STOCKHOLM UNIVERSITY
Department of Mathematics
Examiner: Gregory Arone

Examination for
MM3001: Matematiska metoder för ekonomer
26th November 2020

Time: 13:00-18:00
Instructions:
- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Use natural language when appropriate, not just mathematical symbols.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting $\boxed{\text{A BOX}}$ around it.

Grades: There are 6 problems. A maximum of 5 points can be awarded for each problem solved. At least 15 points are necessary for the grade E. The problems are not ordered according to difficulty.

---

1. Let A be the matrix
$$A = \begin{pmatrix} 2 & 5 & -4 \\ -2 & k-8 & 6 \\ 4 & 9-k & k-7 \end{pmatrix}.$$
depending on the parameter $k \in \mathbb{R}$.

   (a) Compute the determinant $|A|$ as a function of k. (2p)
   Sol: $2(k^2 - 1)$.

   (b) Determine the values of k for which the matrix A is not invertible. (1p)
   Sol: $k \in \{-1, 1\}$.

   (c) Solve the system of linear equations
$$\begin{cases} 2x + 5y - 4z &= 1 \\ -2x - 8y + 6z &= 0 \\ 4x + 9y - 7z &= 2 \end{cases}$$
   in the variables x, y and z using Gaussian elimination. (2p)
   Sol: $(1, -1, -1)$.

2. Consider the function $f(x) = \sqrt[3]{x}$.

   (a) Compute the Taylor polynomial $T_2(x)$ of order 2 of $f(x)$ about $x = 8$. (3p)
   Sol: $2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

   (b) Use the answer from part (a) to estimate $\sqrt[3]{6}$. (2p)
   Sol: $1\frac{59}{72} = 1.819\overline{4}$

3. (a) Compute the primitive $\int \frac{\ln(t)}{5t}\sqrt{1+(\ln(t))^2}\,dt$ (as a function of t). (2p)

   Sol: $\frac{1}{15}(1+\ln(t)^2)^{3/2}$.

   (b) Find a number a for which $\int_1^a \frac{1}{\sqrt{x}} + 1\;dx = 5$ (3p).

   Sol: $a = 4$

4. Consider the function $f(x,y) = e^{xy-y}$ defined on the closed, bounded set

$$D = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x^2 - 2x - 1 \leqslant y \leqslant x - 1\right\}.$$

   (a) Determine the coordinates of the points of intersection of $\{y = x-1\}$ with $\{y = x^2-2x-1\}$. (1p)

   Sol: $(0,-1)$ and $(3,2)$.

   (b) Determine the critical points of f and compute the value of f at those points. (2p)

   Sol: $f(1,0) = 1$ already on the boundary of D.

   (c) Determine the maximal and the minimal values of f on D. (2p)

   Sol: the only critical point on $y = x-1$ is (1,0). On $y = x^2-2x-1$, we find $x = 1 \pm \sqrt{2/3}$. Only $1+\sqrt{2/3}$ is in D with corresponding value $e^{-4\sqrt{2}/(3\sqrt{3})} \equiv 0.337$ for $x = 1+\sqrt{2/3}$ and $e^{-4\sqrt{2}/(3\sqrt{3})} \equiv 2.97$ for $x = 1-\sqrt{2/3}$. $f(0,-1) = e$, $f(3,2) = e^4$. We obtain max $= e^4$ and min $= 0.337$.

5. (a) Compute

$$\lim_{x\to+\infty}\left(\frac{x^2}{x+2} - \frac{x^2+1}{x-3}\right) \quad (3p)$$

   Sol: -5

   (b) Find a number a for which the following function is continuous for all x:

$$f(x) = \begin{cases} ax & x \geq 2 \\ x^2 + a & x < 2 \end{cases} \quad (2p)$$

Sol: $a = 4$

6. Consider the function $f(x) = \dfrac{x}{x^2-3x+2}$.

   (a) Determine the domain of definition of f. (1p)

   Sol: $x \neq 1, 2$.

   (b) Determine the local extreme points of f. (3p)

   Sol: $x = \pm\sqrt{2}$

(c) Determine where f is increasing and where f is decreasing. (1p)
Sol: Decreasing: $(-\infty, -\sqrt{2}), (\sqrt{2}, 2), (2, \infty)$. Increasing: $(-\sqrt{2}, 1), (1, \sqrt{2})$

Formulas

The Taylor polynomial $T_n(x)$ of order n of the function $f(x)$ at $x = x_0$ is

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

The solutions of the equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if $b^2 - 4ac \geqslant 0$.

GOOD LUCK!