STOCKHOLMS UNIVERSITET
Matematiska institutionen
Erik Palmgren/Peter LeFanu Lumsdaine
Menno de Boer

Omtentamen/Re-exam
Logic II, MM7022
Fall term 2019
Friday 2020-02-15

Written Exam Logic II

The maximum score on this written exam is 40 points. Grading (after inclusion of bonus points): A requires at least 32/40, B at least 28/40, C at least 22/40, D at least 18/40 and E at least 16/40. The maximum score for each problem is indicated below.

The allowed time for the exam is five hours. No aids are permitted except paper and pen. Write clearly and justify all answers carefully. You may make use of any theorems from the course.

1. (4p) Let the function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = \lfloor \frac{n}{2} \rfloor$; so $f(0) = f(1) = 0$, $f(2) = f(3) = 1$, $f(4) = f(5) = 2$, and so on.

   Show that f is primitive recursive.

   (You may make use of any primitive recursive functions constructed in the course, and facts about them.)

2. (3p) Is the set of recursively enumerable subsets of $\mathbb{N}$ countable? Justify your answer.

3. (6p)

   (a) Let $\mathcal{L}$ be a language, $\mathcal{A}$ an $\mathcal{L}$-structure, and R a binary relation on the underlying set of $\mathcal{A}$. What does it mean to say that R is *definable* in $\mathcal{A}$?

   (b) Show that the standard order relation $<$ is definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$.

   (c) Show that $<$ is not definable in the structure $\langle \mathbb{R}; 0, + \rangle$.

4. (5p) Let $\mathcal{N} = \langle \mathbb{N}; 0, +, \cdot \rangle$ be the standard model of PA. Write $\mathrm{Th}(\mathcal{N})$ for the set of all closed formulas φ of L such that $\mathcal{N} \vDash \varphi$.

   (a) Show that $\mathrm{Th}(\mathcal{N})$ is consistent.

   (b) Show that $\mathrm{Th}(\mathcal{N})$ is complete.

   (c) Show that $\mathrm{Th}(\mathcal{N})$ is not recursively enumerable.

5. (5p) Let $\mathcal{L}$ be any language.

   (a) Show that for any infinite $\mathcal{L}$-structure $\mathcal{A}$, there is some $\mathcal{L}$-structure $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent, but not isomorphic.

   (b) Show that the assumption "$\mathcal{A}$ is infinite" is necessary in part (a). That is, give some language $\mathcal{L}$ and finite $\mathcal{L}$-structure $\mathcal{A}$ such that there is no $\mathcal{B}$ as above exists.

6. (7p) Recall that ordinal exponentiation α^β is defined by induction on β: $\alpha^0 = 1$, $\alpha^{(\gamma^+)} = \alpha^\gamma \cdot \alpha$, and for β non-zero limit, $\alpha^\beta = \sup_{\gamma<\beta} \alpha^\gamma$.

   By induction, one can show that it is (non-strictly) monotone in both arguments: if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, then ${\alpha_1}^{\beta_1} \leq {\alpha_2}^{\beta_2}$.

   (Note: alternatively, α^β is sometimes equivalently defined as the order-type of a certain set of functions. The alternative definition is not required here.)

   (a) Show that $2^\omega = \omega$.

   (b) Show that ω^ω is countable.

   (c) Define a sequence α_i, for $i \in \mathbb{N}$, by setting $\alpha_0 = 0$, $\alpha_{i+1} = \omega^{\alpha_i}$. Define $\varepsilon_0 = \sup_i \alpha_i$. Show that $\omega^{\varepsilon_0} = \varepsilon_0$.

7. (4p) An *antichain* in a poset P is a subset $A \subseteq P$ such that for any $x, y \in A$, neither $x \leq y$ nor $y \leq x$. An antichain is *maximal* if for every $x \in P$, there is some $a \in A$ such that either $x \leq a$ or $a \leq x$.

   Show that every poset P has some maximal antichain.

   (Hint: think about the poset $\mathrm{Ant}(P)$ *of antichains in* P*, ordered by* $\subseteq$*.)*

8. (6p) Show that if ZFC is consistent, then it has some countable model.

   (Not for credit: Notice that ZFC proves the existence of uncountable sets. Think about why that doesn't contradict the existence of countable models of ZFC. This surprising situation is known as Skolem's Paradox.)