MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sven Raum

Tentamensskrivning i
Combinatorics
7.5 hp
13th January 2020

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.
- The exam is returned on 20th January at 11 o'clock in my office 110 in hus 6.

GOOD LUCK!

1. Recursions and generating series (7 points)

   (a) Define the term **generating series**.

   (b) Assume that the generating series of a sequence $(a_n)_n$ of real numbers has positive radius of convergence and denote its generating function by $f = f(x)$. Prove that
   $$\frac{f(x)}{1-x}$$
   is the generating function of
   $$\left(\sum_{k\leq n} a_k\right)_n .$$

   (c) Find the generating function for the sequence $(n^3)_n$. You may freely use knowledge about the generating function for $(n^2)_n$.

   (d) Use the generating series methods to find the generating function $f = f(x)$ of the unique sequence $(a_n)_n$ satisfying
   $$a_n = 2a_{n-1} + n^3 \text{ for } n \geq 1 \qquad \text{and } a_0 = 1.$$

2. Graphs (7 points)

   (a) Define the terms **directed graph** and **undirected graph**.

   (b) Draw a planar depiction of the following graphs:

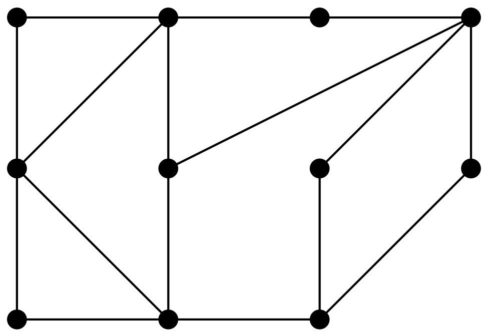
      i. K_4.

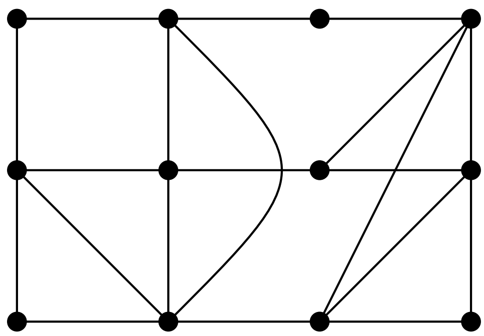
      ii. $K_5 - e$ for an arbitrary edge $e \in E(K_5)$.

      iii. $K_{3,2}$.

      iv. $K_{3,3} - e$ for an arbitrary edge $e \in E(K_{3,3})$.

   (c) Find an Euler circuit in each of the following graphs





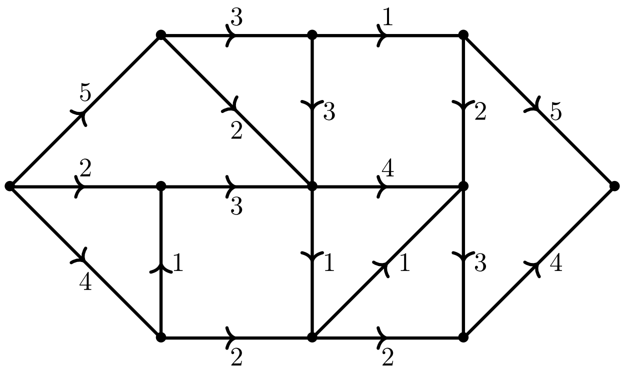
(d) Let G be a graph admitting an Euler circuit. Prove that $\deg(v)$ is even for all $v \in V(G)$.

(e) Calculate the chromatic polynomial of the n-cycle graph for all $n \in \mathbb{N}_{\geq 3}$.

3. Networks (6 points)

(a) Define the term **flow** and the **value of a flow** on a transport network.

(b) Find a maximal flow and a minimal cut of the following transport network:



(c) Let $N = (G, c)$ be a transport network and $f : E(G) \to \mathbb{N}$ a flow on N. Show that for every cut (P, P^{c}) of N the following equality holds:

$$\mathrm{val}(f) = \sum_{v \in P, w \in P^{\mathrm{c}}} f(v, w) - f(w, v).$$

4. Algorithms (4 points)

(a) Define the terms **tree** and **spanning tree**.

(b) Describe how the depth-first algorithm starting at vertex $(0, 0, 0, 0)$ runs on the 4-cube with the lexicographical ordering of vertices.

5. Finite geometry (6 points)

(a) Define the term **finite affine plane**.

(b) Define formally and illustrate with a graphic the examples of the affine planes of rank 2 and 3.

(c) Show that every finite affine plane admits at least three parallelity classes of lines.