MM5013 HT24 (January 16th)

(1) Generating functions.

(a) We have done this in class:

We start from the generating function of $b_n = 2$ for all $n \geq 0$

$$\sum_{n=0}^{\infty} 2x^n = 2\sum_{n=0}^{\infty} x^n = \frac{2}{1-x}$$

Now $\frac{2x}{1-x}$ correspond to $c_n = \begin{cases} 0 & n=0 \\ 2 & n>0 \end{cases}$

Observe that $a_n = \sum_{k=0}^{n} c_k$

Thus the generating function of a_n is $\frac{1}{1-x} \cdot \frac{2x}{1-x} = \frac{2x}{(1-x)^2}$

(b) a_n is the sum of the sequence $b_n = \begin{cases} 0 & \text{for } n=0 \\ 1 & \text{for } n=1 \\ 3 & \text{for } n>1 \end{cases}$

the generating function of b_n

is $$x + \frac{3x^2}{(1-x)}$$

thus the generating function of a_n is

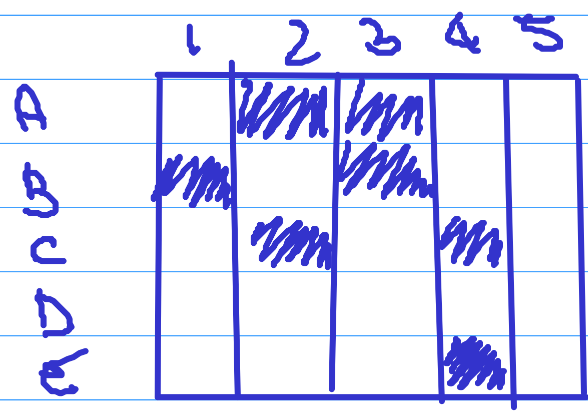
$$\left(x + \frac{3x^2}{(1-x}\right) \frac{1}{1-x}$$

(c) $$c_k = \sum_{n=0}^{k} a_k$$

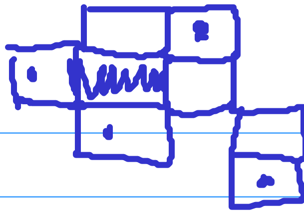
So has gf $$\left(x + \frac{3x^2}{(1-x)}\right) \frac{1}{1-x}$$

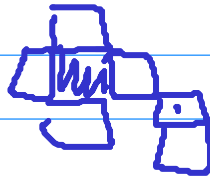
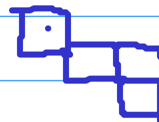
Problem 2 rook poly.

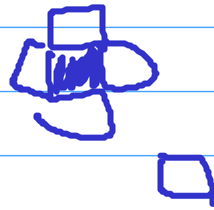
(a)

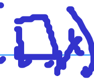
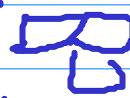


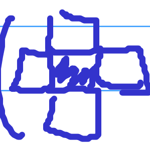
(b) Chessboard of forbidden cells

 $= C$

$r(C) = r\Big($  $x\Big) + x\,r\Big($  $, x\Big)$

$= r\Big($  $x\Big) + x\,r\Big($  $, x\Big)$

$+ x\Big(r($  $) \cdot r\big($  $, x\big)\Big)$

$= (1+x)\,r\Big($  $x\Big) + x\left(1+3x+2x^2\right)$

$+ x\Big((1+x)\cdot\left(1+3x+x^2\right)\Big)$

$$= (1+x)\left(1+4x+6x^2\right) + x+3x^2+2x^3$$

$$+ x\left(1+3x+x^2+x+3x^2+2x^3\right)$$

$$= 1+4x+6x^2+x+4x^2+6x^3+x$$

$$+ \underbrace{3x^2}_{} + \underbrace{2x^3}_{6} + x(1+4x+4x^2+2x^3)$$

$$= 1+6x+13x^2+8x^3+x+4x^2+4x^3+2x^4$$

$$= 1+7x+17x^2+12x^3+2x^4$$

Let C_x be the condition

x is given a wrong job

We want

$|C'_A \cup \cdots \cup C'_E|$ that we can find

with inclusion exclusion

$\Sigma_1 |C_x| = 7 \cdot 4!$

7 ways to assign a wrong job. $4!$ to assign the other jobs

$\Sigma_1 |C_x \cap C_y| = 17 \, 3!$

17 ways to assign 2 wrong jobs and $3!$ ways

to place the othe

$\sum_{t} |C_{x_{i_1}} \cdots C_{x_{i_k}}| = r_k(\text{forbiden } C) \cdot (n-k)!$

the # of ways to assign the jobs is

$5! - 7(4!) + 17(3!) - 12(2!) + 2$

(3) Recursion

Homogeneous probe

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 2, 1$

General solution

$a_n^{(h)} = A \cdot 2^n + B$

For the non homogeneous prob

we have that that solutions

one of the form

$$a_n^{(p)} = Xn\cdot 2^n$$

$$Xn\cdot 2^n = 3X(n-1)2^{n-1} - 2X(n-2)2^{n-2} + 2^n$$

$$2^{n-2}\left(\cancel{4nX} - \cancel{6Xn} + 6X + \cancel{2Xn} - 4\right) = 2^n$$

$$+6X - 4 = 4$$

$$X = \frac{4}{3}$$

General solution of we have then

$$a_n = A\cdot 2^n + B + \frac{4}{3}n\cdot 2^n$$

$$1 = a_0 = A + B$$

$$1 = a_1 = 2A + B + \frac{8}{3}$$

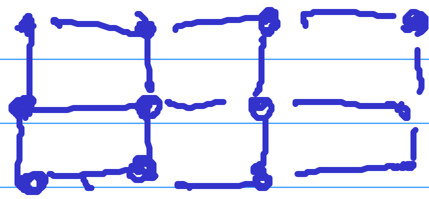
$$-\frac{5}{3} = 2A + B$$

$$-\frac{8}{3} = A \qquad B = \frac{11}{3}$$

The final solution is

$$a_n = -\frac{8}{3} 2^n + \frac{11}{3} + \frac{4}{3} n \cdot 2^n$$

Problem 4

(a) 

(b) there are $q-1$ edges in every row and p rows
$p-1$ edges in cols and q cols

$$|E| = q(p-1) + p(q-1)$$
$$= 2pq - p - q$$

(c) let $v = (1\ a)$ $\quad 1 < a < q$

then there are 3 vertices adjacent to that

$(1\ a+1)$ $\quad (1, a-1)$ and
$(2, a)$

Similarly for $(p\ a)$

and $(b,1)$ $(b\ q)$ for

$1 < b < 1$

Thus we need $q \le 3$ $p \le 3$

and if $q = 3$ then $p = 2$

and if $p = 3$ then $q = 2$

in order for an Euler trail

For the circuit $p, q \le 2$

(c) $(1\,1)$ $(1\,q)$ $(p\,1)$ (p,q)

have degree 2

& $(1,a)$ $(p\ a)$ $(b\ 1)$ $(b\ q)$

have degree 3

all the other ~~one~~ have deg 4

(d) We know that

$$|E(G)| \geq \binom{|V(G)|-1}{2} + 2$$

ensures that G has an hamilton C.

$$pq - p - q \geq \frac{(pq-1)(pq-2)}{2} + 2.$$

Problem 5

Prym with input $\{a\}$

$P = \{a\}$

$P = \{a, d\}$

$P = \{a, d, b\}$

$P = \{a, d, b, c\}$

$P = \{a, d, b, c, f\}$

$P = \{a d b\ c f e\}$

$P = \{a d b\ c f e\ h\}$

$P = \{a d b\ c f e h\ i\}$

$P = \{a$ ———— $i, g\}$

Kruskal {ef}

{ef} {ad}

{ef} {ad} {hi}

{ef} {ad} {hi} {cf}

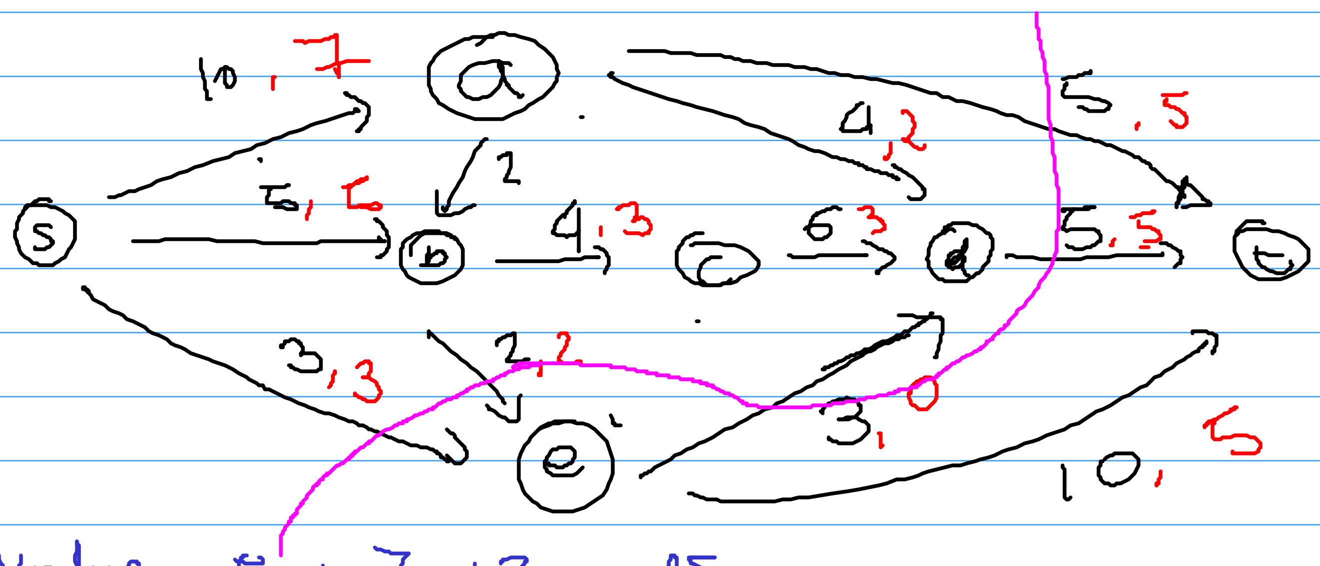
{eh}

{d,b}

{a,b}

{gh}

Problem 6



Value = 5 + 7 + 3 = 15

The cut P = {s, a, b, c, d}

The capacity of the cut

$c(a,e) + c(d,t) + c(b,e) + c(a,t)$

$= 3 + 5 + 2 + 5 = 15$

the capacity of the cut and value of the flow match.