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Solutions to written exam
Logic II
Fall term 2019
2020-01-15

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1. *Solution*:
   Observe that in the empty language an elementary embedding between two infinite structures is the same as an injective function.

   (a) An example is the inclusion of $\mathbb{Q}$ into $\mathbb{R}$ in the empty language. A more exotic example can be found using the upward Löwenheim-Skolem Theorem.

   (b) As mentioned before, any elementary embedding is injective. Hence if $f : M \to N$ is an elementary embedding, $|M| \leq |N|$, so this case is impossible.

   (c) Taking the empty language again, the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ suffices. A more exotic example is the subset inclusion of $\langle \mathbb{Q} \setminus \{0\}, < \rangle$ into $\langle \mathbb{Q}, < \rangle$ in the language $\mathcal{L} = \{<\}$.

2. *Solution*:

   (a) $|\mathbb{R}^{<\omega}| = |\cup_{n<\omega} \mathbb{R}^n| = \sup(\sup\{|\mathbb{R}^n| \mid n < \omega\}, |\omega|) = |\mathbb{R}|$.

   (b) Consider the map $\varphi : P((0,1)) \to P(\mathbb{R})$ given by $A \mapsto A \cup (1, \infty)$. It is injective and has the property that for all $A \in P((0,1))$, $|\varphi(A)| = |\overline{\varphi(A)}| = |\mathbb{R}|$, since one contains $(1, \infty)$, while the other contains $(-\infty, 0)$. For any $A \in P((0,1))$, let θ_A be a bijection from $\varphi(A) \to \overline{\varphi(A)}$. Now define a map $f : P((0,1)) \to \mathrm{Bij}(\mathbb{R}, \mathbb{R})$ as by sending A to the function $g : \mathbb{R} \to \mathbb{R}$ given by swapping $\varphi(A)$ with $\overline{\varphi(A)}$, i.e. $g(x) = \theta_A(x)$ if $x \in \varphi(A)$ and $g(x) = \theta_A^{-1}(x)$ if $x \in \overline{\varphi(A)}$. This shows that $|\mathbb{R}| < |P(\mathbb{R})| = |P((0,1))| \leq |\mathrm{Bij}(\mathbb{R}, \mathbb{R})|$.

   (c) A monotone function is a function that either never increases are never decreases. Given any subset of $\mathbb{N}$, we can define a monotone increasing function that enumerates its elements (if the subset is finite, we can remain constant once we have hit the last element). This means that $\mathrm{Mon}(\mathbb{N}, \mathbb{N})$ is bounded from below by $2^{\aleph_0}$. $\mathrm{Mon}(\mathbb{N}, \mathbb{N})$ is also bounded from above by the set of all function from $\mathbb{N}$ to $\mathbb{N}$, which has cardinality $|\mathbb{N}^{\mathbb{N}}| = |2^{\mathbb{N}}| = 2^{\aleph_0}$. We conclude that $|\mathrm{Mon}(\mathbb{N}, \mathbb{N})| = 2^{\aleph_0} = |\mathbb{R}|$.

3. *Solution*:

   (a) Suppose that $\mathbb{N} \models \exists v \, \mathrm{Drv}(\underline{\#\varphi}, v)$. Hence there exists a number $n \in \mathbb{N}$ such that $\mathbb{N} \models \mathrm{Drv}(\underline{\#\varphi}, \underline{n})$. By virtue of representability, n must be a code of a derivation D whose conclusion is φ. We conclude that $PA \vdash \varphi$.

   (b) Suppose that $\mathbb{N} \not\models \mathrm{Con}(PA)$. Then $\mathbb{N} \models \exists v \, \mathrm{Drv}(\underline{\#\bot}, v)$ and so by (a), $PA \vdash \bot$. But PA is consistent, thus we reach a contradiction. Hence $\mathbb{N} \models \mathrm{Con}(PA)$.

(c) Consider the theory $T := PA \cup \{\mathrm{Con}(PA)\}$. It is consistent, because by (b) we have a model $\mathbb{N}$. It is also clearly recursively enumerable (as PA is) and extends P_0. Furthermore, we have $T \vdash \mathrm{Con}(PA)$.

4. *Solution*:
(=>) Let a be set whose elements are non-empty and pairwise disjoint. Consider the family of maps $(x)_{x\in A}$. By assumption, this is a family of non-empty sets, so by AC we conclude that $\Pi_{x\in a} x = \{f : a \to \cup_{x\in a} x \mid f(x) \in x\}$ is non-empty. Let f be any such map. Define the set $b := f(a)(= \mathrm{ran}(f))$. Then for any element $x \in a$, we see that $f(x) \in b \cap x$. Moreover, if $y \in b \cap x$, we must have an x' such that $y = f(x')$. But then $y \in x'$, and since all elements of a are pairwise disjoint, we must have $x = x'$. Hence $b \cap x = \{f(x)\}$.
(<=) Let $(a_i)_{i\in I}$ be a family of non-empty sets. Consider the set $a := \{\{i\} \times a_i \mid i \in I\}$. All elements of a are non-empty and pairwise disjoint. Therefore there exists a set b such that $b \cap (\{i\} \times a_i)$ is a singleton, for all $i \in I$. Because b might contain some unwanted 'noise', we define $b' = \{(i, x) \in b \mid x \in a_i\}$. Then we see that $b' \subseteq (I \times (\cup_{i\in I} a_i))$ and, for any i, there exists one and only one element in b' such that its first projection is i. Moreover, the second projection of this unique element will be in a_i. We conclude that b' defines a function $I \to \cup_{i\in I} a_i$ such that for all $i \in I$, $b'(i) \in a_i$.

5. *Solution*:

(a) Composition of recursive function is recursive.

(b) Let T be any Turing machine with a states and b bands ($b \geq 1$), and k any number. Let S be the following Turing machine with $a + k + 1$ states and b bands. It uses the first $k - 1$ states (including e_i) to write down the input $k - 1$ on the first band; on a fully blank position, these states will write down a stroke on the first band, move to the right and increase the state by one or $e_i \mapsto e_1$. It uses e_k to write a stroke on the first band, but stay there. If there is already a stroke on the first band it will not change anything and move to the left, remaining in the same state. If the machine is back at the starting position, it changes to state e_{k+1}. Then it will use remaining a states to work as T, where e_i will be replaced by e_{k+1}. If we run this machine on empty input it cannot become stuck as long while it is in its first $k+1$ states, hence it halts if and only if T halts on input k and if either do, it will produce the same output by construction.

(c) Let $f : \mathbb{N} \to \mathbb{N}$ be any recursive function. By (a), we know that $g(n) = f(2n + 1) + 1$ is recursive. Let T be a Turing machine that computes g. Let a be the number of states of T and b be the number of bands. Observe that we can always add dummy states/bands so assume without loss of generality that $a = b$. Because T computes g, it must have at least 1 band. Using (b) with $k = a$ we find a Turing machine S that has $2a + 1$ states and a bands such that $S(0) = T(a)$, because T computes g, which is total. By construction $\mathrm{BB}(2a+1) \geq S(0) = g(a) = f(2a+1)+1 > f(2a+1)$. We conclude that BB cannot be recursive.

6. *Solution*:

(a) Consider the formula

$$F_n := \forall x_1 \dots \forall x_n (\neg(x_n \sim x_1) \vee (\bigvee_{1 \le i < n} (\neg(x_i \sim x_{i+1})))).$$

Then $M \models F_n$ if and only if M does not contain a cycle of length n. Consider the theory $T := \{F_n \mid n \geq 1\}$. We see that $M \models T$ if and only if M is a directed graph that does not contain cycles.

(b) Suppose that the class of directed graphs was axiomatisable by the theory T', i.e. $M \models T'$ iff M is a directed graph that contains at least one cycle. Consider the theory $T'' = T' \cup T$. Then T'' must be inconsistent, as a model would have no cycles as well as at least one. However, any finite subset of T'' is contained in the theory $T' \cup \{F_n \mid n < k\}$ for some k. This theory is consistent, because we have a model with base set $M := \{1, 2, \dots, k\}$ and $\sim^M := \{(n, m) \subseteq M^2 \mid (n = k \wedge m = 1) \vee (m = n + 1)\}$. This contradicts the compactness theorem.

7. *Solution*:

(a) (=>) Suppose that x is a hc set. Because $x \subseteq \mathrm{cl}(x)$ we see that x must by countable. Let $t \in x$. Observe that $t \subseteq \mathrm{cl}(x)$. Since $\mathrm{cl}(t)$ is the smallest transitive set containing t, we must have $\mathrm{cl}(t) \subseteq \mathrm{cl}(x)$. Hence t is also hc.
(<=) Suppose now that x is countable and all its elements are hc. Recall the construction of $\mathrm{cl}(x) := \cup_{n\in\omega} x_n$ where

$$x_0 := x \text{ and } x_{n+1} := x_n \cup \left(\bigcup_{t \in x_n} t \right).$$

We prove by induction on n that for all n, x_n is countable and all its elements are hc. The base case is our initial assumption. Suppose now that x_k is countable and all its elements are hc. Then x_{k+1} is a union of a countable set together with a countable union of sets all of which must be countable, because of (=>). Hence x_{k+1} is countable. Any element $s \in x_{k+1}$ is either in x_k and thus hc or is an element of some $t \in x_k$ and thus must be hc, again because of (=>). Hence the statement indeed holds for all n by induction and we conclude that $\mathrm{cl}(x)$ is countable as it is a countable union of countable sets.

(b) Define the following set

$$z := \{A \in HC \mid \forall \alpha \leq \omega_1,\, A \notin V_\alpha\}.$$

Observe that if $z = \emptyset$, this implies that $HC \subseteq V_{\omega_1}$. Suppose that z is non-empty. By the axiom of foundation we find a set $A \in z$ such that $A \cap z = \emptyset$. In particular, $A \in HC$ and $\forall \alpha \leq \omega_1$, $A \notin V_\alpha$. Let $B \in A$. By (a), we know that $B \in HC$ and hence there must exist some ordinal $\alpha_B \leq \omega_1$ such that $B \in V_{\alpha_B}$. In fact, since $V_{\omega_1} = \bigcup_{\alpha<\omega_1} P(V_\alpha) = \bigcup_{\alpha<\omega_1} V_\alpha$ we can see that we can find $\alpha_B < \omega_1$. However, in (a) we found that A must be countable, so the ordinal $\alpha := \bigcup_{B\in A} \alpha_B$ must be countable. We conclude that $A \subseteq V_\alpha$, and thus $A \in V_{\alpha+1}$ and thus reach a contradiction.

(c) The axiom that does not hold in HC is the axiom of subsets. Observe that $\omega \in HC$ as it is a countable transitive set. If the axiom of subsets would hold in HC, there would be a set $A \in HC$ such that for any $B \in HC$: If for all $t \in B$ we have $t \in \omega$, then $B \in A$. Note that in fact any subset of ω is in HC. Indeed, they are countable and all of their elements are hc. This means that $P(\omega) \subseteq A$, a contradiction as A must be countable.