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Maximum Matchings in Graph Products

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Sammanfattning

En matchning inom grafteorin är en delmängd av en grafs kanter, sådan att inga två kanter i matchningen delar någon ändpunkt. En grafs matchningstal definieras som storleken av den största matchningen i grafen. I denna uppsats behandlas största matchningar i relation till tre olika grafprodukter, med fokus på den Cartesiska samt den starka produkten. Vi introducerar faktorinducerade matchningar av en produkt och visar att de alltid har samma kardinalitet, vilket leder till en undre begränsning av matchningstalet av en produkt. Detta leder till en karaktärisering av de produkter vars största matchingar kan erhållas från dess faktorer. De faktorinducerade matchningarna är enbart definierade för den Cartesiska och den starka produkten, men en undre gräns för matchningstalet av en direktprodukt etableras likväl. Slutligen generaliseras resultaten för k-matchningar, vilka definieras som kantmängden till en k-reguljär delgraf av en given graf.

Abstract

A matching of a graph is a subset of vertex-disjoint edges, and a matching is a maximum matching if there is no matching of greater cardinality. The cardinality of a maximum matching is called the matching number of the graph. In this thesis, we investigate the structural properties of maximum matchings with respect to three graph products. We introduce factor induced matchings of a product and show that they always have the same cardinality, which implies a lower bound of the matching number of a product. This results, in particular, in a characterization of maximum matchings in products that can be derived from their factors. The factor induced matchings are only defined for the Cartesian and the strong product, nevertheless a lower bound of the matching number of a direct product is established. As a consequence, we additionally derive a tight lower bound of the matching numbers for all graphs that contain a Cartesian product as (spanning) subgraph. Lastly, the results are generalized for k-matchings, defined as the edge set of a k-regular subgraph.

Contents

1 Introduction

A common problem in graph theory is the study of graph invariants, that is, structural properites of the underlying graphs that can be described in terms of some number. As virtually any other mathematical object, graphs can be combined with several operations, and it is of theoretical and practical interest to understand how some given operation interacts with a specified invariant. This thesis will deal with this kind of problem, where the graph invariant are the so-called maximum matchings and the operation is graph products.

We say that we match a vertex of a graph by choosing one of the vertex's incident edges. Then a maximum matching of a graph is a selection of edges such that as many vertices as possible are matched by exactly one edge. In particular, the matching is perfect if all vertices are matched precisely once. A helpful way to think about it is to see the existence of an edge in the graph as indication that its endpoints are possible to pair up in some sense; then a maximum matching may be interpreted as a way to pair up as many vertices as possible. By way of example, suppose that we have a group of people that we need to pair up for, say, accommodation in 2-person bedrooms. To respect the individual wishes we let everyone hand in a list of persons they are willing to share room with. A suitable model of the problem of finding a way to distribute the group into as few rooms as possible is then a graph where each vertex represent a person, and two vertices are connected if the two persons both agree to share a room with the other. A maximal matching is then a solution to our accommodation problem.

The second component of this thesis are graph products. There are several different ones — Cartesian, direct and strong products — but the underlying idea is the same, namely that there'll be a vertex in the product for each pair of vertices from the factors. Edges in the product are then defined in terms of slightly different conditions. The Cartesian and strong products will be the main focus here, for reasons that will be discussed further on, but the direct product is included briefly as well.

The remainder of this thesis is divided into four sections. In Section 2 we introduce necessary definitions of graphs, graph products and matchings, as well as one earlier known characterization of matchings. The first part of Section 3 states and proves a lower bound of the cardinality of a maximum matching of a Cartesian or strong product of two graphs which, in turn, provides sufficient conditions for perfect matchings. After this, we find necessary conditions and discuss maximum matchings in direct products. In Section 4 a generalization of matchings into k-matchings is made, and corresponding results stated and proved. The last section contains a summary and outlook.

2 Preliminaries

2.1 Basic definitions and properties

We begin with basic definitions of graphs needed throughout the thesis. For further reference, see e.g. [2], [3] or the first chapter of [5].

Definition 2.1. A *simple, undirected graph* G consists of a vertex set V and an edge set

$$E = \{\{v, u\} \mid v \neq u, \text{ and } u, v \in V\},$$

i.e. $E \subseteq \binom{V}{2}$, where $\binom{V}{2}$ denotes the two-element subsets of V. When not explicitly defined, we will use $V(G)$ and $E(G)$ to refer to the vertex and edge set of the graph G. Furthermore, $|V(G)|$ is called the *order* of G, while $|E(G)|$ is called the *size* of G.

We will only consider graphs of finite order. It is also worth noting that the definition above does not allow so-called *loops*, so edges such as $\{u, u\}$ are not allowed. No other types of graphs will be considered, so when G is said to be a graph we always mean a loop-free, undirected and simple graph of finite order.

Graphs with at least two vertices are called *non-trivial*, and the graphs with zero or one vertices are called *trivial*. Notice that there exists an upper bound to the size m of a graph G of order n, namely $m \leq \binom{n}{2}$. The graphs where $m = \binom{n}{2}$ are called *complete* graphs, and they are denoted by K_n.

Definition 2.2. Given a graph $G = (V, E)$, the vertices u and v are *incident* to the edge $\{u, v\}$, and *adjacent* to each other. Two edges e and f are *incident* if they share a vertex, i.e. $e \cap f \neq \emptyset$. The *degree* of a vertex v in V, denoted $\deg_G(v)$ or $\deg(v)$, equals the number of vertices that are adjacent to v. If a vertex has degree zero, it is said to be *isolated*.

A classical observation in graph theory is the following lemma.

Lemma 2.3 The Handshake Lemma**.** *Let $G = (V, E)$ be a graph. The sum of the degree of each vertex in G equals twice the size of G, that is*

$$\sum_{v \in V} \deg v = 2|E|.$$

Suppose we are given the two graphs $G = \big(\{a, b\}, \{\{a, b\}\}\big)$ and $H = \big(\{1, 2\}, \{\{1, 2\}\}\big)$. Is it true that $G = H$? Not really, since the vertices are named differently, but they are still both graphs with two vertices joined by an edge. To address this issue we introduce the following definition.

Definition 2.4. Given two graphs G and H, a bijective mapping $\varphi : V(G) \to V(H)$ is called an *isomorphism* if it satisfies that

$$\{u, v\} \in E(G) \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E(H).$$

If such an isomorphism exists we say that G and H are *isomorphic* and denote this by $G \cong H$.

Definition 2.5. If the graph H satisfies that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ for some graph G we say that H is a *subgraph* of G, denoted as $H \subseteq G$. If H is a subgraph of G such that $V(H) = V(G)$ we call it a *spanning* subgraph.

When there is no risk of ambiguity we sometimes relax the definition of subgraphs somewhat, and state that $H \subseteq G$ if there exists a graph K such that $K \subseteq G$ and $K \cong H$.

Definition 2.6. Let $G = (V, E)$ be a graph. Given a set of vertices $U \subseteq V$ we define the *vertex-induced subgraph* $\langle U \rangle_G$ as the subgraph of G with vertex set U and edge set $\{\{u, v\} \mid \{u, v\} \in E, \text{ and } u, v \in U\}$. Analogously, if $F \subseteq E$ define the *edge-induced subgraph* $\langle F \rangle_G$ as the subgraph of G with edge set F and vertex set $\{v \mid v \in V, \text{ and } \exists e \in F : v \in e\}$. If the context is clear and there is no risk of confusion, we call an edge induced and vertex induced subgraph simply an *induced subgraph* and write $\langle U \rangle$ respectively $\langle F \rangle$, instead of $\langle U \rangle_G$ and $\langle F \rangle_G$.

Definition 2.7. The *path graph* $P_n = (V, E)$ is defined with $V = \{1, 2, \ldots, n\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$. A*path* P in the graph G is a subgraph of G that is isomorphic to some P_n. We may identify P by the sequence of distinct vertices in P, and write $P = v_0 v_1 \ldots v_n$. If $P = v_0 v_1 \ldots v_n$, $v_0 = x$ and $v_n = y$ we call P an *x-y-path*.

The graph $G = (V, E)$ is said to be *connected* if there exists a u-v-path between any pair of distinct vertices $u, v \in V$. If there are at least two distinct vertices where no such path exist, G is *disconnected*. If $H \subseteq G$ is connected and there exist no connected subgraph K such that $H \subsetneq K$ (i.e. H is connected and inclusion-maximal) then H is said to be a *component* of G. An *odd component* (*even component*) is a component where the order of the subgraph is of odd (even) parity.

The path graph P_n will appear in examples. Two other such graphs are the *cycle graph* C_n and the *star graph* S_n. The former have the vertex set $V(C_n) = \{1, 2, \ldots, n\}$ and the latter the vertex set $V(S_n) = \{1, 2, \ldots, n, n+1\}$. The edge sets are

$$E(C_n) = E(P_n) \cup \big\{\{1, n\}\big\} \quad \text{and}$$
$$E(S_n) = \big\{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}, \{1, n+1\}\big\}.$$

Before we introduce the more elaborate graph products, we end this subsection with two other operations on graphs.

Definition 2.8. Given two graphs G and H such that $V(G) \cap V(H) = \emptyset$ we define their *disjoint union* $G + H$ as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

If the vertex sets aren't disjoint, we may, in practise, find a graph K isomorphic to H with $V(G) \cap V(K) = \emptyset$ and define $G + H$ as $G + K$.

Definition 2.9. Given a graph $G = (V, E)$ and a set $X \subseteq V$ we define the graph $G - X$ as the subgraph induced by the vertex set $V \setminus X$, that is $\langle V \setminus X \rangle_G$. If $X = \{v\}$ we simply write $G - v$ instead.

2.2 Graph Products

As mentioned in the introduction, there are several ways to define the product of two graphs. For further reference of the results in this section, see for example [5] or [6].

Definition 2.10. Let G and H be graphs of finite order. We define the *Cartesian product* $G \,\Box\, H$, the *direct product* $G \times H$ and the *strong product* $G \boxtimes H$ as the respective graphs with the vertex sets

$$V(G \,\Box\, H) = V(G \times H) = V(G \boxtimes H) = \{(g, h) \mid g \in V(G),\, h \in V(H)\} = V(G) \times V(H)$$

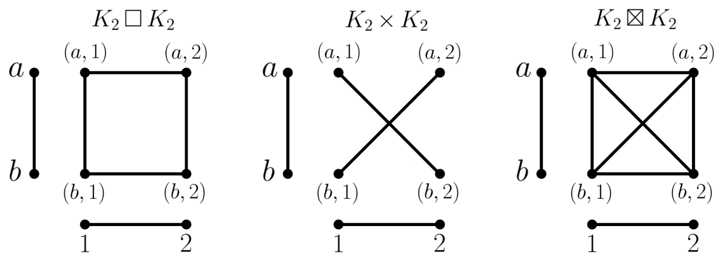

Figure 1: Rationale behind the symbols $\square$, $\times$ and $\boxtimes$.

and the edge sets

$$\begin{aligned} E(G \square H) = \ & \big\{\{(g,h),(g,h')\} \mid \{h,h'\} \in E(H)\big\} \\ & \cup \big\{\{(g,h),(g',h)\} \mid \{g,g'\} \in E(G)\big\}, \end{aligned}$$

$$E(G \times H) = \big\{\{(g,h),(g',h')\} \mid \{h,h'\} \in E(H) \text{ and } \{g,g'\} \in E(G)\big\}$$

and

$$E(G \boxtimes H) = E(G \square H) \cup E(G \times H).$$

Observe that the definition imply that $|V(G \star H)| = |V(G)| \cdot |V(H)|$ for $\star \in \{\square, \times, \boxtimes\}$, something we will use without explicit reference throughout this text. As remarked in [5], the use of the symbols $\square$, $\times$ and $\boxtimes$ has a clear justification, see Figure 1.

Although the graph products are quite elaborate constructions in comparison to, for example, multiplication of numbers it still satisfies some of the algebraic properties we are used to.

Proposition 2.11 [5]**.** *The graph products are commutative and associative up to isomorphism. That is, for* $\star \in \{\square, \times, \boxtimes\}$

$$G_1 \star G_2 \cong G_2 \star G_1, \text{ and}$$
$$(G_1 \star G_2) \star G_3 \cong G_1 \star (G_2 \star G_3).$$

Furthermore, they are distributative over disjoint unions, i.e.

$$G \star (H_1 + H_2) = G \star H_1 + G \star H_2.$$

Proofs of associativity can be found in Proposition 4.1 of [5], whereas commutativity and distributativity are more or less trivial consequences of the respective definitions. A short discussion of why this is the case can be found in section 4.2 and sections 5.1–5.3 of [5].

The associativity allows us to omit the parentheses and write

$$\mathop{\square}_{i=1}^{k} G_i = G_1 \square G_2 \square \ldots \square G_k$$

for the graph with vertex set $V(G_1) \times V(G_2) \times \ldots \times V(G_k)$. The vertices $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ are adjacent in this graph if $\{x_i, y_i\} \in E(G_i)$ for some $i \in \{1, 2, \ldots, k\}$

and $x_j = y_j$ for all $j \in \{1, 2, \ldots, k\} \setminus \{i\}$. Another crucial consequence of the associativity of the Cartesian product is that we can, in most cases, limit ourselves to studying the product of exactly two factors. For example, we may write

$$\mathop{\Box}_{i=1}^{k} G_i = H \,\Box\, K$$

for $H = G_1 \,\Box\, G_2 \,\Box \ldots \Box\, G_l$ and $K = G_{l+1} \,\Box\, G_{l+2} \,\Box \ldots \Box\, G_k$, without loss of generality. The same arguments can be applied to the other two products.

By definition, the vertex set of the graph $G \star H$, with $\star \in \{\Box, \times, \boxtimes\}$, consists of ordered tuples, that is if $v \in V(G \star H)$ then there exists a $g \in V(G)$ and a $h \in V(H)$ such that $v = (g, h)$. We refer to g (h) as the *G-coordinate (H-coordinate) of v*. Fixing either g or h, a certain subgraph of $G \star H$ is obtained.

Definition 2.12. Let $\star \in \{\Box, \times, \boxtimes\}$. Given a vertex $v = (g, h) \in V(G \star H)$, the subgraph

$$G^v = \langle \{(x, h) \mid x \in V(G)\} \rangle_{G \star H}$$

is called the *G-layer through v*. Similarly, the subgraph

$${}^v H = \langle \{(g, y) \mid y \in V(H)\} \rangle_{G \star H}$$

is called the *H-layer through v*.

Immediately from this definition it follows that the vertex $v = (g, h)$ is not unique. In fact, any G-layer is uniquely determined by which vertex $h \in V(H)$ is chosen, but not on $g \in V(G)$. For convenience, we thus simplify the notation from G^v to G^h and from ${}^v H$ to ${}^g H$. This convenient notation is due to [6], where the term fiber is used synonymously to the above definition of a layer.

We also note these important properties of the layers in Cartesian and strong products, for later use.

Lemma 2.13. *Consider the graph $G \star H$, where $\star \in \{\Box, \boxtimes\}$.*

i) ([5, Sec. 4.3]) Any G-layer is isomorphic to G and any H-layer is isomorphic to H.

ii) Let u and v be vertices of G. The layers ${}^u H$ and ${}^v H$ are disjoint (in respect to both vertices and edges) if and only if $u \neq v$.

iii) Let x and y be vertices of H. The layers G^x and G^y are disjoint (in respect to both vertices and edges) if and only if $x \neq y$.

iv) Fix a G-layer and an H-layer. The edge sets of these layers are disjoint, and there is a unique vertex $v \in V(G \,\Box\, H)$ such that the intersection of the layers' vertex sets equals $\{v\}$.

Proof of ii) and iii). By definition of H-layers we have $V({}^u H) = \{(u, h) \mid h \in V(H)\}$ and $V({}^v H) = \{(v, h) \mid h \in V(H)\}$ which clearly are disjoint if and only if $v \neq u$. Since the vertex sets are disjoint, so are the edge sets. Part *iii*) is proven analogously. $\Box$

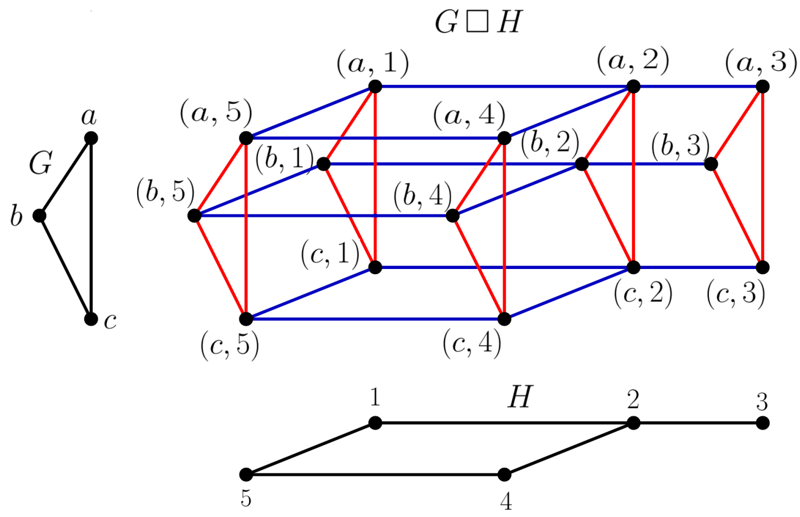

Figure 2: The Cartesian product of two graphs. The factors G and H are depicted alongside their product, and their corresponding layers are highlighted in red and blue, respectively.

Proof of iv). Let $u \in V(G)$ respectively $v \in V(H)$ and consider the layers G^v respectively ${}^uH$. By definition

$$V(G^v) \cap V({}^uH) = \{(g, v) \mid g \in V(G)\} \cap \{(u, h) \mid h \in V(H)\} = \{(u, v)\}.$$

Since the layers do not share more than one vertex, they have no edges in common. $\square$

None of the properties in Lemma 2.13 are true for the direct product $G \times H$ simply because its layers are *totally disconnected* — that is, each G-layer (H-layer) of $G \times H$ consists of $|V(G)|$ ($|V(H)|$) isolated vertices (see [5, Sec. 5.3]).

The concept of layers may also be used when drawing the Cartesian product of two graphs. Draw a copy of G — a G-layer of $G \square H$ —for each vertex in H, then connect corresponding vertices in each layer as edges appear in H. An example of a Cartesian product and its layers is given in Figure 2. The strong product $G \boxtimes H$ can be drawn in a similar manner, then adding in the edges from $G \times H$. However, the drawings of direct and strong products often contain many overlapping edges, resulting in cluttered figures. For this reason, most of our examples will involve Cartesian products.

Another property that distinguishes the direct product from the strong product is this theorem.

Theorem 2.14 [5, Cor. 5.3 and Cor. 5.6]**.** *Let $\star \in \{\square, \boxtimes\}$. The graph $G \star H$ is connected if and only if both G and H are connected.*

Analogously to the case of multiplication of integers we consider *prime graphs*, that is, non-trivial graphs (i.e. graphs with at least two vertices) that cannot be represented as the product of two non-trivial graphs.

Theorem 2.15. *Let G be a non-trivial graph and let $\star \in \{\Box, \times, \boxtimes\}$. Then*

i) there exists a prime factor decomposition of G in regards to the three products, that is, $G = G_1 \star G_2 \star \ldots \star G_k$ where the $k \in \mathbb{Z}_{\geq 1}$ factors G_i are prime graphs. It is not necessarily unique.

ii) If G is connected, then the prime factor decomposition over the Cartesian and strong product is unique, up to isomorphism and the order of the factors.

iii) The respective prime factorizations can be computed in polynomial time.

The existence of (non-unique) prime factorization for the Cartesian product can be found in Theorem 6.1 of [5], and the same proof may be repeated for the direct and strong products. Part ii) is then stated and proved in Theorem 6.6 and Theorem 7.14 of [5]. Note that the uniqueness is not necessary in regards to the direct product. The algorithms for computing the respective prime factorizations are discussed at great length in chapters 23 and 24 of [5].

It is clear that the Cartesian and strong products share some characteristics, whereas the direct product does not. We will rely heavily on Lemma 2.13 in Section 3 and Section 4, which motivates the main focus of the Cartesian and strong products.

A direct consequence of the respective definitions is that both $G \Box H$ and $G \times H$ are spanning subgraphs of $G \boxtimes H$, and $E(G \Box H) \cap E(G \times H) = \emptyset$ for any graphs G and H. Note that this latter fact is true only since we consider graphs without loops, otherwise the edge sets may not be disjoint. We will see how this relationship gives us a reason to still consider the direct product, but first we must introduce the second important concept of this thesis: the maximum matchings.

2.3 Matchings

We introduced the main idea of a matching in the introduction and now make the formal definition, as done in [9].

Definition 2.16. A subset of edges M in a graph G is called a *matching* if no two edges in M share a common vertex. A vertex v is said to be *matched* if some edge in M is incident to v. Similarly, if no edge in M is incident to v, then the vertex is *unmatched*.

Definition 2.17. Suppose M is a matching in G. M is called a *maximum matching* if there are no matchings in G which contain a larger number of edges. The cardinality of such a matching is called the *matching number* of G and is denoted by $m(G)$. Furthermore, M is said to be a *perfect matching* if each vertex in G is matched by M, and it is a *near-perfect matching* if all vertices but one is matched by M. Lastly, define the *unmatching number* of the graph G as the number of vertices left unmatched by any maximum matching. Denote this by $u(G)$.

Note that one should not confuse maximum matchings for *maximal* matchings, since the latter refers to inclusion-maximal matchings, which are not necessarily maximum in terms of cardinality. Observe that maximum matchings does not need to be unique.

Remark 2.18. Only graphs of even order can have a perfect matching. If G satisfies this, then $m(G) = |V(G)|/2$. Similarly, a near-perfect matching can only occur in graphs of odd order, and if such is the case we have that $m(G) = (|V(G)| - 1)/2$.

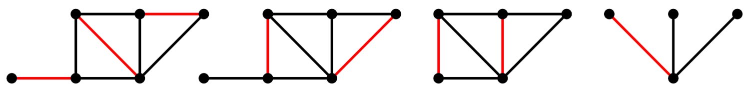

Figure 3: A perfect matching (A), a matching which is not a maximum matching (B), a near-perfect matching (C) and a maximum matching in a graph which has no (near-)perfect matching (D). The graph in A and B has matching number 3 and unmatching number 0. The graph in C has matching number 2 and unmatching number 1. The graph in D has matching number 1 and unmatching number 2.

We use the letters $\mathfrak{m}$ and $\mathfrak{u}$ to allude to the first letter of 'matching number' respectively 'unmatching number'. This is not a standard notation, rather [9] use $\nu(G)$ for the matching number of G and introduce what we call the unmatching number as the *deficiency* of G, denoted as $def(G)$. Furthermore, they use the following relationship as a definition, whereas we state it as a lemma.

Lemma 2.19. *For all graphs G of order n we have*

$$\mathfrak{m}(G) = \frac{n - \mathfrak{u}(G)}{2}.$$

Proof. Put $\mathfrak{m} = \mathfrak{m}(G)$ and $\mathfrak{u} = \mathfrak{u}(G)$. Let M be a maximum matching of G, i.e. $|M| = \mathfrak{m}$. Observe that each edge in M is incident to two distinct vertices in G, all which are matched by M. The other vertices are unmatched by M and thus counted in $\mathfrak{u}$. Therefore

$$n = 2|M| + \mathfrak{u} \quad \Longrightarrow \quad |M| = \mathfrak{m} = \frac{n - \mathfrak{u}}{2}.$$

□

The definitions above are visualized in Figure 3. Other examples include the path graph P_n and the cycle graph C_n which have perfect (near-perfect) matchings when n is even (odd). If $n \geq 3$ the star graph S_n have no perfect nor near-perfect matching.

Let us now make a trivial observation.

Observation 2.20. Let H be a subgraph of G. By definition each matching in H is a matching of G, thus $\mathfrak{m}(G) \geq \mathfrak{m}(H)$.

In particular this means that given the two graphs G and H, we will have that

$$\mathfrak{m}(G \boxtimes H) \geq \max\big(\mathfrak{m}(G \,\square\, H),\, \mathfrak{m}(G \times H)\big).$$

We will discuss possible use of this relationship in Section 3.3.

The existence of perfect matchings in graphs have a characterization stated and proved by W. Tutte in [14].

Theorem 2.21 Tutte's Theorem**.** *Let $c_o(G)$ denote the number of odd components of the graph G. G has a perfect matching if and only if $c_o(G - S) \leq |S|$, for all $S \subseteq V(G)$.*

To show that a graph has a perfect matching we may simply provide one. A clear advantage of Tutte's Theorem is that it implies that it is sufficient to give a subset of vertices S such that the above formula hold, whenever we need to show that a perfect matching does not exist. Such a set is called a *Tutte set*. Obviously, Tutte's Theorem may be applied directly to the product of two graphs but this gives no direct link to the factors. Examples of applications of the theorem can be found in Section 3.2, and we will discuss possible links between Tutte sets in the factors and Tutte sets of their product in Section 5.3.

Using Tutte's Theorem or other methods one may obtain a multitude of sufficient conditions for a graph to have a perfect matching. We will not try to include a comprehensive list of such conditions, but one will be of particular interest later on.

Proposition 2.22 [13, Cor. 2]**.** *If G is a connected graph of even order with no vertex induced subgraph isomorphic to the star graph S_3, then G has a perfect matching.*

Sometimes the star graph S_3 is called a *claw*. With that terminology, it is possible to state the proposition above as "*Every connected, claw-free graph of even order has a perfect matching*".

Lastly, it is reasonable to at least mention that there exists a polynomial-time algorithm for finding a maximum matching of a graph. It is called the *Blossom algorithm*, and its correctness was proved by J. Edmonds in [4].

3 Results

In this section we will explore the possibility of constructing a matching in a graph product from the maximum matchings of the two factors. The existence of such a construction implies both a lower bound of the matching number, and sufficient conditions for perfect and near-perfect matchings of a graph product. We begin with the Cartesian and strong products in Section 3.1, and continue with the direct product in Section 3.3. In section 3.2 we discuss certain necessary conditions for perfect and near-perfect matchings of Cartesian and strong products.

3.1 Lower bound for Cartesian and strong products

Lemma 3.1. *Let G be a graph of order n_G and H be a graph of order n_H. Suppose $\star \in \{\Box, \boxtimes\}$. Then*

$$\mathfrak{m}(G \star H) \geq \max\big(\mathfrak{m}(G)n_H + \mathfrak{m}(H)\mathfrak{u}(G),\ \mathfrak{m}(H)n_G + \mathfrak{m}(G)\mathfrak{u}(H)\big).$$

Proof. We begin by constructing a matching in $G \star H$ with cardinality $\mathfrak{m}(G)n_H + \mathfrak{m}(H)\mathfrak{u}(G)$. Firstly, let M_G and M_H be maximum matchings of G respectively H. Recall that $|M_G| = \mathfrak{m}(G)$ and $|M_H| = \mathfrak{m}(H)$. Define, for all $h \in V(H)$,

$$A_h = \big\{\{(g,h),\, (g',h)\} \mid \{g,g'\} \in M_G\big\}.$$

By construction, each A_h is a matching of the layer G^h and since the layers are subgraphs of $G \star H$, each A_h is a matching of $G \star H$. As observed in Lemma 2.13 the G-layers are pairwisely edge-disjoint, i.e. $E(G^u) \cap E(G^v) = \emptyset$ for all $u, v \in V(H)$ such that $u \neq v$. Since $A_u \subset E(G^u)$ and $A_v \subset E(G^v)$ this means that $V(\langle A_u\rangle_{G\star H}) \cap V(\langle A_v\rangle_{G\star H}) = \emptyset$ for all vertices $u \neq v$ as well. It is easy to verify that the union of two vertex disjoint matchings is another matching, and so the set

$$A = \bigcup_{h \in V(H)} A_h = \big\{\{(g,h),\, (g',h)\} \mid \{g,g'\} \in M_G,\, h \in V(H)\big\}$$

is a matching in $G \star H$. By construction, $|A_h| = |M_G|$ for all $h \in V(H)$. The latter two arguments imply

$$|A| = \sum_{h \in V(H)} |A_h| = \sum_{1}^{n_H} |M_G| = n_H \cdot |M_G| = n_H \mathfrak{m}(G).$$

Now, let U be the subset of vertices of G that are unmatched by M_G, so that $|U| = \mathfrak{u}(G)$. Define, for all $u \in U$,

$$B_u = \big\{\{(u,h),\, (u,h')\} \mid \{h,h'\} \in M_H\big\}.$$

Each B_u is a matching of the layer ${}^uH$, and thus a matching of $G \star H$. As before, the H-layers are disjoint, implying that the set

$$B = \bigcup_{u \in U} B_u = \big\{\{(u,h),\, (u,h')\} \mid u \in U, \{h,h'\} \in M_H\big\}$$

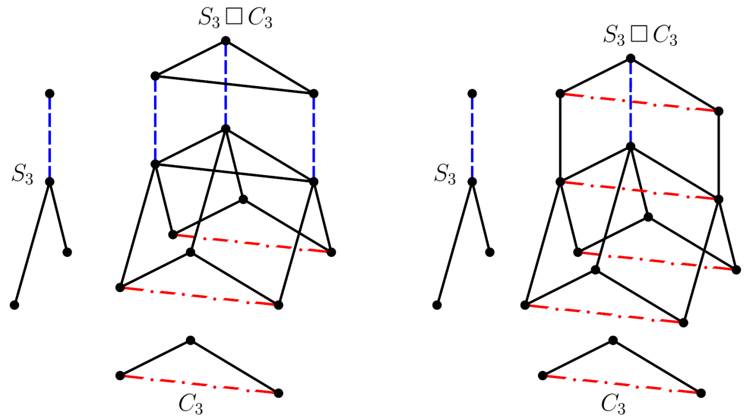

Figure 4: A S_3-induced matching of $S_3 \,\square\, C_3$ with cardinality $m(S_3)n_{C_3} + m(C_3)u(S_4) = 1 \cdot 3 + 1 \cdot 2 = 5$ (left) and a C_3-induced matching of $S_3 \,\square\, C_3$ with cardinality $m(C_3)n_{S_3} + m(S_3)u(C_3) = 1 \cdot 4 + 1 \cdot 1 = 5$ (right).

is another matching of $G \star H$. Its cardinality is

$$|B| = \sum_{u \in U} |B_i| = \sum_{1}^{u(G)} |M_H| = u(G)m(H).$$

By Lemma 2.13, $V(^uH) \cap V(G^h) = \{(u,h)\}$ and $E(^uH) \cap E(G^h) = \emptyset$ for each $u \in U$ and $h \in V(H)$. The vertex u in G is unmatched by M_G, so the vertex $\{(u,h)\}$ is unmatched by A in $G \star H$. In extension, this means that there is no edge in A which is incident to an edge in B. Hence, $A \cup B$ is a matching of $G \,\square\, H$ of cardinality $m(G)n_H + m(H)u(G)$.

By interchanging the roles of M_G and M_H above we obtain another matching, which has size $m(H)n_G + m(G)u(H)$. A maximum matching in $G \star H$ must contain at least as many edges as the largest of these two matchings. Hence

$$m(G \star H) \geq \max\left(m(G)n_H + m(H)u(G),\ m(H)n_G + m(G)u(H)\right).$$

□

In the proof above we constructed a matching in $G \star H$ using the maximum matchings of G and H in two possible ways. One way is to select the edges from each G-layer corresponding to the edges of the matching of the factor G and the edges from the H-layers with a G-coordinate which is not matched by the matching of G. The other option is to start with the H-layers, and then choose edges from the G-layers with a H-coordinate which is not matched by the maximum matching of H. We will refer to the respective construction as a *G-induced matching of $G \star H$* respectively an *H-induced matching of $G \star H$*. An example is clarifying; consider the product $S_3 \,\square\, C_3$ in Figure 4 where both the G-induced matching and the H-induced matching is highlighted. Notice that these matchings have the same cardinality and somewhat surprisingly, this is always the case.

Proposition 3.2. *Let G be a graph of order n_G and H be a graph of order n_H. Then*

$$m(G)n_H + m(H)u(G) = m(H)n_G + m(G)u(H) = \frac{n_G n_H - u(G)u(H)}{2}.$$

Proof. By Lemma 2.19

$$m(G) = \frac{n_G - u(G)}{2} \quad \text{and} \quad m(H) = \frac{n_H - u(H)}{2}.$$

Thus, we obtain

$$\begin{aligned} m(G)n_H + m(H)u(G) &= \frac{n_G - u(G)}{2} \cdot n_H + \frac{n_H - u(H)}{2} \cdot u(G) \\ &= \frac{1}{2}\left(n_G n_H - u(G)n_H + n_H u(G) - u(H)u(G)\right) \\ &= \frac{1}{2}\left(n_G n_H - u(H)u(G)\right). \end{aligned}$$

Similarly,

$$m(H)n_G + m(G)u(H) = \frac{1}{2}\left(n_H n_G - u(H)u(G)\right),$$

and the result follows. □

Since any G- and H-induced matchings of $G \star H$ are equally large, we allow ourselves to refer to any such matching as a *factor induced matching of* $G \star H$*.* These matchings trivially bound the matching number of a product from below.

Corollary 3.2.1. *Let G be a graph of order n_G and H be a graph of order n_H. Suppose $\star \in \{\Box, \boxtimes\}$. Then*

$$m(G \star H) \geq \frac{n_G n_H - u(G)u(H)}{2}.$$

Proof. This is an immediate consequence of Lemma 3.1 and Proposition 3.2. □

The upper bound of the matching number trivially depend on the number of vertices of the product, as seen in Remark 2.18. Thus

$$\frac{n_G n_H - u(G)u(H)}{2} \leq m(G \star H) \leq \frac{n_G n_H}{2}$$

if the order of $G \star H$ is even, and

$$\frac{n_G n_H - u(G)u(H)}{2} \leq m(G \star H) \leq \frac{n_G n_H - 1}{2}$$

if the order of $G \star H$ is odd.

Corollary 3.2.2. *Let G and H be graphs and suppose $\star \in \{\Box, \boxtimes\}$. Then*

$$u(G \star H) \leq u(G)u(H).$$

Proof. By Lemma 2.19

$$m(G \star H) = \frac{n_{G \star H} - u(G \star H)}{2} = \frac{n_G n_H - u(G \star H)}{2}.$$

Inserting this in the inequality from Corollary 3.2.1 we obtain

$$\frac{n_G n_H - u(G \star H)}{2} \geq \frac{n_G n_H - u(G)u(H)}{2} \quad \Longleftrightarrow \quad u(G \star H) \leq u(G)u(H).$$

□

In [6] the kind of relationship occurring in Corollary 3.2.2 is called submultiplicative.

Corollary 3.2.1 also imply the next two propositions, stated and proved in [12], but only in the context of Cartesian products. We state them here and show alternative and simplified proofs.

Proposition 3.3 [12, Lemma 7.]. *Suppose $\star \in \{\Box, \boxtimes\}$. If G has a perfect matching, then $G \star H$ has a perfect matching for all graphs H.*

Proof. If G has a perfect matching, then $u(G) = 0$. Thus Corollary 3.2.1 imply that $m(G \star H) \geq (n_G n_H - 0 \cdot u(H))/2 = n_G n_H/2 = n_{G\star H}/2$, the size of a perfect matching. □

Proposition 3.4 [12, Prop. 17.]. *Suppose $\star \in \{\Box, \boxtimes\}$. If both G and H have near-perfect matchings, then $G \star H$ has a near-perfect matching.*

Proof. If both G and H have near-perfect matchings, then $u(G) = u(H) = 1$, and Corollary 3.2.1 imply that $m(G \star H) \geq (n_G n_H - 1)/2 = (n_{G\star H} - 1)/2$, the size of a near-perfect matching. □

3.2 Characterization of some (near-)perfect matchings

In Section 3.1 we obtained sufficient conditions for (near-)perfect matchings in a Cartesian or strong product. We will now find necessary conditions for a subset of Cartesian and strong products, and thus characterizations of certain (near-)perfect matchings.

Theorem 3.5. *Let $\star \in \{\Box, \boxtimes\}$. G or H has a perfect matching if and only if $G \star H$ has a perfect matching and $u(G \star H) = u(G)u(H)$.*

Proof. By commutativity, we may assume that G has a perfect matching without loss of generality, and so $u(G) = 0$. By Proposition 3.3 there exists a perfect matching of $G \star H$, and thus $u(G \star H) = 0$. Hence

$$u(G \star H) = 0 = 0 \cdot u(H) = u(G)u(H).$$

For the converse, assume $G \star H$ has a perfect matching and that $u(G \star H) = u(G)u(H)$. By assumption $u(G \star H) = 0$, and so $u(G)u(H) = 0$. This means at least one of G and H has a maximum matching with no unmatched vertices, that is, a perfect matching. □

Theorem 3.6. *Let $\star \in \{\Box, \boxtimes\}$. G and H have near-perfect matchings if and only if $G \star H$ has a near-perfect matching and $u(G \star H) = u(G)u(H)$.*

Proof. Assume G and H have near-perfect matchings, so $u(G) = u(H) = 1$. By Proposition 3.4, $G \star H$ has a near-perfect matching, and thus $u(G \star H) = 1$. Hence $u(G \star H) = u(G)u(H)$.

Conversely, assume $G \star H$ has a near-perfect matching and $u(G \star H) = u(G)u(H)$. Thus, $u(G)u(H) = 1$. Recall that the unmatching number is always a non-negative integer and so $u(G) = 1$ and $u(H) = 1$, i.e. both G and H have near-perfect matchings. □

It is not possible to omit the condition $u(G \star H) = u(G)u(H)$. We emphasize this with a couple of examples.

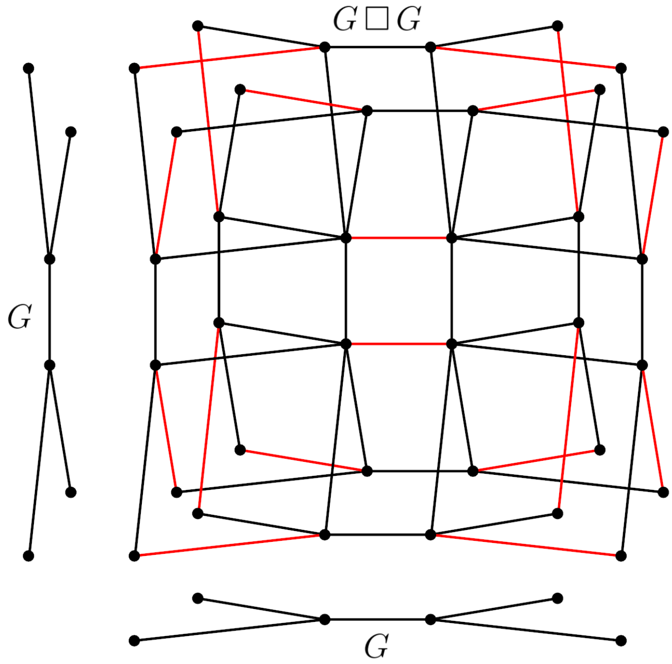

Figure 5: The graph $G \,\square\, G$ and its perfect matching highlighted in red. The two factors G, drawn to the left and below of $G \,\square\, G$, has no perfect matching.

Example 3.1. We may obtain a perfect matching in a product where both factors have even order but no perfect matchings. Consider the graphs shown in Figure 5, where the product $G \,\square\, G$ is drawn with its perfect matching highlighted. Formally, let $G = (V, E)$ be the graph with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and edge set

$$E = \big\{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{4, 6\}\big\}.$$

That G has no perfect matching follows from Tutte's theorem, by choosing S as the set of the two vertices of order 3, obtaining $c_o(G - S) = 4 > |S|$.

Example 3.2. In Figure 6 we see the product $S_3 \,\square\, C_3$ which has a perfect matching, although the star graph S_3 has even order and no perfect matching and the cycle graph C_3 has a near-perfect matching. Compare the perfect matching shown here with the maximal (but not maximum) matchings in Figure 4.

Example 3.3. In Figure 7 we see the product $S_4 \,\square\, C_3$ which has a near-perfect matching. Both factors have odd order, the star graph S_4 has no near-perfect matching and the cycle graph C_3 has a near-perfect matching.

Note that the respective (near-)perfect matchings of the Cartesian products in the three examples are (near-)perfect matchings of the corresponding strong products as well. For example, the matching in Figure 5 is a perfect matching of $G \boxtimes G$ as well.

Furthermore, the examples above highlight that G or (and) H having a perfect (near-perfect) matching is not a *necessary* condition for $G \star H$ to have a perfect (near-perfect)

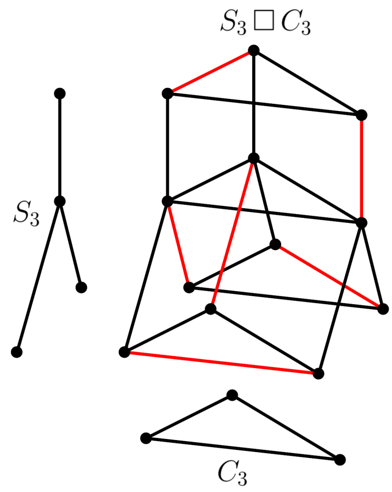

Figure 6: The graph $S_3 \square C_3$ and its perfect matching highlighted in red. The factor S_3 has even order but no perfect matching. The factor C_3 has odd order and a near-perfect matching.

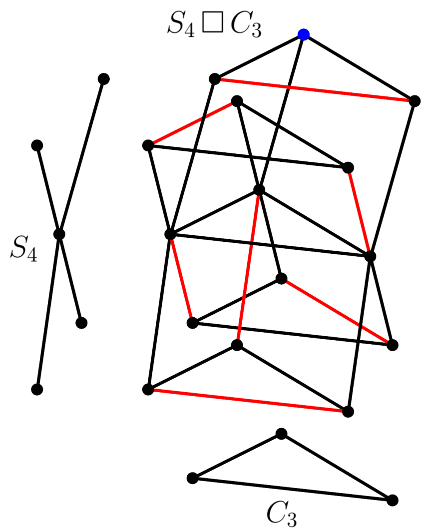

Figure 7: The graph $S_4 \square C_3$ and its near-perfect matching highlighted in red. The factor S_4 has odd order but no near-perfect matching. The factor C_3 has odd order and a near-perfect matching. The unmatched vertex of $S_4 \square C_3$ is highlighted in blue.

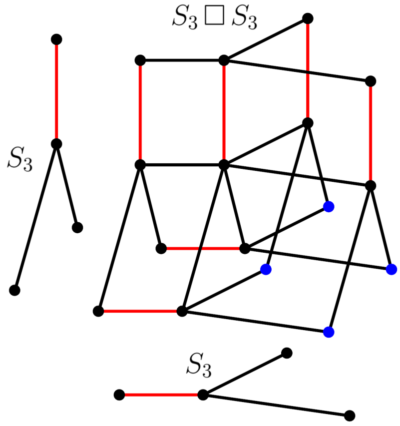

Figure 8: A maximum matching of $S_3 \square S_3$. Matchings of the product and factors are highlighted in red, and the unmatched vertices of $S_3 \square S_3$ are marked with blue. Note that $u(S_3 \square S_3) = 4 = u(S_3)u(S_3)$.

matching, although it is a sufficient condition. Likewise, we cannot keep the condition $u(G \star H) = u(G)u(H)$ and omit that $G \star H$ has a perfect matching. An example suffices — take the product $S_3 \square S_3$ in Figure 8. Here $u(S_3 \square S_3) = 4 = u(S_3)u(S_3)$, but neither S_3 nor $S_3 \square S_3$ have perfect matchings. However, there is more structure to the condition $u(G \star H) = u(G)u(H)$ than one may think.

Proposition 3.7. *Let G and H be graphs and let $\star \in \{\square, \boxtimes\}$. The following statements are equivalent.*

(1) Any factor induced matching of $G \star H$ is a maximum matching of $G \star H$.

(2) $m(G \star H) = m(G)n_H + m(H)u(G)$

(3) $m(G \star H) = m(H)n_G + m(G)u(H)$

(4) $m(G \star H) = \frac{1}{2}\left(n_G n_H - u(G)u(H)\right)$

(5) $u(G \star H) = u(G)u(H)$

Proof. Let M be a factor induced matching of $G \star H$. By Proposition 3.2 we have

$$|M| = m(G)n_H + m(H)u(G) = m(G)n_H + m(H)u(G) = \frac{1}{2}\left(n_G n_H - u(G)u(H)\right).$$

Clearly M is a maximum matching if and only if $m(G \star H) = |M|$ and thus the first four conditions are equivalent.

With no assumptions made on G and H, Lemma 2.19 implies that

$$m(G \star H) = \frac{n_{G \star H} - u(G \star H)}{2} = \frac{n_G n_H - u(G \star H)}{2}.$$

Trivially, statement (4) hold if and only if $u(G \star H) = u(G)u(H)$. $\square$

We will discuss this further in Section 5.4.

The two examples $S_3 \,\square\, S_3$ and $S_3 \,\square\, C_3$ (see figures 4, 6 and 8) are interesting in a shared context as well. Why does $S_3 \,\square\, C_3$ have a larger matching than a factor induced matching, while $S_3 \,\square\, S_3$ does not? We will not answer this question fully, but make an attempt to generalize the situation somewhat.

Proposition 3.8. *Let G be a graph of odd order $n \geq 3$ with $V(G) = \{u_1, u_2, \ldots, u_n\}$. Suppose that G has n distinct near-perfect matchings M_1, M_2, ..., M_n such that u_i is the vertex that is unmatched by M_i for each $i \in \{1, 2, \ldots, n\}$. Let S_n denote the star graph on $n+1$ vertices. Then $S_n \,\square\, G$ has a perfect matching and $S_{n+1} \,\square\, G$ has a near-perfect matching.*

Proof. Let G be a graph satisfying the conditions above. Let S_n be the star graph with $V(S_n) = \{0, 1, \ldots, n\}$ and $E(S_n) = \big\{\{0,1\}, \{0,2\}, \ldots, \{0,n\}\big\}$. Define

$$e_i = \{(0, u_i), (i, u_i)\}$$

and

$$A_i = \{\{(i,u),(i,v)\} \mid \{u,v\} \in M_i\}$$

for all $i \in \{1, 2, \ldots, n\}$. See Figure 9 for a principal sketch of the situation. We claim that the set

$$\left(\bigcup_{i=1}^{n} A_i\right) \cup \{e_i\}_{i=1}^n = A \cup \{e_i\}_{i=1}^n$$

is a perfect matching of $S_n \,\square\, G$. To verify this, first note that both $\{e_i\}_{i=1}^n$ and each A_i are subsets of $E(S_n \,\square\, C_n)$. By definition, each A_i is a near-perfect matching of the layer ${}^{i}G$. Thus Lemma 2.13 imply that the union A is a matching of $S_n \,\square\, G$. Furthermore, the edges of $\{e_i\}_{i=1}^n$ are vertex disjoint, since $(0, u_i) \neq (0, u_j)$ and $(i, u_i) \neq (j, u_j)$ for all $i \neq j$, so $\{e_i\}_{i=1}^n$ is also a matching of $S_n \,\square\, G$.

Now, $(0, u_i) \notin V(\langle A_j \rangle_{S_n \,\square\, G})$ for each $i, j \in \{1, 2, \ldots, n\}$ since no edge of A_j is incident to a vertex with G-coordinate 0. Since the vertex u_i is unmatched by M_i no edge in A_i is incident to the vertex (i, u_i) of e_i either. Using Lemma 2.13 again, we have verified that $A \cup \{e_i\}_{i=1}^n$ is a matching of $S_n \,\square\, G$. Since $|A_i| = |M_i| = (n-1)/2$ by assumption, its cardinality is

$$|\{e_i\}_{i=1}^n| + \sum_{i=1}^{n} |A_i| = n + n \cdot |M_i| = \frac{n(n+1)}{2} = \frac{|V(G)||V(S_n)|}{2} = \frac{|V(S_n \,\square\, G)|}{2},$$

the size of a perfect matching.

The proof of the existence of a near-perfect matching in $S_{n+1} \,\square\, G$ is so similar we omit it.

□

Corollary 3.8.1. *Proposition 3.8 holds if all occurrences of □ is exchanged for ⊠.*

Proof. This follows immediately from Observation 2.20, since $S_n \,\square\, G$ is a spanning subgraph of $S_n \boxtimes G$ and $S_{n+1} \,\square\, G$ is a spanning subgraph of $S_{n+1} \boxtimes G$. □

Corollary 3.8.2. *Let $n \geq 3$ be an odd integer, let $\star \in \{\square, \boxtimes\}$ and consider the cycle graph C_n. Then $S_n \star C_n$ ($S_{n+1} \star C_n$) has a perfect (near-perfect) matching.*

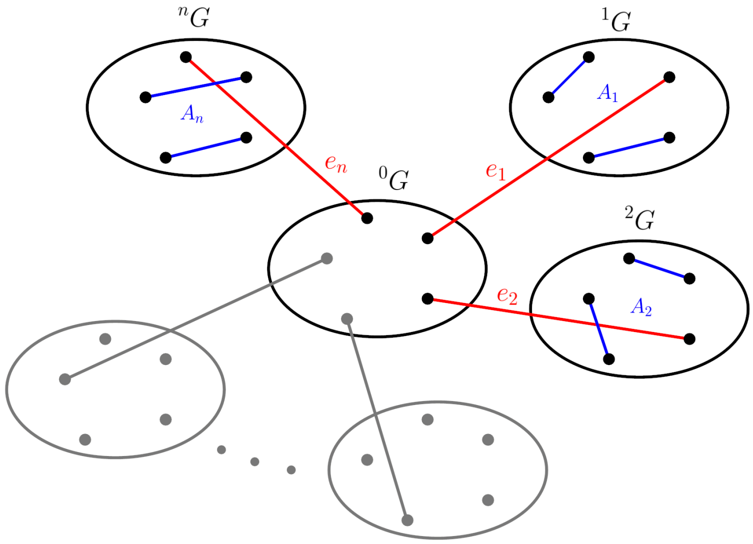

Figure 9: Construction of the perfect matching in the proof of Proposition 3.8. No edges apart from e_1, e_2, ..., e_n and those of A_1, A_2, ..., A_n are shown.

Proof. It suffices to show C_n has n near-perfect matchings that unmatches distinct vertices. This is fairly obvious; given that $V(C_n) = \{1, 2, \dots, n\}$ and $E(C_n) = \big\{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\big\}$ we can, for example, define

$$\begin{aligned}
M_1 &= \big\{\{2,3\}, \{4,5\}, \dots, \{n-1, n\}\big\} \\
M_2 &= \big\{\{3,4\}, \{5,6\}, \dots, \{n-2, n-1\}, \{n, 1\}\big\} \\
M_3 &= \big\{\{1,2\}, \{4,5\}, \dots, \{n-1, n\}\big\} \\
&\ \vdots \\
M_n &= \big\{\{1,2\}, \{3,4\}, \dots, \{n-2, n-1\}\big\}
\end{aligned}$$

which clearly are n near-perfect matchings such that M_i unmatches the vertex i. $\square$

Note that if n is even, then C_n has a perfect matching and so Proposition 3.3 imply that both $S_n \,\square\, C_n$ and $S_{n+1} \,\square\, C_n$ has perfect matchings.

3.3 Lower bound for direct product

It is possible to construct the factor induced matchings of $G \,\square\, H$ and $G \boxtimes H$ only because the respective layers of the products are isomorphic to the respective factor. This is not the case in the direct product $G \times H$; thus we need a different approach.

Lemma 3.9. *Let M_G respectively M_H be matchings of the graphs G and H. The set*

$$M = \big\{\{(g,h),\, (g',h')\} \mid \{g,g'\} \in M_G,\, \{h,h'\} \in M_H\big\}$$

is a matching of $G \times H$.

Proof. By construction, $M \subseteq E(G \times H)$. Let $e = \{(g,h),(u,v)\}$ be an edge in M. We show that if the edge $f \in E(G \times H)$ is incident to e, it is not a member of M. Without loss of generality, assume that e and f share the vertex (g,h) and thus $f = \{(g,h),(x,y)\}$ for some $(x,y) \in V(G \times H)$ with $(x,y) \neq (u,v)$. By definition of the direct product both $\{g,u\}$ and $\{x,g\}$ are edges of G. Since $e \in M$ we must have that $\{g,u\} \in M_G$ which means $\{g,x\} \notin M_G$ (otherwise M_G would not be a matching of G) and therefore $f \notin M$, by construction of M. In extension this means that if e and e' are edges of M, then $e \cap e' = \emptyset$, and so M is a matching of $G \times H$. □

Theorem 3.10. *For all graphs G and H*

$$m(G \times H) \geq 2m(G)m(H).$$

Proof. Let M_G respectively M_H be maximum matchings of the graphs G and H. Define the matching M as in Lemma 3.9. For each $\{g,g'\} \in M_G$ and $\{h,h'\} \in M_H$ there are two corresponding edges in M, namely $\{(g,h),(g',h')\}$ and $\{(g,h'),(g',h)\}$. Thus the cardinality of M equals $2|M_G||M_H|$, and so $|M| = 2m(G)m(H)$. A maximum matching of $G \times H$ must be at least as large as M and so the result follows. □

Corollary 3.10.1. *If both G and H have perfect matchings, then $G \times H$ has a perfect matching.*

Proof. Let G be a graph of order n_G and H a graph of order n_H. If G and H have perfect matchings, then $m(G) = n_G/2$ and $m(H) = n_H/2$. By Theorem 3.10

$$m(G \times H) \geq 2 \cdot \frac{n_G}{2} \cdot \frac{n_H}{2} = \frac{n_G n_H}{2} = \frac{n_{G \times H}}{2},$$

the size of a perfect matching in $G \times H$. □

It is noteworthy that we cannot omit the condition of both factors to have a perfect matching, unlike the case of the Cartesian and strong products (see Proposition 3.3). Take, for example, the graph $P_2 \times P_3$ in Figure 10 which does not have a perfect matching, although the path graph P_2 does. In contrast, $P_2 \,\square\, P_3$ (and $P_2 \boxtimes P_3$) does have a perfect matching.

Seemingly, the lower bound of the direct product is, in some sense, lower than that of the other two products. This is not altogether that surprising, considering that the construction of the matching M in $G \times H$ won't match any vertex where the G-coordinate is unmatched by M_G or where the H-coordinate is unmatched by M_H whereas the factor induced matchings of the Cartesian and strong product at least involves matching a subset of these vertices. The following lemma solidifies this argument.

Lemma 3.11. *Let G be a graph of order n_G and H be a graph of order n_H. Then*

$$\frac{n_G n_H - u(G)u(H)}{2} \geq 2m(G)m(H).$$

Proof. By Proposition 3.2 we have that

$$\frac{n_H n_G - u(H)u(G)}{2} = m(G)n_H + m(H)u(G).$$

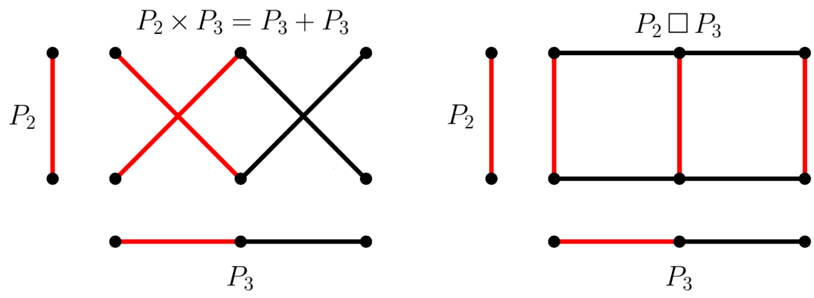

Figure 10: The products $P_2 \times P_3$ and $P_2 \,\square\, P_3$. Maximum matchings are highlighted in red in both products and factors. $P_2 \times P_3$ has no perfect matching, whereas $P_2 \,\square\, P_3$ does.

Both matching number and unmatching number are non-negative integers, so $m(G)n_H + m(H)u(G) \geq m(G)n_H$. By Lemma 2.19 we have $n_H = 2m(H) + u(H)$, and so

$$m(G)n_H = 2m(G)m(H) + m(G)u(H) \geq 2m(G)m(H)$$

concluding that

$$\frac{n_G n_H - u(G)u(H)}{2} \geq 2m(G)m(H).$$

□

Recall from Observation 2.20 that we may bound the matching number of a strong product in terms of the Cartesian and direct product:

$$m(G \boxtimes H) \geq \max\big(m(G \,\square\, H),\ m(G \times H)\big).$$

Thus Lemma 3.11 shows that the lower bound of $m(G \times H)$ does not improve the lower bound

$$m(G \boxtimes H) \geq \frac{n_G n_H - u(G)u(H)}{2}$$

obtained in Corollary 3.2.1. We end this section by generalizing the lower bound of the matching number to a larger class of graphs.

Theorem 3.12. *Let H and K be graphs. Define $\mathcal{G}_{H \square K}$ as the set of all graphs G such that $H \,\square\, K$ is a subgraph of G. Define $\mathcal{G}^s_{H \square K}$ as the subset of $\mathcal{G}_{H \square K}$ where $H \,\square\, K$ is a spanning subgraph of G. If $G \in \mathcal{G}_{H \square K}$, then*

$$m(G) \geq \frac{n_H n_K - u(H)u(K)}{2},$$

where n_H and n_K are the orders of H respectively K. In particular, if $G \in \mathcal{G}^s_{H \square K}$ and $H \,\square\, K$ has a perfect matching, then G has a perfect matching.

Proof. By Corollary 3.2.1 there exists a matching M of $H \,\square\, K$ such that $|M| \geq \frac{1}{2}(n_H n_K - u(H)u(K))$. Now, since $M \subseteq E(H \,\square\, K)$ and $E(H \,\square\, K) \subseteq E(G)$ the set M must be a matching of G as well. Thus the matching number of G is at least as large as the cardinality of M.

If $H \,\square\, K$ is a spanning subgraph with a perfect matching M, then $|M| = n_H n_K/2$ and $|V(H \,\square\, K)| = |V(G)|$. Hence $n_H n_K = n_G$, the order of G, and M is a perfect matching of G as well as of $H \,\square\, K$. □

4 Generalization to k-matchings

To be able to generalize the notion of matchings, we first need the following concept.

Definition 4.1. Let k be a positive integer. A graph G is said to be *k-regular* if $\deg(v) = k$ for all vertices v in G.

Now observe that if we have a matching M of some graph G, and consider the graph $H = \langle M \rangle_G$, then H is a 1-regular subgraph of G. If M happens to be a perfect matching then, by definition, we have that $V(H) = V(G)$ and H is thus a 1-regular *spanning* subgraph of G. This motivates the following definition.

Definition 4.2. Let k be a positive integer. Suppose G is a graph that has some k-regular subgraph H. We define the set $M = E(H)$ to be a k-matching of G. We still say that a vertex v of G is *matched* by M if there are k edges in M incident to v, otherwise v is *unmatched* by M.

As before, M is a *maximum k-matching* if there's no k-matching in G that contains a larger number of edges. If M matches all vertices in G then M is a *perfect k-matching.*

Let M be a maximum k-matching of G, and let $U \subseteq V(G)$ be the set of vertices unmatched by M. Define the *k-matching number* of G as $m_k(G) = |M|$ and the *k-unmatching number* of G as $u_k(G) = |U|$.

If G has no k-matching, then it is reasonable to let $u_k(G) = |V(G)|$ and $m_k(G) = 0$.

Note that by definition, G has a perfect k-matching if and only if it has a k-regular spanning subgraph. In fact, perfect k-matchings are nothing else but what other authors (e.g. [14]) call *k-factors.* Since we are dealing with graph products and their factors the use of the term k-factor would be confusing and ill-advised. There are extensive results on k-factors (see [1] for a comprehensive collection), however they are often concerned with answering the question on whether or not a k-factor exists in a certain setting. We are interested in this as well, but also pose a more general question: at least how large are the maximum k-matchings of the products $G \,\square\, H$ and $G \boxtimes H$? To the best of our knowledge, no such results appeared in literature, so far.

Unlike the matchings in Section 3 — the 1-matchings in our current terminology — there exists perfect k-matchings in graphs of both odd and even order. A couple of examples are provided next.

Example 4.1. As mentioned in Section 3.2 it is easy to see that the cycle graph C_n has a perfect 1-matching if n is even, and a near-perfect matching if n is odd. In contrast, C_n has a perfect 2-matching for all n, namely the full edge set $E(C_n)$.

Example 4.2. The complete graph K_n has a perfect $(n-1)$-matching for each n, namely $E(K_n)$. In particular, K_6 has a perfect k-matching for each $k \in \{1, 2, 3, 4, 5\}$, see Figure 11. K_5, on the other hand, has no perfect 3-matching. If it would, then K_5 would have a 3-regular spanning subgraph and such a graph would have an odd number of vertices of odd degree, violating the Handshaking Lemma.

Example 4.3. Consider the graph G in Figure 12. It is easy to see that $m_1(G) = 2$, $m_2(G) = 4$ and $m_3(G) = 6$. Furthermore, $u_1(G) = u_2(G) = u_3(G) = 1$.

Example 4.4. A trivial necessary condition for the existence of a k-matching in a graph is that each vertex must have degree at least k. For example, $m_k(C_n) = 0$ for all $k \geq 3$, and $m_k(K_n) = 0$ for all $k \geq n$.

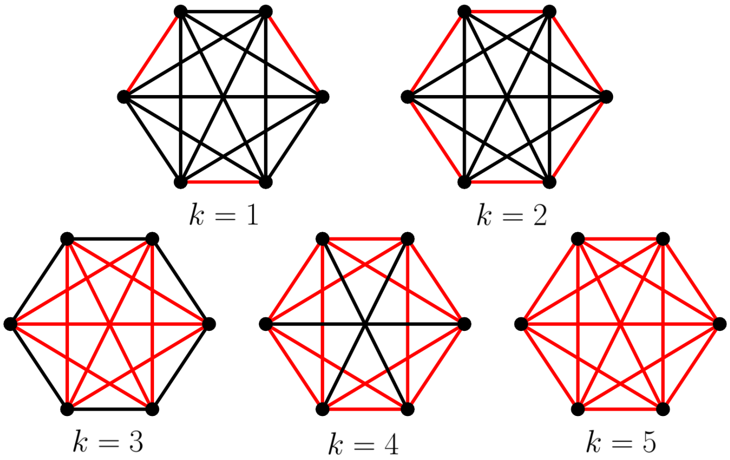

Figure 11: Perfect k-matchings in K_6 for $k \in \{1, 2, 3, 4, 5\}$.

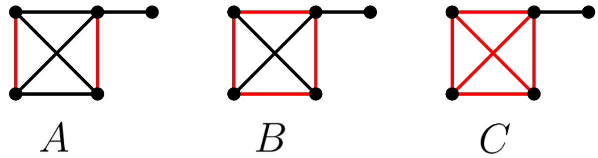

Figure 12: A maximum 1-matching (A), 2-matching (B) and 3-matching (C) of a graph. Neither are perfect.

We begin our discussion of k-matchings with two lemmas.

Lemma 4.3. *For all positive integers k and graphs G of order n we have*

$$m_k(G) = \frac{k(n - u_k(G))}{2}.$$

In particular, G has a perfect k-matching if and only if $m_k(G) = k \cdot n/2$.

Proof. Let M be a maximum k-matching of G and let $H = \langle M \rangle_G$. By definition $m_k(G) = |M| = |E(H)|$. By the Handshaking Lemma

$$2|E(H)| = \sum_{v \in V(H)} \deg v$$

and since H is k-regular $\deg v = k$ for all vertices of H. Hence $2|E(H)| = k|V(H)|$. Now, since vertices of G are either matched or unmatched by M, we have that $|V(G)| = n = |V(H)| + u_k(G)$. Therefore

$$2|E(H)| = k\,(n - u_k(G)) \Leftrightarrow m_k(G) = \frac{k(n - u_k(G))}{2}. \quad (1)$$

By definition, M is a perfect k-matching if and only if $u_k(G) = 0$. Then (1) simplifies to $m_k(G) = k \cdot n/2$.

□

Lemma 4.4. *Suppose that M and M' are k-matchings of the graph G. If the vertex sets of $\langle M\rangle_G$ and $\langle M'\rangle_G$ are disjoint then $M \cup M'$ is a k-matching of G.*

Proof. Assume that the vertex sets of $\langle M\rangle_G$ and $\langle M'\rangle_G$ are disjoint, then their edge sets will also be disjoint. Thus the graph $\langle M\rangle_G + \langle M'\rangle_G$ is a k-regular subgraph of G, and the edge set $M \cup M'$ is a k-matching of G. □

It turns out that the results in Corollary 3.2.1 may be seen as the special case where $k = 1$. We now state this more general version.

Theorem 4.5. *Let G and H be graphs of order n_G and n_H respectively. Let $\star \in \{\square, \boxtimes\}$. For each positive integer k*

$$m_k(G \star H) \geq \frac{k(n_G n_H - u_k(G)u_k(H))}{2}.$$

The proof idea is essentially the same as in Lemma 3.1, where we show the existence of a k-matching of the appropriate size by constructing a sort of factor induced k-matching of $G \star H$. However, the technical details of k-matchings are more extensive. Thus, we first state and prove three lemmas.

Lemma 4.6. *Let G and H be graphs, and let k be a positive integer. Suppose M is a k-matching of G and $v \in V(H)$. Then the set*

$$X = \big\{\{(g, v), (g', v)\} \mid \{g, g'\} \in M\big\}$$

is a k-matching of $G \star H$, where $\star \in \{\square, \boxtimes\}$.

Proof. Since $M \subseteq E(G)$ and $v \in V(H)$ we have, by definition of Cartesian and strong products, that $X \subseteq E(G \star H)$. Thus $\langle X\rangle_{G\star H}$ is a subgraph of $G \star H$. It remains to show that $\langle X\rangle_{G\star H}$ is k-regular. Suppose $(g, v) \in V(\langle X\rangle_{G\star H})$. This vertex (g, v) is adjacent to the vertex (g', v) if and only if $\{g, g'\} \in M$. Hence

$$\deg_{\langle X\rangle_{G\star H}}(g, v) = \deg_{\langle M\rangle_G}(g) = k,$$

where the latter equality holds since $\langle M\rangle_G$ is k-regular. Thus X is a k-matching of $G \star H$. □

We may interchange the roles of G and H in this proof and obtain another k-matching in $G \star H$, since $G \star H \cong H \star G$. This type of k-matching is important enough to deserve its own notation.

Definition 4.7. Let G and H be graphs, let M be a k-matching of G and let $h \in V(H)$. Define the *M-induced k-matching on vertex h* as the set

$$\langle M, h\rangle = \{\{(u, h), (v, h)\} \mid \{u, v\} \in M\}.$$

Analogously, if M is a k-matching of H and $g \in V(G)$, then define the *M-induced k-matching on vertex g* as the set

$$\langle g, M\rangle = \{\{(g, u), (g, v)\} \mid \{u, v\} \in M\}.$$

Notice that these matchings of $G \star H$ are induced from a particular k-matching of G respectively H; they are not equivalent to the factor induced 1-matching introduced in Section 3.1.

Subsequently, we assume that k is a positive integer.

Lemma 4.8. *Let M_G (M_H) be a k-matching of G (H) and suppose $\star \in \{\square, \boxtimes\}$. Let $A \subseteq V(H)$ and $B \subseteq V(G)$. The sets*

$$M' = \bigcup_{v \in A} \langle M_G, v\rangle \quad \text{and} \quad M'' = \bigcup_{v \in B} \langle v, M_H\rangle$$

are both k-matching of $G \star H$. Furthermore, $|M'| = |M_G| \cdot |A|$ and $|M''| = |M_H| \cdot |B|$.

Proof. By Lemma 4.6 $\langle M_G, v\rangle$ is a k-matching of $G \star H$ for each $v \in A$. More so, $\langle M_G, v\rangle$ is a k-matching of the layer G^v. Recall that the G-layers are vertex disjoint, and so the subgraphs of $G \star H$ with edge set $\langle M_G, u\rangle$ and $\langle M_G, v\rangle$ respectively have distinct vertex sets for all $u, v \in A$ such that $u \neq v$. This means that M' is a union of vertex disjoint k-matchings of $G \star H$, so by Lemma 4.4 M' is a k-matching of $G \star H$.

Note that, by definition, $|\langle M_G, v\rangle| = |M_G|$ for each $v \in A$. We thus have that

$$|M'| = \sum_{v \in A} |M_G| = |A| \cdot |M_G|.$$

The proof for M'' is done analogously. □

Lemma 4.9. *Let M_G (M_H) be a k-matching of G (H). Suppose $U \subseteq V(G)$ is the set of vertices that are unmatched by M_G. Let*

$$M' = \bigcup_{v \in V(H)} \langle M_G, v\rangle \quad \text{and} \quad M'' = \bigcup_{v \in U} \langle v, M_H\rangle.$$

Then the set

$$\widetilde{M} = M' \cup M''$$

is a k-matching of $G \star H$, where $\star \in \{\square, \boxtimes\}$.
The cardinality of $\widetilde{M}$ is $|M_G| \cdot |V(H)| + |M_H| \cdot |U|$.

Proof. By Lemma 4.8 both M' and M'' are k-matchings of $G \star H$. We show that $\langle M'\rangle_{G\star H}$ and $\langle M''\rangle_{G\star H}$ are vertex disjoint. Assume (x, y) is a vertex of $\langle M'\rangle_{G\star H}$. There exists exactly k edges in M' that matches (x, y) and k edges in M_G that matches x. Most importantly, x is not unmatched by M_G and thus not an element of the set U. Hence $(x, y) \neq (u, v)$ for all vertices (u, v) in $\langle M''\rangle_{G\star H}$, and the two subgraphs have disjoint vertex sets. By Lemma 4.4 $\widetilde{M}$ is a k-matching of $G \star H$.

Lemma 4.8 implies that $|M'| = |V(H)| \cdot |M_G|$ and $|M''| = |U| \cdot |M_H|$. Hence $|\widetilde{M}| = |V(H)| \cdot |M_G| + |U| \cdot |M_H|$. □

We are now in position to prove the main result of this section, that is, theorem 4.5.

Proof of theorem 4.5. We are proving that

$$m_k(G \star H) \geq \frac{k(n_G n_H - u_k(G)u_k(H))}{2},$$

for all graphs G and H. Let M_G be a maximum k-matching of G and M_H a maximum k-matching of H. Note that $|M_G| = m_k(G)$ and $|M_H| = m_k(H)$. Also, with U as the set of unmatched vertices of M_G, then $|U| = u_k(G)$. Denote the order of H by n_H. By Lemma 4.9 there exists a k-matching of $G \star H$ of size $n_H m_k(G) + u_k(G) m_k(H)$. A maximum matching of $G \star H$ must contain at least as many edges, and so

$$m_k(G \star H) \geq n_H m_k(G) + u_k(G) m_k(H).$$

Now, by Lemma 4.3 we have that

$$m_k(G) = \frac{k(n_G - u_k(G))}{2} \quad \text{and} \quad m_k(H) = \frac{k(n_H - u_k(H))}{2},$$

and so

$$\begin{aligned} n_H m_k(G) + u_k(G) m_k(H) &= n_H \frac{k(n_G - u_k(G))}{2} + u_k(G) \frac{k(n_H - u_k(H))}{2} \\ &= \frac{k(n_H n_G - n_H u_k(G) + u_k(G) n_H - u_k(G) u_k(H))}{2} \\ &= \frac{k(n_G n_H - u_k(G) u_k(H))}{2}. \end{aligned}$$

Thus

$$m_k(G \star H) \geq \frac{k(n_G n_H - u_k(G) u_k(H))}{2},$$

which finishes the proof. □

What we constructed in Lemma 4.9 is nothing but a generalized notion of the G-induced 1-matching from Section 3. Due to associativity, we may interchange the roles of the factors G and H and obtain an H-induced k-matching. Once again, the cardinality of any G-induced k-matching equals the cardinality of any H-induced k-matching, that is, if M is a G-induced k-matching and M' an H-induced k-matching of $G \star H$, then

$$|M| = n_H m_k(G) + u_k(G) m_k(H), \quad |M'| = n_G m_k(H) + u_k(H) m_k(G)$$

and $|M| = |M'| = k \cdot (n_G n_H - u_k(G) u_k(H))/2$. We thus allow ourselves to refer to *factor induced k-matchings*, analogous to the case when $k = 1$.

That G having a perfect k-matching is a sufficient condition for $G \,\square\, H$ having a perfect k-matching was stated and proved in Theorem 4 of [8]. We have the following, stronger result.

Theorem 4.10. *Let G and H be graphs, let k be a positive integer and assume $\star \in \{\square, \boxtimes\}$.*

G or H has a perfect k-matching if and only if $G \star H$ has a perfect k-matching and $u_k(G \star H) = u_k(G) \cdot u_k(H)$.

Proof. Without loss of generality, assume that G has a perfect k-matching. Then $u_k(G) = 0$. By theorem 4.5 $G \star H$ will then satisfy that

$$m_k(G \star H) \geq \frac{k n_G n_H}{2} = \frac{k \cdot n_{G \star H}}{2}.$$

By Lemma 4.3, $G \star H$ thus has a perfect k-matching. Trivially, we have that $u_k(G \star H) = 0 = u_k(G) \cdot u_k(H)$.

Now assume that $G \star H$ has a perfect k-matching and that $u_k(G \star H) = u_k(G) \cdot u_k(H)$. Due to the perfect k-matching we have that $u_k(G \star H) = 0$, but then either $u_k(H) = 0$ or $u_k(G) = 0$, and one of the factors must have a perfect k-matching. □

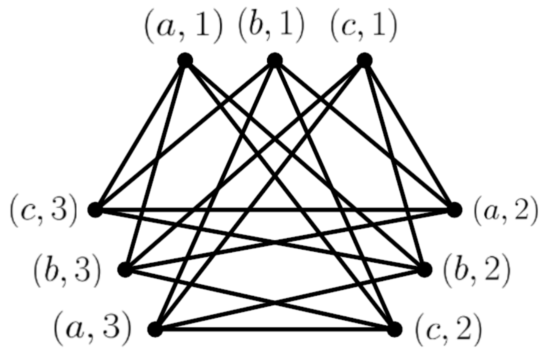

Figure 13: The product $C_3 \times C_3'$. Here C_3 is the graph with vertex set $V(C_3) = \{a, b, c\}$ and edge set $E(C_3) = \{\{a, b\}, \{b, c\}, \{c, a\}\}$ while C_3' is the graph with vertex set $V(C_3') = \{1, 2, 3\}$ and edge set $E(C_3') = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. Clearly, $C_3 \cong C_3'$.

Analogous to Proposition 3.7 we also have this result. The proof is almost identical, so we omit it.

Proposition 4.11. *Let G and H be graphs and let $\star \in \{\square, \boxtimes\}$. The following statements are equivalent.*

(1) Any factor induced k-matching of $G \star H$ is a maximum k-matching of $G \star H$.

(2) $m_k(G \star H) = m_k(G) n_H + m_k(H) u_k(G)$

(3) $m_k(G \star H) = m_k(H) n_G + m_k(G) u_k(H)$

(4) $m_k(G \star H) = \frac{1}{2}\left(n_G n_H - u_k(G) u_k(H)\right)$

(5) $u_k(G \star H) = u_k(G) u_k(H)$

We have now generalized the results for Cartesian and strong products in Section 3. However, the results obtained for the direct product will not necessarily hold for k-matchings when $k \neq 1$. We show this by example. Recall that we constructed the 1-matching $M = \{\{(g, h), (g', h')\} \mid \{g, g'\} \in M_G, \{h, h'\} \in M_H\}$ in $G \times H$ from the 1-matching M_G of G and the 1-matching M_H of H. Consider the product $C_3 \times C_3$ in Figure 13. As shown in Example 4.1, the factor C_3 has a perfect 2-matching, namely $E(C_3)$. Define

$$X = \left\{\{(g, h), (g', h')\} \mid \{g, g'\} \in E(C_3), \{h, h'\} \in E(C_3)\right\}.$$

By definition of direct products, $X = E(C_3 \times C_3)$ — but the full edge set of $C_3 \times C_3$ is not a 2-matching since $C_3 \times C_3$ is not a 2-regular graph.

5 Summary and outlook

In this thesis we've investigated connections between maximum k-matchings of the graphs G and H and the maximum k-matching of their product $G \star H$, where $\star$ primarily denote the Cartesian product $\square$ or the strong product $\boxtimes$. We have mostly considered factor induced k-matchings and showed that

$$m_k(G \star H) \geq \frac{k\,(n_G n_H - u_k(G)u_k(H))}{2}$$

for $\star \in \{\square, \boxtimes\}$. This lower bound of the k-matching number arises from the surprising fact that any G-induced k-matching of $G \star H$ have the same cardinality as any H-induced k-matching. Furthermore, from the lower bound of the k-matching number we've deduced sufficient criteria for the existence of perfect k-matchings in the Cartesian and strong product of G and H. We also derived that $m_1(G \times H) \geq 2m_1(G)m_1(H)$, and remarked on why this is not necessarily the case for $k \neq 1$.

The topic of this thesis raises a couple of follow-up questions, which we will discuss in this section.

5.1 Is there a relation between $m_k(G \square H)$ and $m_k(G \boxtimes H)$?

Virtually all results in Section 3 are stated for the Cartesian product alongside the strong product. We have not discussed how the results for the Cartesian product are interrelated to those for the strong product. Since $G \square H \subseteq G \boxtimes H$ we trivially have $m_k(G \square H) \leq m_k(G \boxtimes H)$. When does equality hold? There are cases when $m_k(G \square H) < m_k(G \boxtimes H)$: see Figure 14. There is no perfect 1-matching in the product $S_3 \square P_3$ whereas the graph $S_3 \boxtimes P_3$ has a perfect 1-matching.

5.2 Claw-free graphs

As mentioned in both the preliminaries and in the beginning of Section 4 there exists a large set of sufficient conditions implying existence of perfect k-matchings (i.e. k-factors) in a graph. Using Theorem 4.10 we may utilize these conditions for all Cartesian and strong products as well. More precisely, suppose C is a condition on G such that

$$C(G) \text{ is satified } \Rightarrow G \text{ has a perfect } k\text{-matching.}$$

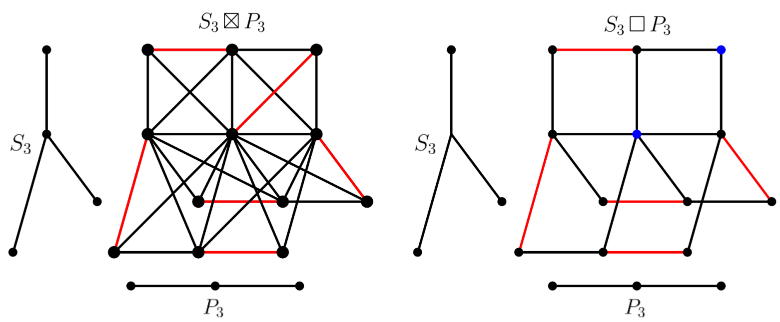

Figure 14: The two graphs $S_3 \boxtimes P_3$ and $S_3 \square P_3$. The former has a perfect 1-matching, as indicated in red. $S_3 \square P_3$ has no perfect 1-matching.

Principally, we then have that

$$C(G) \text{ is satisfied } \Rightarrow \begin{cases} G \,\square\, H \text{ has a perfect } k\text{-matching for all graphs } H \\ G \boxtimes H \text{ has a perfect } k\text{-matching for all graphs } H. \end{cases}$$

Limiting ourselves to 1-matchings, one particular condition comes to mind, namely that in Proposition 2.22. Recall that a graph is claw-free if it does not contain an induced subgraph isomorphic to the star graph S_3 (called the claw graph). The proposition then states that connected, claw-free graphs of even order have a perfect (1-)matching. With the examples $S_3 \,\square\, S_3$ and $S_3 \,\square\, C_3$ (discussed in Section 3.2, see also Example 3.1 in Figure 5) in mind it would indeed be interesting to understand exactly what happens in general. On one hand, what happens if both factors are of even order, connected and contains claws? What if one factor is of even order, connected and contains a claw and the other is of odd order? This could at least clarify the situation for products of connected graphs. Similar case distinctions are possible to study for each existing sufficient condition.

5.3 When is $u_k(G \star H) = u_k(G)u_k(H)$?

Let $\star \in \{\square, \boxtimes\}$. In Theorem 4.10 we concluded that $G \star H$ has a perfect k-matching and $u_k(G \star H) = u_k(G)u_k(H)$ if and only if G or H has a perfect k-matching. It is no obvious task to describe which graphs that satisfy $u_k(G \star H) = u_k(G)u_k(H)$ though. When is the unmatching number multiplicative?

For simplicity, we limit this discussion to 1-matchings, henceforth called matchings. One possible method is to try to use *The Berge Formula*, a theorem tightly connected to Tutte's Theorem. Recall that $c_o(G)$ denotes the number of odd components of the graph G. Berge's formula ([9, Thm. 3.1.14]) then states that

$$u(G) = \max \{c_o(G - S) - |S| \,:\, S \subseteq V(G)\}.$$

Recall that Tutte's Theorem states that G has a perfect matching if and only if $c_o(G-S) \leq |S|$ for all $S \subseteq V(G)$. For the (Cartesian or strong) product of two graphs we then have

$$u(G \star H) = \max \{c_o(G \star H - T) - |T| \,:\, T \subseteq V(G \star H)\},$$

where at least some sets $T \subseteq V(G \star H)$ can be constructed from $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Is it possible to find a connection between $c_o(G \star H - T)$, $c_o(G - S_G)$ and $c_o(G - S_H)$ respectively $|T|$, $|S_G|$ and $|S_H|$? There may not exist such a connection at all and if it does, we suspect it'll be hard to find.

We motivate this suspicion with an example. First consider the graph $S_3 \,\square\, S_3$. Let $T \subseteq V(S_3)$ be the set $T = \{c\}$ where c is the only vertex of degree three, and then define

$$T' = \{(v, w) \mid v \in T,\, w \in V(S_3) \setminus T\} \cup \{(v, w) \mid v \in V(S_3) \setminus T,\, w \in T\}.$$

As shown in Figure 15, $c_o(S_3 - T) = 3 > 1 = |T|$ and $c_o(S_3 \,\square\, S_3 - T') = 10 > 6 = |T'|$, implying that neither S_3 nor $S_3 \,\square\, S_3$ have perfect matchings. Now consider the graph $G \,\square\, G$ in Example 3.1 (see Figure 5). Recall that $G \,\square\, G$ has a perfect matching, although G doesn't. For G we may, for example, choose the set S to contain one of the vertices of degree three. Then $c_o(G - S) = 3 > 1 = |S|$, and Tutte's theorem implies the non-existence of a perfect matching in G. We will not find such a Tutte set for $G \,\square\, G$ (since it has a perfect matching). In particular, the set

$$S' = \{(v, w) \mid v \in S,\, w \in V(G) \setminus S\} \cup \{(v, w) \mid v \in V(G) \setminus S,\, w \in S\}$$

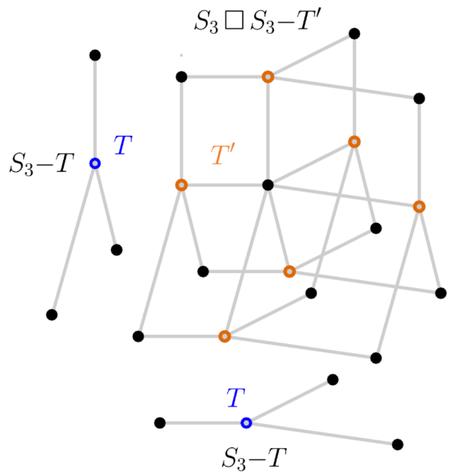

Figure 15: Tutte sets of S_3 and $S_3 \,\square\, S_3$. The edges drawn in gray are those removed from the respective graphs — they are not part of $S_3 - T$ and $S_3 \,\square\, S_3 - T'$. Thus $c_o(S_3 - T) = 3 > 1 = |T|$ and $c_o(S_3 \,\square\, S_3 - T') = 10 > 6 = |T'|$. See the text for further details.

gives $c_o(G \,\square\, G - S') = 10 \leq 10 = |S'|$, as seen in Figure 16. Although S and T resemble each other (same cardinality and $c_o(G - S) = c_o(S_3 - T)$) and T' is defined analogous to S', we still obtain quite different results.

Apart from Tutte's Theorem there is a characterization of matchings which we have not included to keep the thesis compact; it is called the Gallai-Edmonds decomposition. It is indeed more complicated than Tutte's Theorem, which is the reason why we excluded it earlier on. For now we will only state a part of the theorem, as it will suffice for us. See Theorem 3.2.1 in [9] for the full statement and proof. Given a graph G, define $U(G) \subseteq V(G)$ as the set of vertices that are unmatched by at least one maximum matching of G and let

$$A(G) = \{v \mid v \in V(G) \setminus U(G), \exists\, u \in U(G) \text{ s.t. } v \text{ and } u \text{ are adjacent}\}.$$

Denote the number of components of the subgraph $H \subseteq G$ with $c(H)$. Then the *Gallai-Edmonds Structure Theorem* states that

$$m(G) = \frac{|V(G)| + |A(G)| - c(\langle U(G)\rangle_G)}{2}.$$

Possibly, it would be interesting to examine relationships between $A(G)$, $A(H)$, $A(G \star H)$, $c(\langle U(G)\rangle_G)$, $c(\langle U(H)\rangle_H)$ and $c(\langle U(G \star H)\rangle_{G\star H})$, hoping for a characterization of when $u(G \star H) = u(G)u(H)$.

Lastly, by Proposition 4.11 the condition $u_k(G \star H) = u_k(G)u_k(H)$ is equivalent to a number of other conditions. In particular, $u_k(G \star H) = u_k(G)u_k(H)$ if and only if any factor induced matching of a product is a maximum matching, which links the discussion of this subsection to the examples $S_3 \,\square\, S_3$ and $S_3 \,\square\, C_3$ in Section 5.2.

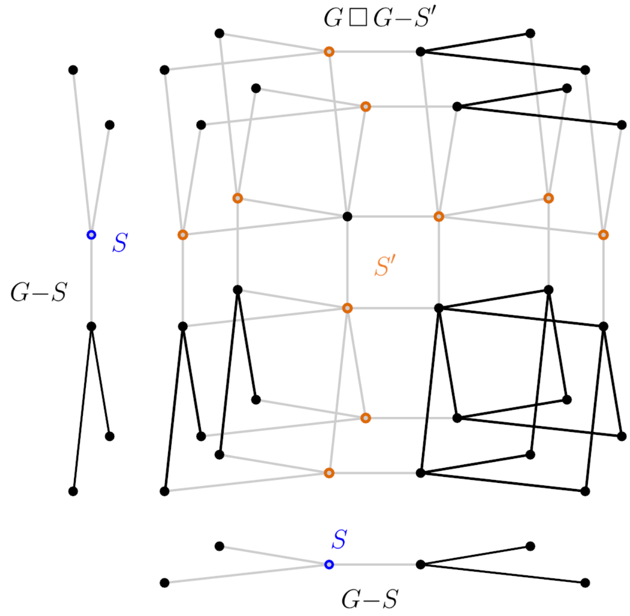

Figure 16: A Tutte set S of the graph G, and a vertex subset S' of $G \Box G$ which is not a Tutte set. The edges drawn in gray are those removed from the respective graphs — they are not part of $G - S$ and $G \Box G - S'$. Notice that $c_o(G - S) = 3 > 1 = |S|$ and $c_o(G \Box G - S') = 10 \leq |S'|$. See the text for further details.

5.4 What algorithmic applications could future results and ours have?

We end this section with a short discussion of algorithmic applications of the topics discussed in this thesis. We assume that the reader has some previous knowledge of algorithms and time complexity.

As mentioned in the preliminaries, there is a polynomial time algorithm for computing a maximum 1-matchings of a graph. The blossom algorithm for finding maximum matchings runs in $O(n^4)$ time (see [9, Thm.9.1.8, Table 9.1.1]) for a graph of order n. This algorithm was improved significantly by in S. Micali and V. Vazirani in [11], and the fastest known runtime is now $O(m\sqrt{n})$ for a graph with n vertices and m edges. For simplicity, let us assume we have two graphs G and H of equal order n and equal size m ($G \cong H$ is not required). To compute a factor induced matching in $G \star H$, where $\star \in \{\Box, \boxtimes\}$, we first find a maximum matching of both G and H in $O(m\sqrt{n}) + O(m\sqrt{n}) = O(m\sqrt{n})$ time. After this, the factor induced matching of $G \star H$ must be constructed by some algorithm running in $f(n)$ time, obtaining a complete runtime of $O(m\sqrt{n}) + f(n)$. Trivially, $f(n) \in O(n^2)$, since such a construction is possible by iterating over the edges in the maximum matching of G and the vertices of H (and vice versa). We omit to construct such an algorithm here.

By definition of the Cartesian product, $|V(G \Box H)| = |V(G)||V(H)| = n^2$ and $|E(G \Box H)| = |V(G)||E(H)| + |V(H)||E(G)| = 2nm$. If a maximum matching of $G \Box H$ is computed with the algorithm we thus obtain the runtime $O(2nm\sqrt{n^2}) = O(n^2 m)$. The expected runtime $O(m\sqrt{n}) + O(n^2) = O(n^2)$ obtained by calculating the factor induced matching is significantly better than this, indicating that it is more efficient to calculate the factor induced matching of the product, than it is to calculate the maximum matching directly. The difference is even greater for $G \boxtimes H$, since this graph has more edges than $G \,\square\, H$. However, the factor induced matching is only maximum if $u(G \star H) = u(G)u(H)$, further motivating the topic discussed in Section 5.3.

In some cases, it would be even more practical if we could first find the factors of a given graph G, and then find a factor induced matching. Recall that there exist polynomial time algorithms for finding prime factorization (over all three products). The fastest known algorithm for recognizing Cartesian products is presented in [7], with a runtime of $O(n \log n)$ for a graph with n vertices[1]. Whether or not this technique is viable depend on how the time complexity of the combined runtime of prime factorization, maximum matching calculations for each prime factor and the construction of a factor induced matching compare to the time complexity of computing a maximum matching of G directly.

Since we introduced k-matchings where $k \neq 1$ in this thesis, there are no known algorithms for computing them. However, there is an algorithm for computing perfect k-matchings described in [10]. It is worth investigating if this algorithm may be extended to an algorithm for computing maximum k-matchings in general, as well as perfect k-matchings.

[1] It is more time consuming to find prime factorizations of strong and direct products, and there are some restrictions that need to be made; see Chapter 24 of [5] for details.

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