MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sven Raum

Tentamensomskrivning i
Combinatorics
7.5 hp
14th February 2020

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.
- The exam is returned on 25th Feburary at 11 o'clock in my office 110 in hus 6.

GOOD LUCK!

1. Enumerative combinatorics (8 points)

   (a) Use the generalised inclusion-exclusion formula to calculate how many integers between 1 and 100 are divisible by exactly three different primes.

   (b) State and prove the pigeon hole principle.

   (c) Show that Euler's ϕ-function satisfies

   $$\phi(n) = n \cdot \prod_{\substack{p \text{ divides } n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right),$$

   for all $n \in \mathbb{N}_{\geq 1}$.

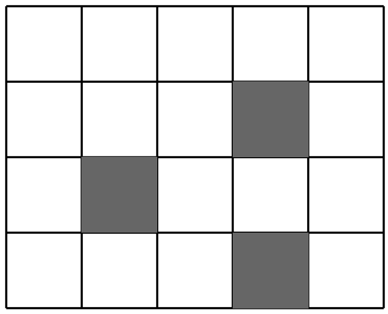
2. Rook polynomials (8 points)

   (a) Let us fix the following formalism for a combinatorial chessboard: a chessboard of size $m \times n$ is a matrix C of size $m \times n$ whose entries are either 0 or 1. We interpret an entry of C equal to 0 as a forbidden field, and an entry equal to 1 as an allowed field.

   Define the **rook numbers** and the **rook polynomial** of a combinatorial chessboard.

   (b) Draw all possible chessboards of size 2×2 and find their rook polynomials.

   (c) Calculate the rook polynomial of the following 4×5 chessboard.

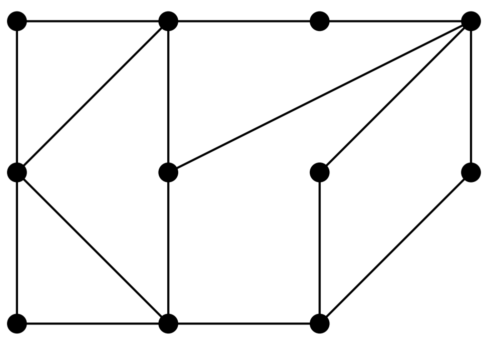


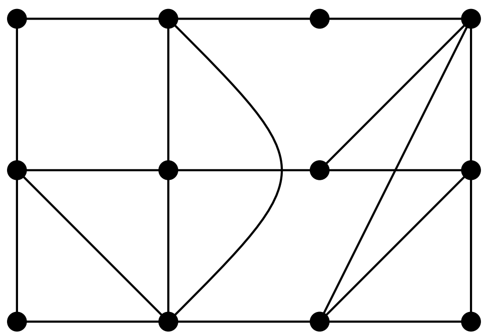
   (d) State formally and prove the fact that the rook polynomial is multiplicative.

3. Graphs (6 points)

   (a) Define a **Hamiltonian cycle** in a graph.

   (b) For each of the following graphs find a Hamiltonian cycle or show that there is none.



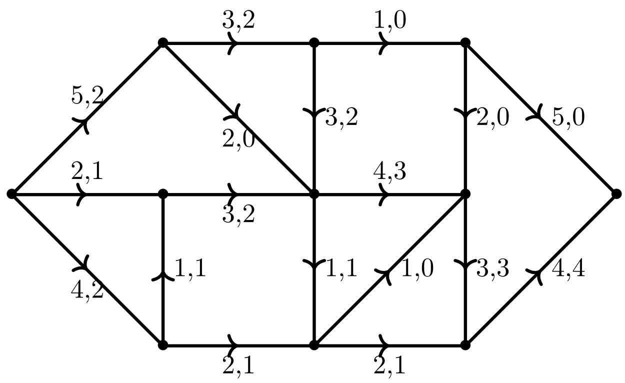


(c) Let G be a graph and $v, w \in V(G)$. Show that if $p = (v_1, \ldots, v_n)$ is a walk from v to w that has minimal length, then p is a path.

4. Networks (4 points)

(a) Let N be a transport network and f a flow on N. Define the term **f-augmenting path**.

(b) For the following flow, find all augmenting paths which are also paths in the underlying directed graph. Explain why you found all.



5. Finite geometry (4 points)

(a) Define the term **parallel** in the context of finite affine planes.

(b) Show that being parallel defines an equivalence relation.