a_0) - x f(x) = \sum_{n=0}^{\infty} 3^n x^n$$

$$f(x)(1-x) = \frac{1}{1-3x} + 1$$

$$f(x) = \frac{1}{(1-3x)(1-x)} + \frac{1}{1-x}$$

We have $\frac{1}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$

by setting $x=1 \rightarrow B = -\frac{1}{2}$
$x = \frac{1}{3} \rightarrow A = \frac{3}{2}$

So $f(x) = \frac{3}{2}\frac{1}{1-3x} + \frac{1}{2}\frac{1}{1-x}$

$$= \sum_{n=0} \left(\frac{3^{n+1}+1}{2}\right) x^n$$

b) $a_{n+1} - a_n = n^2, \ n \geq 0, \ a_0 = 1$

$$(f(x) - 1) - x f(x) = \sum_{n \geq 0} n^2 x^n$$

$$f(x)(1 - x) = \sum_{n \geq 0} n^2 x^n + 1$$

$$f(x) = \frac{1}{1-x} \cdot \sum_{n \geq 0} n^2 x^n + \frac{1}{1-x}$$

sum operator

$$f(x) = \sum_{n \geq 0} \frac{n(n+1)(2n+1)}{6} x^n + \sum_{n \geq 0} x^n$$

$$f(x) = \sum_{n \geq 0} \left[\frac{n(n+1)(2n+1)}{6} + 1 \right] x^n$$

c) $a_{n+2} - 3a_{n+1} + 2a_n = 0, \ n \geq 0, \ a_0 = 2$
$a_1 = 6$

$$(f(x) - a_0 - a_1 x) - 3x(f(x) - a_0) + 2x^2 f(x) = 0$$

$$f(x) - 1 - 6x - 3x f(x) + 3x + 2x^2 f(x) = 0$$

$$f(x)(1 - 3x + 2x^2) = 1 + 3x$$

$$f(x) = \frac{1+3x}{1-3x+2x^2} = \frac{1+3x}{(1-x)(1-2x)}$$

Again, $\frac{1+3x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$

$x = 1 \rightarrow A = -4$

$x = \frac{1}{2} \rightarrow B = 5$

$$f(x) = \frac{5}{1-2x} - \frac{4}{1-x}$$

$$= \sum_{n \geq 0} (5 \cdot 2^n - 4)\, x^n$$

d) $a_{n+2} - 2a_{n+1} + a_n = 2^n, \ n \geq 0, \ a_0 = 1$

$a_1 = 2$

$$(f(x) - a_0 - x a_1) - 2x(f(x) - a_0) + x^2 f(x) = \sum_{n \geq 0} 2^n x^n$$

$$f(x) - 1 - 2x - 2x f(x) + 2x + x^2 f(x) = \frac{1}{1-2x}$$

$$f(x)(1-2x+x^2) = \frac{1}{1-2x} + 1$$

$$f(x)(1-x)^2 = \frac{1}{1-2x} + 1$$

$$f(x) = \frac{1}{(1-2x)(1-x)^2} + \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-2x)(1-x)^2} = \frac{4}{1-2x} - \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

So $f(x) = \dfrac{4}{1-2x} - \dfrac{1}{1-x}$

$$= \sum_{n \geq 0} (4 \cdot 2^n - 1) x^n .$$