1. (a)

$$\begin{aligned} 3y - z + w &= 1 \\ -x + 2y + 2z \quad &= 0 \\ -x + y - 3z + w &= -1 \end{aligned}$$

Use Gaussian elim:

$$\left[\begin{array}{cccc|c} 0 & 3 & -1 & 1 & 1 \\ -1 & 2 & 2 & 0 & 0 \\ -1 & 1 & -3 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 3 & -1 & 1 & 1 \\ 1 & -2 & -2 & 0 & 0 \\ -1 & 1 & -3 & 1 & -1 \end{array}\right]$$ Multiply ② by (-1) to get leading 1

$$\left[\begin{array}{cccc|c} 0 & 3 & -1 & 1 & 1 \\ 1 & -2 & -2 & 0 & 0 \\ 0 & -1 & -5 & 1 & -1 \end{array}\right]$$ Add ② to ③ to clear col. of that leading 1

$$\left[\begin{array}{cccc|c} 0 & 0 & -16 & 4 & -2 \\ 1 & 0 & 8 & -2 & 2 \\ 0 & 1 & 5 & -1 & 1 \end{array}\right]$$ Subtract $3\times$③ from ① & add $2\times$③ to ② to clear 2nd col.

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{8} \\ 1 & 0 & 8 & -2 & 2 \\ 0 & 1 & 5 & -1 & 1 \end{array}\right]$$ Multiply ① by $-\frac{1}{16}$ to get leading 1

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{8} \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{3}{8} \end{array}\right]$$ Subtract $8\times$① from ② & $5\times$① from ③ to clear column 3.

I.e. the original system is equivalent to:

$$\begin{aligned} z - \tfrac{1}{4}w &= \tfrac{1}{8} \\ x \quad &= 1 \\ y \quad + \tfrac{1}{4}w &= \tfrac{3}{8} \end{aligned}$$

1(a) cont'd So the general solution is

$$\boxed{\left.\begin{aligned} x &= 1 \\ y &= 3/8 - 1/4\, t \\ z &= 1/8 + 1/4\, t \\ w &= t \end{aligned}\right\} \text{ for any } t \in \mathbb{R}.}$$

(b) Again, use G.E:

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & -2 \\ 2 & 2 & 4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & -1 \\ 2 & 2 & 4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & -1 \\ 0 & 3 & 6 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\begin{aligned} A \qquad\qquad &= -\tfrac{1}{2} \\ B + 2C &= 1 \end{aligned}$$

$$\boxed{\left.\begin{aligned} A &= \tfrac{1}{2} \\ B &= 1 - 2t \\ C &= t \end{aligned}\right\} \text{ for any } t \in \mathbb{R}}$$

2 (a) $\lim_{x\to 0} \frac{x^2+1}{x+2} = \frac{0^2+1}{0+2} = \boxed{\frac{1}{2}}$

(b) $\lim_{x\to\infty} \frac{x^2+1}{x+2} + \frac{x^2+2}{x+1} = \lim_{x\to\infty} \frac{x^2+1}{x+2} + \lim_{x\to\infty} \frac{x^2+2}{x+1}$

$= \lim_{x\to\infty} \frac{x+1/x}{1+2/x} + \lim_{x\to\infty} \frac{x+2/x}{1+1/x}$

$\text{“}=\text{''} \frac{\infty+\cancel{1/\infty}}{1+2/\infty} + \frac{\infty+2/\infty}{1+2/\infty}$

$\text{“}=\text{''} \frac{\infty}{1} + \frac{\infty}{1}$

$\text{“}=\text{''} \infty + \infty \text{“}=\text{''} \boxed{\infty}$

(c) $\underbrace{\lim_{x\to\infty} \frac{x^2+1}{x+2} - \frac{x^2+2}{x+1}} = \lim_{x\to\infty} \frac{x^2+1}{x+2} - \lim_{x\to\infty} \frac{x^2+2}{x+1}$

found in (b)

$\text{“}=\text{''} \infty - \infty$

<u>Indeterminate!</u>

So simplify the original.

$= \lim_{x\to\infty} \frac{(x^2+1)(x+1) - (x^2+2)(x+2)}{(x+2)(x+1)} = \lim_{x\to\infty} \frac{(x^3+x^2+x+1) - (x^3+2x^2+2x+4)}{(x+2)(x+1)}$

$= \lim_{x\to\infty} \frac{-x^2-x-3}{x^2+3x+2}$

$= \lim_{x\to\infty} \frac{-1-1/x-3/x^2}{1+3/x+2/x^2} = \frac{-1-0-0}{1+0+0} = \boxed{-1}$

3. (a) $$A = \begin{pmatrix} 2 & 0 & 2+k \\ 3 & -1 & 0 \\ k & 0 & -2 \end{pmatrix}$$

Using middle col:

$$\det A = -0\begin{vmatrix} 3 & 0 \\ k & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 2+k \\ k & -2 \end{vmatrix} - 0\begin{vmatrix} 2 & 2+k \\ 3 & 0 \end{vmatrix}$$

$$= (-1)\left(-4 - k(2+k)\right)$$

$$\boxed{\det A = k^2 + 2k + 4}$$

or: $= (k+1)^2 + 3$ (completing the square)

(b) By (a), $\det A = (k+1)^2 + 3$
$\geq 3 > 0$

so $\det A \neq 0$ for any k,
so A is invertible.

(c) Check stat pts + endpoints of $[-2, 2]$:

stat pts: $\frac{d}{dk}(\det A) = 0$
$2k + 2 = 0$
$k = -1$

k	$\det A$
-2	4
-1	3
2	12

so minimum is 3, for $k = -1$
maximum is 12, for $k = 2$.

4. $h(x,y) = (xy + y^2)e^x$ Partial derivs:

$$h_x(x,y) = y e^x + (xy+y^2)e^x = (xy+y^2+y)e^x$$

$$h_y(x,y) = (x+2y)e^x$$

Stat pts: $(xy+y^2+y)e^x = 0$,
$(x+2y)e^x = 0$.

Since $e^x \neq 0$ always, this is equiv to:

① $xy+y^2+y = 0$

② $x+2y = 0$

② gives $x=-2y$. Subst. in ①:

$$(-2y)y + y^2 + y = 0$$

$$-y^2 + y = 0$$

$$(1-y)y = 0$$

$$y=0 \quad \underline{\text{or}} \quad y=1$$

Now $x=-2y$ gives: $x=0$ $\quad$ $x=-2$

So $\boxed{\text{stat pts are } (0,0) \text{ and } (-2,1)}$

4 cont'd. Classify by finding sign of det of Hessian:

$$H(x,y) = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} (xy+y^2+2y)e^x & (x+2y+1)e^x \\ (x+2y+1)e^x & 2e^x \end{pmatrix}$$

$$|H(x,y)| = (e^x)^2 \begin{vmatrix} xy+y^2+2y & x+2y+1 \\ x+2y+1 & 2 \end{vmatrix}$$

$$|H(0,0)| = (e^0)^2 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = (0\cdot 2 - 1\cdot 1) = -1 < 0$$

so $\boxed{(0,0) \text{ is a saddle pt. of } h}$

$$|H(-2,1)| = (e^{-4})^2 \begin{vmatrix} -2+1+2 & -2+2+1 \\ -2+2+1 & 2 \end{vmatrix}$$

$$= (1\cdot 2 - 1\cdot 1)e^{-4} = e^{-4} > 0$$

& $h_{xx}(-2,1) = (-2+1+2)e^{-2}$
$= e^{-2} > 0$

so $\boxed{(-2,1) \text{ is a local minimum of } h}$

5 (a) $\int (t+1)e^{t-2}\,dt$

Int. by parts: $\begin{cases} u = t+1 & \frac{dv}{dt} = e^{t-2} \\ \frac{du}{dt} = 1 & v = e^{t-2} \end{cases}$

$= (t+1)e^{t-2} - \int 1 \cdot e^{t-2}$

$= (t+1)e^{t-2} - e^{t-2} + C$

$\boxed{= te^{t-2} + C}$

(b) $\int_1^2 (x+1)(3x-3)\,dx = \int_1^2 3x^2 - 3\,dx$

$= \left[x^3 - 3x\right]_1^2$

$= (8-6) - (1-3)$

$= 2 - (-2) = \boxed{4}$

6. (a) Profit = revenue − cost

$$P(x,y) = 5x + 5y - C(x,y)$$
$$= -500 + 2x + 3y - \frac{2}{1000}x^2 - \frac{1}{1000}y^2 - \frac{2}{1000}xy$$

(b) Partial derivs 0 at stat pt:

$$P_x(x,y) = 2 - \frac{4}{1000}x - \frac{2}{1000}y = 0 \qquad (1)$$
$$P_y(x,y) = 3 - \frac{2}{1000}y - \frac{2}{1000}x = 0 \qquad (2)$$

$$(1) \Leftrightarrow \quad 2000 = 4x + 2y \qquad (1a)$$
$$(2) \Leftrightarrow \quad 3000 = 2y + 2x \qquad (2a)$$

(2a) gives: $y = 1500 - x$

Subst. in (1a): $2000 = 4x + 2(1500 - x)$

$$-1000 = 2x$$
$$x = -500$$
$$y = 1500 - x = 2000.$$

So $(-500, 2000)$ is the only stat. pt.

(c) At $(0, 1500)$:

$$C_x(x,y) = 3 + \frac{4}{1000}x + \frac{2}{1000}y = 3 + 0 + \frac{2 \cdot 1500}{1000}$$
$$= 3 + 3 = 6$$

$$C_y(x,y) = 2 + \frac{2}{1000}y + \frac{2}{1000}x = 2 + \frac{2 \cdot 1500}{1000} + 0$$
$$= 2 + 3 = 5$$

So marginal cost is 6€ per bowl, 5€ per plate.

7 Curve $5x^2+5y^2+6xy=4$ ①

(a) $x+y=0 \Leftrightarrow y=-x$.

So ① then gives $5x^2+5(-x)^2+6x(-x)=4$

$$(5+5-6)x^2=4$$

$$x^2=1$$

$$x=\pm 1$$

$$(\& \ y=-x)$$

$\boxed{\text{So the curve intersects } x+y=0 \text{ in the points } (1,-1)\,,\ (-1,1).}$

(b) Write $F(x,y)=5x^2+5y^2+6xy$,

so ① is $F(x,y)=4$.

Slope of tangent to level curve is

given by $-\dfrac{F_x(x,y)}{F_y(x,y)}$

Here $F_x(x,y)=10x+6y$

$F_y(x,y)=10y+6x$

So at $(\frac{1}{2},\frac{1}{2})$, slope is $-\dfrac{10\cdot\frac{1}{2}+6\cdot\frac{1}{2}}{10\cdot\frac{1}{2}+6\cdot\frac{1}{2}}$

$$=-1.$$

7 (b) cont'd So tangent is the line thru $(\frac{1}{2},\frac{1}{2})$ with slope -1:

$$y = \frac{1}{2} + (-1)\left(x - \frac{1}{2}\right)$$

$$\boxed{y = 1 - x}$$